

A personal note on infinitesimal deformations

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1. INFINITESIMAL LIFTING PROPERTY AND SMOOTHNESS

Definition 1.1. Let k be a field and A be a k -algebra. We say that A has the *infinitesimal lifting property* if the following is true: given any k -algebra C and a square-zero ideal I of C , any k -algebra homomorphism $f : A \rightarrow C/I$ can be lifted to a k -algebra homomorphism $g : A \rightarrow C$ so that $\bar{g} : A \rightarrow C \rightarrow C/I$ is equal to f .

In other words, the homomorphism

$$\mathrm{Hom}_{k\text{-alg}}(A, C) \rightarrow \mathrm{Hom}_{k\text{-alg}}(A, C/I)$$

is surjective. The following diagram shows the situation of the lifting:

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & C & \longrightarrow & C/I \longrightarrow 0 \\ & & & & \uparrow & \nearrow f & \\ & & & & A & & \end{array}$$

Some mathematicians call this property *quasi-free*. One well-known result (see, for instance, the Appendix E. of J.-L. Loday's Cyclic Homology) is the following equivalence.

Theorem 1.2. *Let A be a k -algebra of finite type. Then, A has the infinitesimal lifting property if and only if A is smooth over k .*

We are not going to give a full proof of this theorem, but as an exercise of the usage of the infinitesimal lifting property (to show the forward direction), let's try to do the Exercise II-8.6 (b) and (c) of Hartshorne. Here is the outline. Let P be a free k -algebra of finite type (i.e. a polynomial ring with finitely many indeterminates) of which A is a quotient with the kernel J .

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & C & \longrightarrow & C/I \longrightarrow 0 \\ & & & & \uparrow & & \uparrow f \\ & & & & h \uparrow & & \\ 0 & \longrightarrow & J & \longrightarrow & P & \longrightarrow & A \longrightarrow 0 \end{array}$$

Since P is a free k -algebra, there certainly is h that gives the above commutative diagram. More precisely, we can define h as follows. Let $P = k[x_1, \dots, x_n]$ for some n . Choose $c_1, \dots, c_n \in C$ such that $f(\bar{x}_i) = \bar{c}_i$. Then for a polynomial $p \in P$, define $h(p(x_1, \dots, x_n)) := p(c_1, \dots, c_n)$.

Claim. $h(J) \subset I$ and $h(J^2) = 0$.

Proof. If $p \in J$, then $\bar{p} = 0$ in A . Thus, $\overline{h(p)} = 0$ in C/I so that $h(p) \in I$. Thus, in particular, if $p \in J^2$, then $h(p) \in I^2 = 0$. □

Thus, the claim shows that h induces a map $\bar{h} : J/J^2 \rightarrow I$. Now, since A is smooth, we have an exact sequence

$$0 \rightarrow J/J^2 \rightarrow \Omega_{P/k} \otimes A \rightarrow \Omega_{A/k} \rightarrow 0.$$

Applying the functor $\mathrm{Hom}_A(\cdot, I)$ we obtain then

$$0 \rightarrow \mathrm{Hom}_A(\Omega_{A/k}, I) \rightarrow \mathrm{Hom}_P(\Omega_{P/k}, I) \rightarrow \mathrm{Hom}_A(J/J^2, I).$$

Claim. *The last map is in fact surjective.*

Proof. It is enough to show that $\text{Ext}_A^1(\Omega_{A/k}, I) = 0$. But, this is obvious: since A is smooth over k , $\Omega_{A/k}$ is projective. \square

Now, for $\bar{h} \in \text{Hom}_A(J/J^2, I)$, choose

$$\theta \in \text{Hom}_P(\Omega_{P/k}, I) \subset \text{Hom}_P(\Omega_{P/k}, C) \simeq \text{der}_k(P, C).$$

Regard this θ as a derivation in the last group. Let $h' = h - \theta$.

Claim. h' is a k -algebra homomorphism from P to C such that $h'(J) = 0$.

This claim is obvious, thus we have the desired lifting $\bar{h}' : A \rightarrow C$.

2. INFINITESIMAL EXTENSIONS

In the note on André-Quillen homology, when k is a ring, A is a k -algebra, W is an A -module, an infinitesimal extension of A by W is a short exact sequence of X -modules

$$0 \rightarrow W \rightarrow X \rightarrow A \rightarrow 0$$

where W serves as a square-zero ideal. We can prove that W is then in fact a A -module. These extensions form the set denoted by $\text{Exalcomm}_k(A, W)$. In Hartshorne, its globalized notion is introduced as follows:

Definition 2.1. Let k be a field, X be a k -scheme of finite type, and \mathcal{F} is a coherent sheaf on X . A scheme X' over k with an ideal sheaf \mathcal{I} with $\mathcal{I}^2 = 0$ is called an *infinitesimal extension of X by \mathcal{F}* if

- (1) $(X', \mathcal{O}_{X'}/\mathcal{I}) \simeq (X, \mathcal{O}_X)$ as ringed spaces (this implies that \mathcal{I} is given a \mathcal{O}_X -module structure),
- (2) $\mathcal{I} \simeq \mathcal{F}$ as \mathcal{O}_X -modules.

In other words, we have a short exact sequence of $\mathcal{O}_{X'}$ -modules on the topological space $|X| = |X'|$

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{X'} \rightarrow \mathcal{O}_X \rightarrow 0$$

with $\mathcal{I} \simeq \mathcal{F}$ and $\mathcal{I}^2 = 0$ in $\mathcal{O}_{X'}$.

Remark. The trivial extension is formed by taking, as in the case of rings,

$$\mathcal{O}_{X'} = \mathcal{O}_X \oplus \mathcal{F}$$

$$(a \oplus f) \cdot (b \oplus g) = (ab \oplus (ag + bf)).$$

Try the following exercise of Hartshorne (Ex. II-8.7)

Exercise 2.2. When X is affine and smooth over k , $\text{Exalcomm}_k(X, \mathcal{F}) = 0$, i.e. all infinitesimal extensions are trivial.

The hint is to use the infinitesimal lifting property.

Using the above result, we can actually extend the above result to non-affine case as follows:

Exercise 2.3 (Hartshorne Ex. III-4.10). Let X be a smooth variety (thus separated) over k , and \mathcal{F} is a coherent sheaf on X . Then, there is a bijection

$$\text{Exalcomm}_k(X, \mathcal{F}) \simeq H^1(X, \mathcal{F} \otimes \mathcal{T}),$$

where \mathcal{T} is the tangent sheaf of X .

Proof. (Everything referred are in Hartshorne.) Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an affine open cover of X . Note that since X is nonsingular, so is each U_i . Let $(X', \mathcal{O}_{X'})$ be an infinitesimal extension of X by \mathcal{F} , that is, there is a scheme X' with an ideal sheaf $\mathcal{I} \simeq \mathcal{F}$ as \mathcal{O}_X -modules, such that $\mathcal{I}^2 = 0$, $(X, \mathcal{O}_{X'}/\mathcal{I}) \simeq (X, \mathcal{O}_X)$ as ringed spaces, and a short exact sequence of \mathcal{O}_X -modules

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{X'} \rightarrow \mathcal{O}_X \rightarrow 0.$$

Since each U_i is nonsingular and affine, by Ex.II-8.7, the above short exact sequence restricts to a split exact sequence on U_i , where the splitting is given by a lifting $\alpha_i : \mathcal{O}_X|_{U_i} \rightarrow \mathcal{O}_{X'}|_{U_i}$.

On each $U_{ij} = U_i \cap U_j$, that is affine since X is separated, we have two liftings $\alpha_i|_{U_{ij}}$, $\alpha_j|_{U_{ij}} : \mathcal{O}_X|_{U_{ij}} \rightarrow \mathcal{O}_{X'}|_{U_{ij}}$, and they differs by a section β_{ij} in $\text{der}_k(\mathcal{O}_X(U_{ij}), \mathcal{I}(U_{ij}))$ so that on U_{ij} we have

$$\alpha_i - \alpha_j = \beta_{ij}.$$

Notice that β_{ij} can be seen as a section in $(\mathcal{F} \otimes \mathcal{T})(U_{ij})$ via isomorphisms

$$\text{der}_k(\mathcal{O}_X(U_{ij}), \mathcal{I}(U_{ij})) \simeq \text{Hom}_{\mathcal{O}_{X/k}(U_{ij})}(\Omega_{X/k}(U_{ij}), \mathcal{I}(U_{ij})) \simeq (\mathcal{F} \otimes \Omega_{X/k}^*)(U_{ij}) \simeq (\mathcal{F} \otimes \mathcal{T})(U_{ij}).$$

Restricting all the above sections onto $U_{ijk} = U_i \cap U_j \cap U_k$, we thus obtain

$$\beta_{ij} + \beta_{jk} + \beta_{ki} = (\alpha_i - \alpha_j) + (\alpha_j - \alpha_k) + (\alpha_k - \alpha_i) = 0,$$

and $\{\beta_{ij}\}$ gives a cocycle of the Čech complex $\check{C}^\bullet(\mathcal{U}, \mathcal{F} \otimes \mathcal{T})$ in degree 1.

For a different choice of liftings $\mu_i : \mathcal{O}_X|_{U_i} \rightarrow \mathcal{O}_{X'}|_{U_i}$ for each U_i , as above we have the corresponding sections β'_{ij} of $\text{der}_k(\mathcal{O}_X(U_{ij}), \mathcal{I}(U_{ij}))$ with $\mu_i - \mu_j = \beta'_{ij}$ on U_{ij} , and with $\beta'_{ij} + \beta'_{jk} + \beta'_{ki} = 0$ on U_{ijk} .

Applying the Ex. II-8.6- (a) again to the pair of liftings α_i and μ_i on U_i , we have sections ξ_i of $\text{der}_k(\mathcal{O}_X(U_i), \mathcal{I}(U_i))$ for each U_i with $\alpha_i - \mu_i = \xi_i$, that can also be seen as a section of $\mathcal{F} \otimes \mathcal{T}$ on U_i . Then, on U_{ij} we have

$$\beta_{ij} - \beta'_{ij} = (\alpha_i - \alpha_j) - (\mu_i - \mu_j) = \xi_i - \xi_j,$$

thus the cocycles $\{\beta_{ij}\}$ and $\{\beta'_{ij}\}$ give the same cohomology class in $\check{H}^1(\mathcal{U}, \mathcal{F} \otimes \mathcal{T})$. This last group is isomorphic to $H^1(X, \mathcal{F} \otimes \mathcal{T})$ by (4.5). The converse is easy. This finishes the proof. \square

3. INFINITESIMAL DEFORMATIONS

Infinitesimal deformations are special cases of infinitesimal extensions.

Definition 3.1. When X is a scheme of finite type over k , an *infinitesimal deformation* of X is a scheme X' flat over $\text{Spec}(k[\epsilon])$ such that $X' \otimes_{k[\epsilon]} k \simeq X$.

Definition 3.2. Let X be a scheme of finite type over k . A *global deformation* of X is a flat morphism $f : \text{cal}X \rightarrow T$ of k -varieties such that for some $t \in T$, $\mathcal{X}_t \simeq X$.

Remark. When we have a global deformation $f \rightarrow \mathcal{X} \rightarrow T$ of X , it induces some infinitesimal deformations as follows. Given a tangent vector at $t \in T$, Hartshorne Ex-II-2.8 shows that there is a morphism $\text{Spec}(k[\epsilon]) \rightarrow T$. By a base change, then $X' = Y \times_T \text{Spec}(k[\epsilon]) \rightarrow \text{Spec}(k[\epsilon])$ is still flat, and it gives its fibre at t , $X'_t \simeq X$, thus, we have obtained an infinitesimal deformation of X . It shows thus the following:

- (1) a global deformation gives various infinitesimal deformations, and
- (2) each of these various infinitesimal deformations are given by each tangent vector at t of the base.

In fact, for smooth varieties, we can classify all infinitesimal deformations using the previous triviality result for affine varieties:

Theorem 3.3. *When X is smooth over k , the set of isomorphism classes of infinitesimal deformations is bijective to $H^1(X, \mathcal{T})$.*

Proof. What we will essentially prove is that infinitesimal deformations are nothing but infinitesimal extensions of \mathcal{O}_X by \mathcal{O}_X . Then we can just use the fact that

$$\text{Exalcomm}_k(\mathcal{O}_X, \mathcal{O}_X) \simeq H^1(X, \mathcal{T}).$$

Suppose that X' is scheme flat over $k[\epsilon]$, that is an infinitesimal deformation of X . Consider a short exact sequence of $k[\epsilon]$ -modules

$$0 \rightarrow k \xrightarrow{\times \epsilon} k[\epsilon] \rightarrow k \rightarrow 0.$$

Since X' is flat over $k[\epsilon]$, by tensoring with $\mathcal{O}_{X'}$ over $k[\epsilon]$, we get

$$0 \rightarrow \mathcal{O}_X \xrightarrow{\times \epsilon} \mathcal{O}_{X'} \rightarrow \mathcal{O}_X \rightarrow 0,$$

and $\mathcal{O}_{X'}$ is thus an infinitesimal extension of \mathcal{O}_X by \mathcal{O}_X . Thus, it gives a class in $H^1(X, \mathcal{T})$.

Conversely, any class of $H^1(X, \mathcal{T})$ gives an infinitesimal extension of \mathcal{O}_X by \mathcal{O}_X

$$0 \rightarrow \mathcal{O}_X \xrightarrow{\phi} \mathcal{O}_{X'} \rightarrow \mathcal{O}_X \rightarrow 0$$

. We need to see that it makes X' a scheme flat over $k[\epsilon]$. It is seen as follows. the underlying topological space of X' is X so that we already have a continuous map $f : X' \rightarrow |\text{Spec}(k[\epsilon])| = \{*\}$. The morphism of sheaves $f^\# : \mathcal{O}_{\text{Spec}(k[\epsilon])} \rightarrow f_*\mathcal{O}_{X'}$ is given by a homomorphism on the global sections

$$k[\epsilon] \rightarrow (f_*\mathcal{O}_{X'})({*}) = \Gamma(X', \mathcal{O}_{X'})$$

that sends $a + b\epsilon$ to $a + \phi(b)$.

To show that this structure morphism is flat, we need to show that for each ideal $I \subset k[\epsilon]$, the map $\Gamma(X', \mathcal{O}_{X'} \otimes I) \rightarrow \Gamma(X', \mathcal{O}_{X'}) \otimes I$ is injective. But, there are only three of them, namely, $I = (0), k[\epsilon], (\epsilon)$. For the first two ideals, the injectivity is obvious. For the third one, it is injective because ϕ is. Thus, $X' \rightarrow \text{Spec}(k[\epsilon])$ is flat. This finishes the proof. \square

4. INFINITESIMAL DEFORMATIONS OF AFFINE SCHEMES OVER k

This is related to Hartshorne Ex. III-9.8 and the previously cited exercise problems. Let A be a finitely generated k -algebra. Recall that when A is smooth over k , there is no nontrivial infinitesimal deformation of A . Thus, if we wish to study a nontrivial one, we must always consider a k -algebra *not smooth* over k . Its answer is essentially given by various people by various groups, that are actually all isomorphic. We will present these results here.

4.1. Method of Lichtenbaum and Schlessinger. Let P be a polynomial ring over k of which A is a quotient, and let J be the kernel so that

$$0 \rightarrow J \rightarrow P \rightarrow A \rightarrow 0$$

is exact. For this situation, we have an exact sequence of (both P and) A -modules

$$J/J^2 \xrightarrow{\delta} \Omega_{P/k} \otimes_P A \rightarrow \Omega_{A/k} \rightarrow 0$$

where $\delta(\bar{b}) = db \otimes 1$ for $b \in J$. Apply $\text{Hom}_A(\cdot, A)$ and we let $T^1(A)$ be the cokernel:

$$\text{Hom}_A(\Omega_{P/k} \otimes_P A, A) \rightarrow \text{Hom}_A(J/J^2, A) \rightarrow T^1(A) \rightarrow 0.$$

Then the result is the following:

Theorem 4.1. *The set of infinitesimal deformations of A up to isomorphism is bijective to $T^1(A)$.*

Since infinitesimal deformations are simply infinitesimal extensions of A by A , thus, we can say that

Corollary 4.2. $\text{Exalcomm}_k(A, A) \simeq T^1(A)$.

The proof of the above theorem can be given in various ways. One way could be to identify this group with the appropriate André-Quillen cohomology group, and to use the Jacobi-Zariski exact sequence. And use the discussion on the cohomology. However, for that case as well, we have to do some acrobatics to establish the identification between the infinitesimal extensions and the D^1 (see the first theorem in the next subsection), so, let's just prove it using elementary methods here.

Proof. Suppose that $P = k[x_1, \dots, x_n]$ is a polynomial k -algebra of which A is a quotient with the kernel J . Let $P_2 := k[x_1, \dots, x_n, y_1, \dots, y_n]$.

For each infinitesimal deformation A' of A , we can define a k -algebra homomorphism $f : P_2 \rightarrow A'$ so that we obtain the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & J & \xrightarrow{t} & K & \longrightarrow & J \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & P & \xrightarrow{t} & P_2 & \longrightarrow & P \longrightarrow 0 \\
 & & \downarrow & & \downarrow f & & \downarrow \\
 0 & \longrightarrow & A & \xrightarrow{t} & A' & \longrightarrow & A \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

where K is an ideal of $P - 2$.

Notice that to give a k -algebra A' with the required properties is equivalent to give an ideal K , and the ambiguity is given by the choice of the k -algebra homomorphism f . Thus, the set of equivalence classes of infinitesimal deformations A' of A is equal to

$$\frac{\{\text{choices of an ideal } K\}}{\{\text{choices of } f\}}.$$

We will identify the numerator and the denominator.

Claim. $\{\text{choices of an ideal } K\} \simeq \text{Hom}_P(J, A)$ as sets.

Notice that the middle row splits via the natural inclusion $P \rightarrow P_2$ of the right hand side P . So that as modules, $P_2 = P \oplus tP$.

Suppose an ideal K was chosen. For each $x \in J$, lift it to $\tilde{x} \in K$. Since $P_2 = P \oplus tP \supset K$, $\tilde{x} = x + t(y)$ for some $y \in P$. Two liftings of x differ by an image of tz for some $z \in I$, thus, $y \in P$ is not uniquely determined by x , but $\bar{y} \in A$ is uniquely determined. Thus, it defines a map in $\text{Hom}_P(J, A)$ that sends $x \mapsto \bar{y}$.

Conversely, suppose that $\phi \in \text{Hom}_P(J, A)$. Define an ideal K of P_2 by

$$K = \{x + ty | x \in J, y \in P \text{ such that } \bar{y} = \phi(x) \text{ in } A\}.$$

It is easy to see that K is an ideal of P_2 , and the image of K in P is J so that

$$0 \rightarrow J \rightarrow K \rightarrow J \rightarrow 0$$

is exact. It defines $A' := P_2/K$, and here f is the canonical quotient map. Thus, it shows the claim.

Claim. $\{ \text{choices of } f \} \simeq \text{der}_k(P, A)$ as sets.

A choice of $f : P_2 \rightarrow A'$ gives after composing with $t : P \rightarrow P_2$, a lifting of $P \rightarrow A$ to $P \rightarrow A'$. Thus, Hartshorne Ex. II-8.6-(a) shows the assertion. This proves the claim.

Hence, the obvious identities

$$\text{Hom}_P(J, A) \simeq \text{Hom}_A(J/J^2, A), \text{ and}$$

$$\text{der}_k(P, A) \simeq \text{Hom}_P(\Omega_{P/k}, A)$$

show that the set of isomorphism classes of infinitesimal deformations are in one-to-one correspondence with the coker $(\text{Hom}_P(\Omega_{P/k}, A) \rightarrow \text{Hom}_A(J/J^2, A))$, which is by definition $T^1(A)$. This finishes the proof. \square

Remark. In fact, via a natural map

$$T_1(A) \supset \text{Ext}_A^1(\Omega_{A/k}, A),$$

where the natural map will be apparent from the following discussion.

For the exact sequence

$$J/J^2 \rightarrow \Omega_{P/k} \otimes_P A \rightarrow \Omega_{A/k} \rightarrow 0,$$

let L be the kernel of the second map so that we have a natural projection $J/J^2 \rightarrow L$ and a commutative diagram

$$\begin{array}{ccccccc} & & J/J^2 & & & & \\ & & \downarrow & \searrow & & & \\ 0 & \longrightarrow & L & \longrightarrow & \Omega_{P/k} \otimes_P A & \longrightarrow & \Omega_{A/k} \longrightarrow 0. \end{array}$$

Then, it induces a commutative diagram with exact rows

$$\begin{array}{ccccccc} \text{Hom}_P(\Omega_{P/k}, A) & \longrightarrow & \text{Hom}_P(J/J^2, A) & \longrightarrow & T^1(A) & \longrightarrow & 0 \\ \uparrow = & & \uparrow & & & & \\ \text{Hom}_P(\Omega_{P/k}, A) & \longrightarrow & \text{Hom}_P(L, A) & \longrightarrow & \text{Ext}_A^1(\Omega_{A/k}, A) & \longrightarrow & \text{Ext}_P^1(\Omega_{P/k}, A) \end{array}$$

First of all, since P is smooth over k , $\text{Ext}_P^1(\Omega_{P/k}, A) \simeq 0$, $\Omega_{P/k}$ being projective. Hence, by diagram chasing we can define a map

$$\text{Ext}_A^1(\Omega_{A/k}, A) \rightarrow T^1(A)$$

and furthermore, the diagram implies that it must be injective.

It is known that this map becomes an isomorphism when

- (1) k is a perfect field, and
- (2) A is a reduced k -algebra of finite type,

according to Lichtenbaum and Schlessinger.

4.2. Method of André-Quillen. We don't prove things here. From the theory of André-Quillen (co)homology, what we know is the following:

Theorem 4.3. *For a field k , a k -algebra A , and all A -module W ,*

$$\text{Exalcomm}_k(A, W) \simeq D^1(A/k, W).$$

Thus, in particular when $W = A$ for which $D^1(A/k) := D^1(A/k, W)$,

Corollary 4.4.

$$\begin{aligned} \text{Exalcomm}_k(A, A) &\simeq T^1(A) \simeq D^1(A/k) \\ &\simeq \{ \text{the class of infinitesimal deformations of } A \}. \end{aligned}$$

In fact, also from the theory of André-Quillen (co)homology, we have seen that A is smooth if and only if $D^1(A/k) = 0$. Thus, we actually make the following statement: *The vanishing of $D^1(A/k)$ tells the smoothness of A : if it is zero, then A is smooth, and if it is not zero, then A is not smooth. But having a singular algebra is not too bad, because the group classifies all infinitesimal deformations of A instead.*

4.3. Method of Gerstenhaber using Hochschild cohomology. We follow the presentation by Charles Weibel's *An introduction to homological algebra*. This approach, that I believe first done by M. Gerstenhaber, is more general than the previous ones, since the Hochschild cohomologies can be defined for associative rings that are not necessarily commutative, where the previous methods for commutative rings do not work.

Let k be a commutative ring with unity, R be a k -associative algebra, not necessarily commutative, and M be a $R - R$ -bimodule. The definition of *infinitesimal extensions* is same as the commutative case: it is a k -associative algebra E with a surjective k -algebra homomorphism $\epsilon : E \rightarrow R$ such that $\ker(\epsilon)$ is square-zero, and with an isomorphism of R -modules from M to $\ker(\epsilon)$. We call it a *Hochschild extension* if the short exact sequence $0 \rightarrow M \rightarrow E \rightarrow R \rightarrow 0$ is k -split.

Suppose now that k is a field, thus any infinitesimal extensions are k -split. Let $\sigma : R \rightarrow E$ be a k -splitting that gives a k -module decomposition $E \simeq R \oplus E$. Using the product structure of E , suppose that the product of two pairs

$$(r_1 + m_1) \cdot (r_2 + m_2) := r_1 r_2 + r_1 m_2 + m_1 r_2 + f(r_1, r_2),$$

where f measures how far R is from being a subalgebra of E . More precisely, the morphism $f \in \text{Hom}_R(R \otimes_k R \rightarrow M)$ is given by

$$f(r_1, r_2) = \sigma(r_1) \cdot \sigma(r_2) - \sigma(r_1 r_2).$$

Since E is associative, in particular the associativity of products of three elements $(r_0, 0), (r_1, 0), (r_2, 0)$ reads

$$\begin{aligned} (r_1 r_2, f(r_0, r_1)) \cdot (r_2, 0) &= (r_0, 0) \cdot (r_1 r_2, f(r_1, r_2)) \\ \Leftrightarrow (r_0 r_1) r_2 + f(r_0, r_1) r_2 + f(r_0 r_1, r_2) &= r_0 (r_1 r_2) + r_0 f(r_1, r_2) + f(r_0, r_1 r_2) \\ r_0 f(r_1, r_2) - f(r_0 r_1, r_2) + f(r_0, r_1 r_2) - f(r_0, r_1) r_2 &= 0. \end{aligned}$$

Notice that this is precisely the 2-cycle condition for the Hochschild cocomplex $\text{Hom}_R(R^{\otimes*}, M)$. Conversely, any such f gives an associative product structure on the set $R \oplus M$.

For a different lifting $\sigma' : R \rightarrow E$, we have a map $f' : R \otimes_k R \rightarrow M$ with $\sigma'(r_1) \cdot \sigma'(r_2) - \sigma'(r_1 r_2) = f'(r_1, r_2)$, and,

$$\begin{aligned} f'(r_1, r_2) - f(r_1, r_2) &= \sigma'(r_1) \cdot \sigma'(r_2) - \sigma'(r_1 r_2) - \sigma(r_1) \sigma(r_2) + \sigma(r_1 r_2) \\ &= \sigma'(r_1) \cdot (\sigma'(r_2) - \sigma(r_2)) - (\sigma'(r_1 r_2) - \sigma(r_1 r_2)) + (\sigma'(r_1) - \sigma(r_1)) \sigma(r_2) \end{aligned}$$

which is the coboundary of $\sigma' - \sigma \in \text{Hom}_R(R, M)$. Thus, an infinitesimal extension determines a unique cohomology class of $HH^2(R, M)$ and vice versa. In other words, we proved the following theorem:

Theorem 4.5. *Let k be a field, R be a k -associative algebra, not necessarily commutative, M be a (R, R) -bimodule. Then the equivalence classes of infinitesimal extensions of R by M are in one-to-one correspondence with the elements of $HH^2(R, M)$.*

Corollary 4.6. *In addition to the assumptions of the above theorem, suppose that $M = R$ and R is commutative. Then, the equivalence classes of infinitesimal deformations of R are in one-to-one correspondence with the elements of $HH^2(R)$.*

Remark. There is an approach using formal deformations. See. M. Gerstenhaber's paper, for instance.

Thus, we have proved the following summary:

Theorem 4.7. *Let k be a field, A be a commutative k -algebra. Then, the following sets are all bijective with each other:*

- (1) { classes of infinitesimal deformations of A },
- (2) $\text{Exalcomm}_k(A, A)$,
- (3) $T^1(A)$,
- (4) $D^1(A/k)$,
- (5) $HH^2(A)$,

and (3), (4), (5) are isomorphic as groups. Furthermore, any one of the above sets is a singleton (or a trivial group) if and only if A is smooth over k .

4.4. So where do we go? Recall that when X is a general nonsingular variety, its infinitesimal deformations are measured by the group $H^1(X, \mathcal{T})$. But, then, what can we say about general *singular* varieties? In particular, what groups classify all infinitesimal deformations? Can we find a suitable group that does the job? Of course it must be, by definition, $\text{Exalcomm}_k(\mathcal{O}_X, \mathcal{O}_X)$, but what else can we say?

A general answer does not seem to come easily. At this moment I do not know the answer, but there seem to be several ideas: we could probably form a local to global spectral sequence for the André-Quillen (co)homology groups, and they could at least tell something about the case... But I don't know. I will make a separate note on this remaining case.