

Personal notes on Grothendieck spectral sequence

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1. GROTHENDIECK SPECTRAL SEQUENCE

We prove the following theorem of Grothendieck:

Theorem 1.1 (Grothendieck spectral sequence). *Let \mathcal{A} , \mathcal{B} , \mathcal{C} be abelian categories and assume that \mathcal{A} , \mathcal{B} have enough injectives. Let $F : \mathcal{A} \rightarrow \mathcal{B}$, $G : \mathcal{B} \rightarrow \mathcal{C}$ be left exact functors, and suppose that for any injective object I of \mathcal{A} , $F(I)$ is G -acyclic, i.e. $R^p G(F(I)) = 0$ for $p > 0$. Then, for any object A in \mathcal{A} , there is a spectral sequence $\{E_r(A)\}$ such that*

$$E_2^{p,q}(A) = (R^p G)(R^q F)(A) \Rightarrow R^{p+q}(GF)(A).$$

To prove this theorem, we need a *fully injective* resolution of a complex, not just an injective resolution. Let's recall the definition of it first.

Definition 1.2 (Fully injective resolution). *Let $C = \oplus C^p$ be a complex. An injective resolution $I_C = (I^{p,q})$ of the complex C is called fully injective if for any row q ,*

$$Z^{p,q} = \ker(I^{p,q} \rightarrow I^{p+1,q})$$

$$B^{p,q} = \text{im}(I^{p-1,q} \rightarrow I^{p,q})$$

$$H^{p,q} = Z^{p,q} / B^{p,q}$$

are all injective and

$$0 \rightarrow Z^p(C) \rightarrow Z^{p+1}(C)$$

$$0 \rightarrow B^p(C) \rightarrow B^{p+1}(C)$$

$$0 \rightarrow H^p(C) \rightarrow H^{p+1}(C)$$

are all injective resolutions as well.

The existence of such an injective resolution can be proven as follows:

Lemma 1.3. *Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be exact, and let $0 \rightarrow M' \rightarrow I'$, $0 \rightarrow M'' \rightarrow I''$ be injective resolutions. Then, there is an injective resolution $0 \rightarrow M \rightarrow I$ such that*

$$\begin{array}{ccccccc} 0 & \longrightarrow & I' & \longrightarrow & I & \longrightarrow & I'' \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ & & 0 & & 0 & & 0 \end{array}$$

is exact and commutative.

Proof. This is a typical lemma so-called the *Horseshoe lemma*. □

Lemma 1.4 (Existence of fully injective resolution). *Given a complex C , there exists a fully injective resolution of C .*

Proof. First, we consider the kernels and cokernels in \mathcal{C} :

$$(1) \quad 0 \rightarrow B^p(C) \rightarrow Z^p(C) \rightarrow H^p(C) \rightarrow 0$$

$$(2) \quad 0 \rightarrow Z^{p-1}(C) \rightarrow C^{p-1}(C) \rightarrow B^p(C) \rightarrow 0.$$

We prove it inductively. Begin with an injective resolution of (2). Then, we choose any injective resolution of $H^p(C)$ in (1), and then by the Horseshoe lemma, we can find a resolution for $Z^p(C)$ that fits into (1). This completes the proof. \square

Now we are ready to prove the theorem.

Proof of the theorem. Let A be an object of \mathcal{A} , and choose an injective resolution $0 \rightarrow A \rightarrow C \rightarrow \dots$ in \mathcal{A} . Then, by applying F , we have

$$FC : 0 \rightarrow FC^0 \rightarrow FC^1 \rightarrow FC^2 \rightarrow \dots$$

which is a complex in \mathcal{B} .

Now, apply the lemma to obtain a fully injective resolution

$$0 \rightarrow FC \rightarrow I$$

in \mathcal{B} where I is a double complex of injective objects. It can be written as follows:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow d & & \uparrow d & & \uparrow d \\
 0 & \longrightarrow & I^{0,1} & \xrightarrow{\delta} & I^{1,1} & \xrightarrow{\delta} & I^{2,1} \longrightarrow \dots \\
 & & \uparrow d & & \uparrow d & & \uparrow d \\
 0 & \longrightarrow & I^{0,0} & \xrightarrow{\delta} & I^{1,0} & \xrightarrow{\delta} & I^{2,0} \longrightarrow \dots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & FC^0 & \longrightarrow & FC^1 & \longrightarrow & FC^2 \longrightarrow \dots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

It is commutative, and its columns are exact.

Now apply G to I and consider the double complex $GI = GI^{\cdot, \cdot}$. We have two filtrations on it:

$${}'F^p(GI) = \bigoplus_{p' \geq p} GI^{p', q}$$

$${}''F^q(GI) = \bigoplus_{q' \geq q} GI^{p, q'}$$

so that we have two spectral sequences ${}'E_r$ and ${}''E_r$. Let's try to compute them separately. We do the easier one first, which is ${}'E_r$.

$${}'E_0^{p, q} = ({}'F^p(GI))^{p, q} / ({}'F^{p+1}(GI))^{p, q} \simeq GI^{p, q} \text{ and } d_0 = d, \text{ so,}$$

$${}'E_1^{p, q} = H_d^q(GI^{p, \cdot}).$$

By the definition of $R^q G$, above ${}'E_1^{p, q}$ is in fact $R^q G(FC^p)$, and by assumption FC^p was G -acyclic, so we have

$${}'E_1^{p, q} = H_d^q(GI^{p, \cdot}) = R^q G(FC^p) = \begin{cases} 0 & \text{if } q > 0 \\ GFC^p & \text{if } q = 0 \end{cases}.$$

Pictorially, ${}'E_1$ with $d_1 = \delta$ looks as follows:

$$\begin{array}{ccccccc}
q = 2 & & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
q = 1 & & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
q = 0 & & GFC^0 & \longrightarrow & GFC^1 & \longrightarrow & GFC^2 \\
& & p = 0 & & p = 1 & & p = 2
\end{array}$$

Hence, by taking the cohomology for $d_1 = \delta$, we obtain

$${}'E_2^{p,q} = \begin{cases} 0 & \text{if } q > 0 \\ R^p(GF)(A) & \text{if } q = 0 \end{cases}.$$

Of course, now we see that $d_2 = 0$ so that $'E_r$ degenerates at $r = 2$. But, in fact, from above computation, we can say much more about the cohomology of the total complex:

Claim. $H^n(\text{tot}(GI)) = R^n(GF)(A)$

Proof. When $n = 0$ it is obvious. So, assume that $n > 0$. Then, from

$$'E_2^{p,n} = 0 = F^p H^{p+n}(\text{tot}(GI)) / F^{p+1} H^{p+n}(\text{tot}(GI)),$$

for all p and $n > 0$, we have

$$F^p H^{p+n}(\text{tot}(GI)) = F^{p+1} H^{p+n}(\text{tot}(GI)).$$

We apply above identity over and over again, until we cannot do it any further.

$$\begin{aligned}
H^n(\text{tot}(GI)) &= F^1 H^n(\text{tot}(GI)) = F^1 H^{1+(n-1)}(\text{tot}(GI)) \\
&= F^2 H^{2+(n-2)}(\text{tot}(GI)) = \dots = F^n H^{n+0}(\text{tot}(GI)).
\end{aligned}$$

Since, obviously, $F^{n+1} H^n(\text{tot}(GI)) = 0$, we have

$$'E_2^{n,0} = R^n(GF)(A) = F^n H^n(\text{tot}(GI)) / F^{n+1} H^n(\text{tot}(GI)) = H^n(\text{tot}(GI))$$

which proves the claim. □

Now we compute $''E_r$. Since the resolution $0 \rightarrow FC \rightarrow I$ is fully injective we have injective resolutions

$$\begin{aligned}
0 &\rightarrow Z^q(FC) \rightarrow Z^q \cdot \\
0 &\rightarrow B^q(FC) \rightarrow B^q \cdot \\
0 &\rightarrow H^q(FC) \rightarrow H^q \cdot
\end{aligned}$$

and, split exact sequences

$$\begin{aligned}
0 &\rightarrow Z^{q,p} \rightarrow I^{q,p} \rightarrow B^{q+1,p} \rightarrow 0 \\
0 &\rightarrow B^{q,p} \rightarrow Z^{q,p} \rightarrow H^{q,p} \rightarrow 0
\end{aligned}$$

which are still split after applying G .

Then,

$$''E_1^{p,q} = H_\delta^q(GI^{\cdot,p}) = GZ^{q,p} / GB^{q,p} = GH^{q,p}$$

so that

$$''E_2^{p,q} = H_d^p(GH^{q,\cdot}).$$

Notice that $H^{q\cdot}$ was an injective resolution of $H^q(FC\cdot) = R^qF(A)$, so that by the definition of R^pG ,

$${}''E_2^{p,q} = H_d^p(GH^{q\cdot}) = R^pG(R^qF(A)),$$

and by the spectral sequence theorem, we know that it abuts to $H^{p+q}(\text{tot}(GI))$ which was $R^{p+q}(GF)(A)$. This proves the theorem. \square

2. APPLICATIONS

Corollary 2.1 (Leray spectral sequence). *Let $\pi : Y \rightarrow X$ be a morphism between spaces equipped with suitable Grothendieck topologies, and let \mathcal{F} be a sheaf on Y . Then, there is a spectral sequence*

$$H^r(X, R^s\pi_*\mathcal{F}) \Rightarrow H^{r+s}(Y, \mathcal{F}).$$

Proof. We have the following functors:

$$Sh(Y) \xrightarrow{\pi_*} Sh(X) \xrightarrow{\Gamma(X,-)} Ab$$

and obviously $\Gamma(X, -) \circ \pi_* = \Gamma(Y, -)$. Since π_* preserve injectives we can apply the Grothendieck spectral sequence to above left exact functors. \square

Remark. When the given Grothendieck topologies are really a topology in the usual sense, including Zariski topology, or étale topology, then we obtain the usual classical Leray spectral sequence.