A personal survey on Gersten resolutions and Bloch's formula

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1. Introduction

Let X be a quasiprojective smooth variety over k. S. Bloch proved that $CH^2(X) \simeq H^2(X, \mathcal{K}_2)$ and later D. Qullien generalized this result for all $CH^n(X)$ using his K-theory, to $CH^n(X) \simeq H^n(X, \mathcal{K}_n)$. The idea of the proof was to use a certain resolution of the K-sheaf $\mathcal{K}_n$ by flasque (in fact, constant) sheaves, and from the resolution, he read off the n-th cohomology which was exactly the Chow groups. The resolution is nowadays called the Gersten resolution (or Gersten-Quillen resolution) of $\mathcal{K}_n$. From the proof, many similar resolutions were invented. The purpose of this personal survey note is to collect all informations about those machines and some generalizations of Bloch's formula to Bloch's higher Chow groups. Unless otherwise specified, X is always a quasiprojective smooth variety over k. This note may be updated constantly.

2. Review on exact couples

2.1. **Brief review of definition.** Let $\mathcal{A}$ be an abelian category and $D_1, E_1 \in \mathrm{Ob}\mathcal{A}$. Recall that a diagram

$$\begin{tikzcd}
D_1 \arrow[rr, "b_1"] & & D_1 \arrow[dl, "c_1"] \\
& E_1 \arrow[ul, "a_1"] &
\end{tikzcd}$$

ica called an *exact couple* if it is exact at each vertex. We define $d_1 = c_1 \circ a_1 : E_1 \to E_1$ then, $d_1^2 = 0$. Let $D_2 = \mathrm{im} b_1$ and induce a_2, b_2, c_2 naturally on D_2 and $E_2 = \ker d_1 / \mathrm{coker} d_1$ then, we have another exact couple

$$\begin{tikzcd}
D_2 \arrow[rr, "b_2"] & & D_2 \arrow[dl, "c_2"] \\
& E_2 \arrow[ul, "a_2"] &
\end{tikzcd}$$

which is called the *derived couple* of the 1st one. Inductively, we let $D_n = \mathrm{im} b_1^{n-1} \simeq D_1 / \ker b_1^{n-1}$ and $E_n = a_1^{-1}(\mathrm{im} b_1^{n-1}) / c_1(\ker b_1^{n-1})$, then it forms an exact couple.

2.2. **Bigraded exact couple and the spectral sequence.** Let $\{A^{m,n}\}$, $\{E^{m,n}\}$ be indexed by $\mathbb{Z} \times \mathbb{Z}$ with long exact sequences for all $p \in \mathbb{Z}$:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{p+1,q-1} \arrow[r, "f", "(-1{,}1)"'] & A^{p,q} \arrow[r, "g", "(0{,}0)"'] & E^{p,q} \arrow[r, "h", "(1{,}0)"'] & A^{p+1,q} \arrow[r, "f", "(-1{,}1)"'] & A^{p,q+1} \arrow[r] & \cdots .
\end{tikzcd}$$

Let $D_1 = \oplus A^{p,q}$, $E_1 = \oplus E^{p,q}$ with $a_1 = h$, $b_1 = f$, $c_1 = g$. Obviously, a_1 has bidegree $(1,0)$, b_1 has $(-1,1)$, and c_1 has $(0,0)$. Hence for D_n and E_n, a_n has bidgree $(1,0)$, b_n has $(-1,1)$ and c_n has $(n-1, 1-n)$. and

$$\begin{tikzcd}
\cdots \arrow[r] & D_n^{p-n+2,q+n-2} \arrow[r] & D_n^{p-n+1,q+n-1} \arrow[r] & E_n^{p,q} \arrow[r] & D_n^{p+1,q} \arrow[r] & D_n^{p,q+1} \arrow[r] & \cdots .
\end{tikzcd}$$

Notice that we have

$$A^{n-q,q} \to A^{n-q-1,q+1} \to A^{n-q-2,q+2} \to \cdots .$$

Let $A^n = \varinjlim_q A^{n-q,q}$ and let $F^pA^n = \mathrm{im}(A^{p,n-p} \to A^n)$. Assume that $A^{n-q+1,q-1} \to A^{n-q,q}$ is stable for all sufficiently big q and $A^{n-q,q} = 0$ for all q sufficiently small. Then, for a sufficiently large n, $E_n^{p,q} \simeq E_\infty^{p,q} \simeq F^pA^{p+q}/F^{p+1}A^{p+q}$ and $E_r^{p,q} \Rightarrow A^{p+q}$. We can prove it easily using the boundedness on $A^{p,q}$.

3. BGQ-spectral sequence, Gersten resolution and K-theory

In this section, we begin with a scheme X.

Theorem 3.1 (BGQ(Brown-Gersten-Quillen)-spectral sequence)**.** *Let X be a scheme and let X^p be the set of points of codimension p in X. Then, there is a spectral sequence*

$$E_1^{p,q}(X) = \coprod_{x \in X^p} K_{-p-q}k(x) \Rightarrow K'_{-n}(X)$$

which is convergent when X has a finite dimension.

Remark. we interpret $K_n = 0$ for $n < 0$, so that the spectral sequence is concentrated in the rage $p \geq 0$ and $p + q \leq 0$. It lies in the 4-th quadrant.

Proof. Let $\mathcal{M}$ be the category of coherent $\mathcal{O}_X$-modules. let $\mathcal{M}^i \subset \mathcal{M}$ be the full subcategory of sheaves whose support has codim $\geq i$. Then,

$$\mathcal{M}^i/\mathcal{M}^{i+1} \simeq \coprod_{x \in X^i} \bigcup_n Mod(\mathcal{O}_{X,x}/m_{X,x}^n).$$

Any finite $\mathcal{O}_{X,x}/m_{X,x}^n$-module admits a finite filtration with successive quotient $\mathcal{O}_{X,x}/m_{X,x} \simeq k(x)$, so that by the Dévissage theorem (of Jordan-Hölder type), we have

$$K_n(\mathcal{M}^i/\mathcal{M}^{i+1}) \simeq \coprod_{x \in X^i} K_n(k(x)).$$

Now, from the localization sequence, we have

$$\begin{tikzcd}
\cdots \arrow[r] & K_n(\mathcal{M}^{i+1}) \arrow[r] & K_n(\mathcal{M}^i) \arrow[r] & \coprod_{x \in X^i} K_n(k(x)) \arrow[r] \arrow[d, "=" swap] & K_{n-1}(\mathcal{M}^{i+1}) \arrow[r] \arrow[dll] & \cdots . \\
\cdots \arrow[r] & K_{n-1}(\mathcal{M}^{i+2}) \arrow[r] & K_{n-1}(\mathcal{M}^{i+1}) \arrow[r] & \coprod_{x \in X^{i+1}} K_n(k(x)) \arrow[r] & K_{n-2}(\mathcal{M}^{i+2}) \arrow[r] & \cdots
\end{tikzcd}$$

We want the localization sequence to give a spectral sequence through an exact couple of double complexes, i.e. we want it to look like:

$$\begin{tikzcd}
A^{p+1,q-1} \arrow[r] \arrow[d, "=" swap] & A^{p,q} \arrow[r] \arrow[d, "=" swap] & E_1^{p,q} \arrow[r] \arrow[d, "=" swap] & A^{p+1,q} \arrow[r] \arrow[d, "=" swap] & A^{p,q+1} \arrow[r] \arrow[d, "=" swap] & \cdots . \\
K_n(\mathcal{M}^{i+1}) \arrow[r] & K_n(\mathcal{M}^i) \arrow[r] & \coprod_{x \in X^i} K_n(k(x)) \arrow[r] & K_{n-1}(\mathcal{M}^{i+1}) \arrow[r] & K_{n-1}(\mathcal{M}^i) \arrow[r] & \cdots
\end{tikzcd}$$

Taking $i = p$ looks like a natural choice. We want $A^{p,q} = K_{n(p,q)}(\mathcal{M}^p)$. Let $n(p,q) = ap + bq$ with $a, b \in \mathbb{Z}$. We want to determine a and b.

Compare the 1st and the 2nd columns. Then, $n(p+1, q-1) = n(p,q) \Leftrightarrow ap + a + bq - b = ap + bq \Leftrightarrow a = b$.

Now, compare the 3rd and the 4th columns. Then, $n(p+1, q) = n(p,q) - 1 \Leftrightarrow ap + a + aq = ap + aq - 1 \Leftrightarrow a = -1$. Hence $n = -p - q$ so that

$$\left\{ \begin{array}{l} A^{p,q} = K_{-p-q}(\mathcal{M}^p(X)) \\ E_1^{p,q} = \coprod_{x \in X^p} K_{-p-q}(k(x)) \end{array} \right. .$$

Then, $A^{-n} = \varinjlim_q A^{-n-q,q} = \varinjlim_q K_n(\mathcal{M}^{-n-q}(X)) = K_n(\mathcal{M}(X)) = K'_n(X)$. Hence, via the technic of exact couples, we have

$$E_1^{p,q} = \coprod_{x \in X^p} K_{-p-q}(k(x)) \Rightarrow K'_{p+q}(X)$$

as desired.

□

Remark. The filtration on $K'_{p+q}(X)$ was defined as

$$F^p K'_{p+q}(X) = F^p A^{-n} = \operatorname{im}(A^{p,n-p} \to A) = \operatorname{im}(K_{-n}(\mathcal{M}^p(X)) \to K_{-n}(X))$$

which is the filtration by the codimension of the support, or the coniveau filtration.

Remark. The above procedure can be written in terms of the language of spectral sequence for filtered objects as well. Instead of working with exact couples, we could have begun with the coniveau filtration of categories:

$$\mathcal{M}(X) \supset \mathcal{M}^1(X) \supset \mathcal{M}^2(X) \supset \mathcal{M}^3(X) \supset \cdots$$

and follow the usual trick with K_n playing the role of the cohomology functor. Then, we will have $E_0 \simeq gr\mathcal{M}(X) = \mathcal{M}^p(X)/\mathcal{M}^{p+1}(X)$ and $E_1 \simeq K_{-n}(gr\mathcal{M}(X))$ and $E_\infty = grK_{-n}(\mathcal{M}(X))$, as usual.

Corollary 3.2. *The followings are equivelent:*

1. *For all $p \geq 0$, $\mathcal{M}^{p+1}(X) \hookrightarrow \mathcal{M}^p(X)$ induces 0 on K-groups.*
2. *For all q, $E_2^{p,q}(X) = 0$ if $p \neq 0$ and the edge homomorphism $K'_{-q}(X) \to E_1^{0,q}(X)$ is an isomorphism.*
3. *For all n,*
$$0 \to K'_n(X) \xrightarrow{e} \coprod_{x \in X^0} K_n k(x) \xrightarrow{d_1} \coprod_{x \in X^1} K_{n-1} k(x) \xrightarrow{d_1} \cdots$$
is exact, where e is obtained by the pullback along $\operatorname{Spec} k(x) \to X$.

The proof is obvious. The point is that, however, very rarely, one of the above cases is true, even for affine cases. But, we use it as a *local condition* so that if it is true locally, then we can expect the corresponding sequence for sheafified one might be true, and indeed they are for some cases:

Theorem 3.3 (Gersten)**.** *Assume that above Corollary is true for* $\operatorname{Spec}(\mathcal{O}_{X,x})$ *for all $x \in X$. Then, there is a canonical isomorphism*

$$E_2^{p,q}(X) \simeq H^p(X, \mathcal{K}'_{-q})$$

where $\mathcal{K}'_{-q}$ is the sheaf associated to the presheaf $U \mapsto K'_{-q}(U)$.

Proof. We have a sequence of sheaves

$$0 \to \mathcal{K}'_n(X) \to \coprod_{x \in X^0} i_* K_n(k(x)) \to \coprod_{x \in X^1} i_* K_{n-1}(k(x)) \to \cdots$$

where $i_* : \operatorname{Spec} k(x) \hookrightarrow X$. The stalk

$$\mathcal{K}'_n(X)_x = \varinjlim_{x \in U} \mathcal{K}'_n(U) = \mathcal{K}'_n(\varprojlim_{x \in U} U) = \mathcal{K}'_n(\mathcal{O}_{X,x}),$$

and so, by assumption, the above sequence is an exact sequence of sheaves. But, it is a flasque resolution, so that we can use it to compute the cohomologies of the sheaf $\mathcal{K}'_n(X)$:

$$H^p(X, \mathcal{K}'_n(X)) \simeq H^p(\Gamma(X, \coprod_{x \in X^*} i_* K_{n-*}(k(x)))) = H^p(E_1^{*,-n}(X)) = E_2^{p,-n}(X).$$

□

Hence, whether local rings $\mathcal{O}_{X,x}$ satisfy them or not matters. The Gersten's conjecture claims that it is true for any regular local rings. Quillen gave a proof for some special cases:

Theorem 3.4 (Quillen)**.** *Let k be a field, and let R be a regular local k-algebra of geometric type, i.e. a localization of a k-algebra of finite type. Then, the Gersten's conjecture is true for R.*

Corollary 3.5. *Let X be a regular k-scheme of finite type. Then, we have the Gersten resolution of the K-sheaf.*

Remark. Note that in case X is regular, $K'_n(X) \simeq K_n(X)$, so that we can replace K' by K.

The proof of Quillen's result uses some kind of normalization lemma for smooth k-algebras. I'll do it later.

Theorem 3.6 (Quillen)**.** *For $k[[X_1, \cdots, X_n]]$ we have the Gersten resolution. In case k has a nontrivial valuation and k is complete with respect to the valuation, then, for the ring $k\{X_1, \cdots, X_n\}$ of convergeant power series, we also have the Gersten resolution.*

4. BGQ-spectral sequence, Gersten resolution and various cohomology theories

Let $H^*(X)$ be any of the following cohomology theories:

1. $k = \mathbb{C}$ and $H^*(X) = H^*(X, A)$, the singular cohomology (with complex topology) with coefficient in A, an abelian group.
2. char $k = 0$ and $H^*(X) = H^*_{DR}(X)$ the de Rham cohomology in the sense of Hartshorne, *On the cohomology of algebraic varieties*, Publ. Math. IHES, 1976.
3. n is prime to char k, $H^*(X) = H^*_{et}(X, \mu_n^{\otimes r})$, where $\mu_n^{\otimes r}$ is the r-th twist of the sheaf μ_n of n-th roots of unity. we will choose a n-th root ξ and regard $\mu_n^{\otimes r} \simeq \mathbb{Z}/n\mathbb{Z}$, then, later we state the result keeping track of the twist by r.

Recall the notion of cohomology with support in Z, or the relative cohomology $H^r_Z(X)$ or $X^*(X, X-Z)$, and the relative long exact sequence:

$$\cdots H^*_Z(X) \to H^r(X) \to H^r(X-Z) \to X^{r+1}_Z(X) \to \cdots,$$

which exists for all of above cases.

Remark. Typical application of Grothendieck duality theory (see Verdier, J.L. *A duality theorem in the étale cohomology of schemes*, Proc. of a Conference on local fields, Springer 1967, and Verdier, J.L *Dualitité dans la cohomologie des espaces localement compacts*, Sém. Bourbaki. exposé, no. 300 (1965)) is that, if X is smooth over k of dimension n, then for any of above cohomology theories, if we define

$$H_r(Z) := H^{2n-r}_Z(X),$$

then, $H_r(Z)$ is independent of X and contravariant for proper maps $Z \to Z'$.

In fact, $H_*(Z)$ is a *Borel-Moore homology theory.* (See Verdier's 2nd one), in the sense that if Z is *smooth* itself of dimension d (not necessarily complete), then, $H_r(Z) \simeq H^{2d-r}(Z)$ (simply taking $Z = X$ in above). It is also called the Thom isomorphism theorem for the complex case, and the Purity for the étale case.

Exercise 4.1. *Let $Z' \subset Z \subset X$ be a sequence of closed subsets, X is smooth. Then, show that*

$$\cdots \to H_r(Z') \to H_r(Z) \to H_r(Z-Z') \to H_{r-1}(Z') \to \cdots$$

is exact.

Define $H^*_{Z^p}(X) := \varinjlim_{Z\subset X, \mathrm{codim} Z\geq p} H^*_Z(X)$.

For the sequence

$$\cdots \to H_{2n-r}(Z') \to H_{2n-r}(Z) \to H_{2n-r}(Z-Z') \to H_{2n-r-1}(Z') \to \cdots$$

we apply the limit

$$\varinjlim_{(Z',Z),Z:\text{pure codim i in } X, \mathrm{codim}_Z Z'\geq 1}.$$

Then, for $H_{2n-r}(Z')$, it can be computed as $\varinjlim_{Z'\subset X, \mathrm{codim}_X Z'\geq i+1}$ so that the limit is $H^r_{Z^{i+1}}(X)$. For $H_{2n-r}(Z)$, Z' doesn't play any role, so that we can see the limit as $\varinjlim_{Z\subset X \mathrm{codim}_X Z\geq i}$ because any codimension $\geq i$ subvarieties are contained in a union of varieties of pure codimension i, so that the limit is $H^r_{Z^i}(X)$. For $H_{2n-r}(Z,Z')$, we can see it as $\varinjlim_{Z\subset X, \text{pure codim}=i} \varinjlim_{Z'\subset Z, \mathrm{codim}\geq 1}$ and for each Z, the singular locus of Z lies in some Z' with codimension ≥ 1, so that as $\dim Z = n-i$,

$$\varinjlim_{Z'\subset Z, \mathrm{codim}\geq 1} H_{2n-r}(Z-Z') = H^{2(n-i)-(2n-r)}(k(Z)) = H^{r-2i}(k(Z)),$$

where $H^*(k(x)) = \varinjlim_{U\subset Z, \text{open}} H^*(U)$, so that over all $Z\subset X$ of pure codimension i, the limit is going to be

$$\coprod_{x\in X^i} H^{r-2i}(k(x)),$$

so that $\varinjlim$ being exact, we have

$$\begin{tikzcd}
\cdots \arrow[r] & H^r_{Z^{i+1}}(X) \arrow[r] & H^r_{Z^i}(X) \arrow[r] & \coprod_{x\in X^i} H^{r-2i}(k(x)) \arrow[r] \arrow[d, "=", swap] & H^{r+1}_{Z^{i+1}}(X) \arrow[r] \arrow[dll] & \cdots . \\
\cdots \arrow[r] & H^{r+1}_{Z^{i+2}}(X) \arrow[r] & H^{r+1}_{Z^{i+1}}(X) \arrow[r] & \coprod_{x\in X^{i+1}} H^{r-2i-1}(k(x)) \arrow[r] & H^{r+2}_{Z^{i+2}}(X) \arrow[r] & \cdots
\end{tikzcd}$$

To see it as a part of an exact couple, compare it as follows:

$$\begin{tikzcd}
A^{p+1,q-1} \arrow[r] \arrow[d, "=", swap] & A^{p,q} \arrow[r] \arrow[d, "=", swap] & E_1^{p,q} \arrow[r] \arrow[d, "=", swap] & A^{p+1,q} \arrow[r] \arrow[d, "=", swap] & A^{p,q+1} \arrow[d, "=", swap] & . \\
H^r_{Z^{i+1}}(x) \arrow[r] & H^r_{Z^i}(X) \arrow[r] & \coprod_{x\in X^i} H^{r-2i}(k(x)) \arrow[r] & H^{r+1}_{Z^{i+1}}(X) \arrow[r] & H^{r+1}_{Z^i}(X) &
\end{tikzcd}$$

Here, it is reasonable to take $i=p$. For r, let $r(p,q) = ap+bq$, $a,b\in\mathbb{Z}$ so that $A^{p,q} = H^{r(p,q)}_{Z^i}(X)$. Then, from the 1st and the 2nd columns, $r(p+1,q-1) = r(p,q)$ implies $a=b$. From the 1st and the 4th columns, we have $r(p+1,q) = r(p+1,q-1)+1$ so that $a = a-a+1$ i.e. $a=1$. Hence, $r(p,q) = p+q$. Hence, we have

$$\begin{cases} A^{p,q} = H^{p+q}_{Z^p}(X) \\ E_1^{p,q} = \coprod_{x\in X^p} H^{q-p}(k(x)) \end{cases}.$$

Note that

$$A^n = \varinjlim_q A^{n-q,q} = \varinjlim_q (H^n_{Z^{n-q}}(X)) = H^n_{Z^0}(X) = H^n(X)$$

and

$$F^pH^n(X) = F^pA^n = \operatorname{im}(A^{p,n-p} \to A^n) = \operatorname{im}(H^n_{Z^p}(X) \to H^n(X))$$

which is the filtration by codimension, i.e. $a \in F^pA^n$ iff there is $Z \subset X$ of codimsion p such that $a \mapsto 0$ in $H^n(X - Z)$. This filtration is called the *coniveau* filtration, and can also be written as $\cup \ker\{H^i(X) \to H^i(X - Z) | Z \subset X, \text{ closed, codim } p\}$.

Hence, by the general theory of exact couples, we have a spectral sequence

$$E_1^{p,q} = \coprod_{x \in X^p} H^{q-p}(k(x)) \Rightarrow H^{p+q}(X)$$

so that we have a similar type of spectral sequence. We state it as follows:

Theorem 4.2. *For any reasonable cohomology theories $H^*(X)$, where the Grothendieck duality theorem holds, we have the following spectral sequence*

$$E_1^{p,q} = \coprod_{x \in X^p} H^{q-p}(k(x)) \Rightarrow H^{p+q}(X)$$

where the target has the coniveau filtration, i.e. the filtration by codimension.

As before, we let $\mathcal{H}^q$ be the sheaf associated to the presheaf $U \mapsto H^q(U)$. Then, do we have the Gersten resolution for $\mathcal{H}^q$? The answer is yes:

Theorem 4.3. *Let X be a smooth variety over a field k. Then,*

$$0 \to \mathcal{H}^q \to \coprod_{x \in X^0} i_*H^q(k(x)) \to \coprod_{x \in X^q} i_*H^{q-1}(k(x)) \to \cdots \to \coprod_{x \in X^q} i_*H^0(k(x)) \to 0$$

is a resolution of $\mathcal{H}^q$ by constant (hence flasque) sheaves.

The proof is similar to Qullen's one for K-theory. See Bloch's Algebraic Cycles. It will be updated later.

Corollary 4.4. *We have*

$$H^p(X, \mathcal{H}^q) \simeq E_2^{p,q}(X).$$

In particular, $E_2^{p,q} = 0$ for $p > q$.

Proof. $H^p(X, \mathcal{H}^q) = H^p(\Gamma(\coprod_{x \in X^*} i_*H^{q-*}(k(x)))) = H^p(E_1^{*,q}(X)) = E_2^{p,q}(X)$ and the resolution of $\mathcal{H}^q$ has length q, so that there is no cohomology for $p > q$. □

Remark. For the étale or the singular cohomologies, we have finer topologies on X than the Zariski topology, so that we have a natural morphism of sites $\pi : X_\cdot \to X_{Zar}$, where $\cdot$ is either the étale or the complex topology. Then, from the Leray spectral sequence, we have

$$E_2^{p,q} = H^p_{Zar}(X, R^q\pi_*) \Rightarrow H^{p+q}_\cdot(X).$$

This spectral sequence coincides with the above one. See Bloch, S., Ogus, A., *Gersten's conjecture and the homology of schemes*

5. BGQ-spectral sequence, Gersten resolution and Higher Chow group

Let X be a quasiprojective scheme over k. Recall that Higher Chow group $CH^r(X, n)$ satisfies the following localization theorem.

Theorem 5.1 (Localization). *Let $Z' \subset Z$ be a closed subset of X, with* $\operatorname{codim}_Z Z' = d$. *Then,*

$$\cdots \to CH^{r-d}(Z', n) \to CH^r(Z, n) \to CH^r(Z - Z', n) \to CH^{r-d}(Z', n-1) \cdots$$

is exact.

We can either give a filtration on the complex $\mathcal{Z}^r(X,n)$ from the begining and form a spectral sequence for the filtration, or, we follow the way did before to use an exact couple. We are going to apply the method we used for cohomology theories. For the filtration, I'll probably update it sometime later, when time permits.

Let $H^r_p(X,n) := \varinjlim_{Z\subset X, \mathrm{codim}_X Z = p} CH^r(Z,n)$. To above localization sequence with $d=1$, we apply $\varinjlim_{(Z',Z), Z:\text{ pure codim p }, \mathrm{codim}_Z Z'=1}$.

Then, for $CH^{r-d}(Z',n)$, the limit acts like taking $\varinjlim_{Z'\subset X, \mathrm{codim}_X Z' = p+1}$ so that, we obtain $H^{r-1}_{p+1}(X,n)$.

For $CH^r(Z,n)$, the limit acts like taking $\varinjlim_{Z\subset X, \mathrm{codim}_X Z = p}$ so that we obtain $H^r_p(X,n)$.

For $CH^r(Z-Z',n)$, the limit acts like $\varinjlim_{Z\subset X, \mathrm{codim}_X Z = p} \varinjlim_{Z'\subset Z, \mathrm{codim}_Z Z'=1}$ and, $\varinjlim_{Z'\subset Z, \mathrm{codim}_Z Z'=1} CH^r(Z-Z',n) = CH^r(\mathrm{Spec} k(Z),n)$ for a fixed irreducible Z of codimension p in X. Hence, the limit is $\coprod_{x\in X^p} CH^r(\mathrm{Spec} k(x),n)$. Hence we have

$$\cdots H^{r-1}_{p+1}(X,n) \to H^r_p(X,n) \to \coprod_{x\in X^p} CH^r(\mathrm{Spec} k(x),n) \to H^{r-1}_{p+1}(X,n-1) \to H^r_p(X,n-1) \to \cdots .$$

We want to compare it with the standard spectral sequence:

$$\cdots A^{p+1,q-1} \to A^{p,q} \to E^{p,q}_1 \to A^{p+1,q} \to A^{p,q+1} \to \cdots .$$

As before, we let $A^{p,q} = H^{r(p,q)}_p(X, n(p,q))$ where we let $r(p,q) = a_1 p + b_1 q + r$, $n(p,q) = a_2 p + b_2 q$. By comparing the 1st and the 2nd columns, we know that $\begin{cases} r(p+1,q-1) = r(p,q) - 1 \\ n(p+1,q-1) = n(p,q) \end{cases}$ i.e. $\begin{cases} a_1 - b_1 = -1 \\ a_2 = b_2 = 0 \end{cases}$. Also, by comparing the 2nd and the 4th columns, we have $\begin{cases} r(p,q) - 1 = r(p+1,q) \\ n(p,q) - 1 = r(p+1,q) \end{cases}$ i.e. $\begin{cases} -1 = a_1 \\ -1 = a_2 \end{cases}$ so that $r(p,q) = r-p, n(p,q) = -p-q$. Hence we have

$$\begin{cases} A^{p,q} = H^{r-p}_p(X, -p-q) \\ E^{p,q}_1 = \coprod_{x\in X^p} CH^{r-p}(\mathrm{Spec} k(x), -p-q) \end{cases}.$$

Note that $A^n = \varinjlim_q (A^{n-q,q}) = \varinjlim_q H^{r-n+q}_{n-q}(X,-n) \overset{n=q}{=} CH^r(X,-n)$, and the filtration is by codimension, i.e. coniveau, so that by the general theory of exact couple, we have a spectral sequence

$$E^{p,q}_1 = \coprod_{x\in X^p} CH^{r-p}(\mathrm{Spec} k(x), -p-q) \Rightarrow CH^r(X,-p-q).$$

We state it as a theorem.

Theorem 5.2. *There is a BGQ-spectral sequence for Higher Chow groups as above.*

As we did before, once we have a BGQ-spectral sequence, we can localize for the Zariski topology on X so that we have a complex of constant sheaves:

$$\mathcal{E}^{\cdot,q}_1 : \coprod_{x\in X^0} i_* CH^r(\mathrm{Spec} k(x), -q) \to \coprod_{x\in X^1} i_* CH^{r-1}(\mathrm{Spec} k(x), -q-1) \to \cdots$$

$$\to \coprod_{x\in X^{-q}} i_* CH^{r+q}(\mathrm{Spec} k(x), 0).$$

Also as before, let $\mathcal{CH}^r_X(-q)$ be the sheaf associated to the presheaf $U \mapsto CH^r(U,-q)$. Then, as before, we have the following theorem for smooth case:

Theorem 5.3. *If X is quasiprojective smooth scheme over k, then above complex $\mathcal{E}_1^{\cdot,q}$ is a flasque resolution of $\mathcal{CH}^r_X(-q)$.*

The proof is similar to the former cases. I'll update it later when time permits.

6. NESTERENKO/SUSLIN AND TOTARO THEOREM

Recall that the Milnor K-theory for a commutative ring R is defined to be $K^M_*(R) = T_{\mathbb{Z}}(R^\times)/ < a \otimes (1-a) >$. In particular, $K^M_0(R) = \mathbb{Z}, K^M_1 = R^\times$. We know that when $R = k$ is a field, $K^M_n(k) = K_n(k)$ for $n \leq 2$.

Theorem 6.1 (Nesterenko/Suslin 1990, Totaro 1992)**.** *For a field k, we have*

$$K^M_n(k) \simeq CH^n(k,n).$$

We also have the following slight generalization of above result:

Theorem 6.2 (Elbaz-Vincent/Müller-Stach 1998)**.** *When R is a smooth (semi) local k-algebra over an infinte field k, the natural map*

$$K^M_n(R) \to CH^n(R,n)$$

is surjective, and isomorphism if it is a PID.

7. GERSTEN RESOLUTION FOR MILNOR $\overline{\mathcal{K}}^M$-THEORY

Let R be a commutative ring with unity. We let $\overline{K}^M_n(R) = \mathrm{im}(K^M_n(R) \to \coprod_{x\in(\mathrm{Spec}R)^0} K_n(k(x)))$, and let $\overline{\mathcal{K}}^M_n$ be the sheafification of $\overline{K}^M_n$ on any scheme X. Then, in particular, the Gersten resolution of $\mathcal{CH}^r_X(r)$ can be written as

$$0 \to \mathcal{CH}^r_X(r) \to \coprod_{x\in X^0} i_*K^M_r(k(x)) \to \coprod_{x\in X^1} i_*K^M_{r-1}(k(x)) \to \cdots \to \coprod_{x\in X^r} i_*K^M_0(k(x)) \to 0$$

which is exact. In particular, the sheaf $\mathcal{CH}^r_X(r) \simeq \overline{\mathcal{K}}^M_r$ and above sequence is a resolution of $\overline{\mathcal{K}}^M_r$.

8. BLOCH'S FORMULA AND ITS VARIATIONS

Now we use the Gersten resolutions of various theories to deduce interesting results.

Theorem 8.1 (Bloch's formula)**.** *Let X be a smooth variety over a field k. Then,*

$$CH^n(X) \simeq H^n(X, \mathcal{K}_n).$$

Proof. Since X is regular, $\mathcal{K}'_n = \mathcal{K}_n$, so that we have a resolution of $\mathcal{K}_n$:

$$0 \to \mathcal{K}_n \to \coprod_{x\in X^0} i_*K_nk(x) \to \cdots \to \coprod_{x\in X^{n-1}} K_1(k(x)) \xrightarrow{\partial} \coprod_{x\in X^n} K_0(k(x)) \to 0.$$

Recall that $K_1(k(x)) \simeq k(x)^\times$ and $K_0(k(x)) \simeq \mathbb{Z}$. Hence $\coprod_{x\in X^n} K_0(k(x)) \simeq \mathcal{Z}^n(X)$. But, ∂ is giving the rational equivalence on $\mathcal{Z}^n(X)$ so that

$$H^n(E_1^{\cdot,-n}) = \mathcal{Z}^n(X)/\mathrm{im}\partial = CH^n(X).$$

□

Theorem 8.2. *When X is smooth over $\mathbb{C}$ and $H^*(X) = H^*(X;\mathbb{Z})$ then,*

$$H^p(X, \mathcal{H}^p) \simeq CH^p(X)/\mathcal{Z}^r_{alg}(X).$$

Theorem 8.3. *When X is smooth over a field k and $H^*(X) = H^*_{et}(X, \mu_l)$ then,*

$$H^p(X, \mathcal{H}^p) \simeq CH^p(X)/l \cdot CH^p(X).$$

Theorem 8.4. *When X is smooth over a field k,*

$$H^p(X, \mathcal{CH}^p_X(p)) \simeq H^p(X, \overline{\mathcal{K}}^M_p) \simeq CH^p(X).$$

Remark. For the étale case, the correct spectral sequence reflecting twists is

$$E_1^{p,q} = \coprod_{x \in X^p} H^{q-p}_{Gal}(k(x), \mu_n^{\otimes m-p}) \Rightarrow H^{p+q}_{et}(X, \mu_n^{\otimes m})$$

and if the sheaf $\mathcal{H}^p(\mu_n^{\otimes m})$ then, the resolution of $\mathcal{H}^p(\mu_n^{\otimes m})$ is, in fact,

$$\coprod_{x \in X^0} i_* H^p_{Gal}(k(x), \mu_n^{\otimes m}) \to \coprod_{x \in X^1} i_* H^{p-1}_{Gal}(k(x), \mu_n^{\otimes m-1}) \to \cdots$$

$$\to \coprod_{x \in X^p} i_* H^0_{Gal}(k(x), \mu_n^{\otimes m-p}) \to 0.$$