A personal note on functors of Artin rings

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Let k be a field. Define the following three categories:
$\bullet(Art/k)$: the category of artin local k-algebras with residue field k
$\bullet(Loc/k)$: the category of local noetherian k-algebras with residue field k
$\bullet(CLoc/k)$: the category of complete local noetherian k-algebras with residue field k

For $\Lambda \in (Loc/k)$, we also consider the following subcategories:
$\bullet(Art/\Lambda)$: the category of artin local Λ-algebras with residue field k
$\bullet(Loc/\Lambda)$: the category of local noetherian Λ-algebras with residue field k
In case $\Lambda \in (CLoc/k)$, we consider the following category as well:
$\bullet(CLoc/\Lambda)$: the category of complete local noetherian Λ-algebras with residue field k

1. Basics on functors of Artin rings

Definition 1.1. (1) A *functor of Artin rings* is a covariant functor

$$F : (Art/\Lambda) \to (Sets)$$

where $\Lambda \in (CLoc/k)$. For $A \in (Art/\Lambda)$, an element $\xi \in F(A)$ is called an *infinitesimal deformation* of $\xi_0 \in F(k)$ if $\xi \mapsto \xi_0$ under the natural map $F(A) \to F(k)$. If $A = k[\epsilon]$, the element ξ is called a *first-order deformation* of ξ_0.

(2) Define $h_{R/\Lambda} : (Art/\Lambda) \to (Sets)$ by

$$h_{R/\Lambda}(A) = \mathrm{Hom}_{(Art/\Lambda)}(R, A)$$

where $R \in (CLov/\Lambda)$. If a functor of Artin rings F is isomorphic to $h_{R/\Lambda}$, then we say that the functor F is *pre-representable*.

Remark. The usage of the word *pre-representable* is not equal to the one used by the original paper of Schlessinger.

Example 1.2. Let M be a scheme, and consider the restriction of the functor h_X : $(Sch)^{op} \to (Sets)$ to $(Art/k) \to (Sets)$ that sends $A \mapsto \mathrm{Hom}_k(\mathrm{Spec}(A), M)$.

Fix $A \in (Art/k)$, and let $\phi : \mathrm{Spec}(A) \to M$ be a morphism of schemes. The composition $\mathrm{Spec}(k) \to \mathrm{Spec}(A) \xrightarrow{\phi} M$ gives a k-rational point $m \in M$, and consider the subfunctor

$$F : (Art/k) \to (Sets)$$

defined as

$$F(A) = \mathrm{Hom}(\mathrm{Spec}(A), M)_m := \{\phi : \mathrm{Spec}(A) \to M | \mathrm{im}(\phi) = \{m\}\}.$$

Actually, then $F = h_R$, where $R = \widehat{\mathcal{O}}_{M,m}$.

Proposition 1.3. *A pro-representable functor F has the following properties:*

H_0) *$F(k)$ consists of one element.*

H_l) *For each diagram in (Art/Λ)*

$$\begin{tikzcd} A' \arrow[dr] & & A'' \arrow[dl] \\ & A, & \end{tikzcd}$$

there is a natural map

$$\alpha : F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'').$$

Then, this α is bijective.

H_f) *$F(k[\epsilon])$ has a structure of a finite dimensional k-vector space, denoted by $t_F = t_{R/\Lambda}$.*

Remark. We single out the following special weaker property of H_l):

H_ϵ) α is bijective for $A = k$ and $A'' = k[\epsilon]$.

Actually, just H_0) and H_ϵ) give a k-vector space structure on $F(k[\epsilon])$ in a functorial way without the pro-representability:

Lemma 1.4. *Let F be a functor of Artin rings with the properties H_0) and H_ϵ). Then the set $F(k[\epsilon])$ has a structure of k-vector space in a functorial way. This space is denoted by t_F.*

Proof. We give some details of the proof, as the full details will be quite boring.

(1) (addition structure) For the evaluation map $k[\epsilon] \to k$, consider the k-vector space homomorphism

$$\begin{aligned} + : k[\epsilon] \times_k k[\epsilon] &\to k[\epsilon] \\ (a + b\epsilon, a + b'\epsilon) &\mapsto a + (b + b')\epsilon. \end{aligned}$$

Claim. *This $+$ is a k-algebra homomorphism.*

Proof. For two elements $(a_i + b_i\epsilon, a_i + b_i'\epsilon)$, $i = 1, 2$, in $k[\epsilon] \times_k k[\epsilon]$,

$$\begin{aligned} & + \big((a_1 + b_1\epsilon, a_1 + b_1'\epsilon) \cdot (a_2 + b_2\epsilon, a_2 + b_2'\epsilon)\big) \\ = \;& + \big(a_1a_2 + (a_1b_2 + a_2b_1)\epsilon, a_1a_2 + (a_1b_2' + a_2b_1')\epsilon\big) \\ = \;& a_1a_2 + (a_1b_2 + a_2b_1 + a_1b_2' + a_2b_1')\epsilon. \end{aligned}$$

On the other hand,

$$\begin{aligned} & +(a_1 + b_1\epsilon, a_1 + b_1'\epsilon) \cdot +(a_2 + b_2\epsilon, a_2 + b_2'\epsilon) \\ = \;& (a_1 + (b_1 + b_1')\epsilon) \cdot (a_2 + (b_2 + b_2')\epsilon) \\ = \;& a_1a_2 + (a_1b_2 + a_1b_2' + a_2b_1 + a_2b_1')\epsilon. \end{aligned}$$

This proves the claim. ☐

Hence, using the identification α^{-1} (by H_ϵ)), and $F(+)$, we have the following composition:

$$F(k[\epsilon]) \times_{F(k)} F(k[\epsilon]) \to F\left(k[\epsilon] \times_k k[\epsilon]\right) \overset{F(+)}{\to} F(k[\epsilon]),$$

where the first term is equal to $F(k[\epsilon]) \times F(k[\epsilon])$ since $F(k)$ is a singleton by H_0). This is the addition structure $+_F$. That it is commutative is obvious since $+$ is obvious.

(2) (zero element) Define the zero element in $F(k[\epsilon])$ to be the image of the map $F(k) \to F(k[\epsilon])$, denoted by 0.

Claim. *For all $a \in F(k[\epsilon])$, $0 +_F a = a +_F 0 = a$.*

Proof. The composition

$$F(k) \times F(k[\epsilon]) \to F(k[\epsilon]) \times F(k[\epsilon]) \to F\left(k[\epsilon] \times_k k[\epsilon]\right) \overset{F(+)}{\to} F(k[\epsilon])$$

is induced by the identity map, thus $0 +_F a = a$. ☐

(3) (associativity) Use H_ϵ) suitably.

(4) (constant multiple) For a constant $c \in k$, define a k-vector space homomorphism

$$c_* : k[\epsilon] \to k[\epsilon]$$
$$a + b\epsilon \mapsto a + (cb)\epsilon.$$

Claim. *This map c_* is a k-algebra homomorphism.*

Proof. For two elements $a_i + b_i\epsilon \in k[\epsilon]$, $i = 1, 2$, note that

$$\begin{aligned} c_*\left((a_1 + b_1\epsilon)(a_2 + b_2\epsilon)\right) &= c_*\left(a_1a_2 + (a_1b_2 + a_2b_1`)\epsilon\right) \\ &= a_1a_2 + c(a_1b_2 + a_2b_1)\epsilon, \end{aligned}$$

and on the other hand,

$$\begin{aligned} c_*(a_1 + b_1\epsilon) \cdot c_*(a_2 + b_2\epsilon) &= (a_1 + cb_1\epsilon)(a_2 + cb_2\epsilon) \\ &= a_1a_2 + c(a_1b_2 + a_2b_1)\epsilon. \end{aligned}$$

This finishes the proof of the claim. □

□

Definition 1.5. Let $f : F \to G$ be a morphism of functors of Artin rings. The induced map $t_F \to t_G$ is called the *differential* of f, and it is denoted by df. Of course, in case both F and G satisfy H_0 and H_ϵ, then the differential df is k-linear.

2. Formal couples

Let F be a functor of Artin rings. Then F extends to a functor

$$\widehat{F} \to (CLoc/\Lambda) \to (Sets)$$

in the following fashion. Let $R \in (CLoc/\Lambda)$, and let m_R be the maximal ideal of R. Note that for each $n \geq 1$, the ring R/m_R^n is in (Art/Λ), and $\{R/m_R^n, n \geq 1\}$ forms a projective system. Define

$$\widehat{F}(R) = \varprojlim F\left(R/m_R^n\right).$$

For $\phi : R \to S$, we let $\widehat{F}(\phi) : \widehat{F}(R) \to \widehat{F}(S)$ be the map induced by the maps

$$F\left(R/m_R^n\right) \to F\left(S/m_S^n\right)$$

for $n \geq 1$.

Definition 2.1.
(1) An element $\widehat{u} \in \widehat{F}(R)$ is called a *formal element* of F. By definition, in fact $\widehat{u}$ is given by a set of elements $\{u_{n-1} \in F\left(R/m_R^n\right)\}_{n\geq 1}$ such that for each $n \geq 1$, the natural map $F\left(R/m_R^{n+1}\right) \to F\left(R/m_R^n\right)$ induced by the projection sends u_n to u_{n-1}.
(2) $\widehat{u}$ is also called a *formal deformation* of u_0.
(3) A pair $(R, \widehat{u})$, where $\widehat{u} \in \widehat{F}(R)$ is called a *formal couple.*

Remark.
(1) When $R \in (Art/\Lambda)$, an ordinary couple (R, u) is a special case of formal couple in which $(R_n, u_n) = (R_{n+1}, u_{n+1})$ for $n >> 0$.
(2) Let $f : F \to G$ be a morphism of functors of Artin rings. It extends to a morphism of functors $\widehat{f} : \widehat{F} \to \widehat{G}$.

The following lemma is the crucial result regarding a formal element.

Lemma 2.2. *Let $R \in (CLoc/\Lambda)$, and F be a functor of Artin rings. Then there is a one-to-one correspondence*

$$\widehat{F}(R) \simeq \left\{ \textit{ morphisms of functors } h_{R/\Lambda} \to F \right\}.$$

Proof. Let $\widehat{u} \in \widehat{F}(R)$. This is given by a set of elements $\left\{u_n \in F\left(R/m_R^{n+1}\right)\right\}_{n\geq 0}$ such that $u_n \mapsto u_{n-1}$. Since $\mathrm{Hom}(R/m_R^{n+1}, F) = F(R/m_R^{n+1})$, each element u_n gives a functor $h_{R/m_R^{n+1}} \to F$ such that the following diagram

$$\begin{tikzcd}
h_{R/m_R^{n+1}} \arrow[r] & F \\
h_{R/m_R^{n}} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

is commutative. For each $A \in (Art/k)$, being finite dimensional, for $n >> 0$,

$$h_{R/m_R^n}(A) \to h_{R/m_R^{n+1}})(A)$$

is an isomorphism. Thus, we can define $h_{R/\Lambda} \to F$ to be the limit

$$\varinjlim \left(h_{R/m_R^{n+1}}(A) \to F(A)\right).$$

Conversely, given $h_{R/\Lambda} \to F$, let u_n be the image of the canonical projection $R \to R/m_R^{n+1}$ via the map $h_{R/\Lambda}\left(R/m_R^{n+1}\right) \to F\left(R/m_R^{n+1}\right)$. It gives $\widehat{u}$ since $u_n \mapsto u_{n-1}$ for $n \geq 1$. $\square$

Definition 2.3. Let F be a functor or Artin rings, and let $(R, \widehat{u})$ be a formal couple for F, where $R \in (CLoc/\Lambda)$ and $\widehat{u} \in \widehat{F}(R)$. If an element $\widehat{u}$ gives an isomorphism $h_{R/\Lambda} \to F$, then certain F is pre-representable, and we say that $\widehat{u}$ is a *universal formal element* for F. The formal couple $(R, \widehat{u})$ is called a *universal formal couple* for F.

Remark. A universal formal couple rarely exists so that we need to introduce some weaker conditions: semiuniversality and versality. To do so, we need the concept of smoothness of functors of Artin rings.

3. Smooth functors

In this sections, all functors F of Artin rings are supposed to satisfy the condition H_0), i.e. $F(k)$ is a singleton.

Definition 3.1. (1) Let $f : F \to G$ be a morphism of functors of Artin rings. It is said to be *smooth* if for every surjection $\mu : B \to A$ in (Art/Λ), the natural map

$$F(B) \to F(A) \times_{G(A)} G(B)$$

induced by the diagram

$$\begin{tikzcd}
F(B) \arrow[r] \arrow[d] & G(B) \arrow[d] \\
F(A) \arrow[r] & G(A)
\end{tikzcd}$$

is surjective.

(2) The functor F is called *smooth* if the morphism from F to the constant singleton functor G, where $G(A) = \{*\}$ for all A, is smooth. Equivalently, the map $F(\mu) : F(B) \to F(A)$ is surjective for all surjection $\mu : B \to A$ in (Art/Λ).

Remark. Any surjection in (Art/Λ) is a compositionof finite small extensions, thus, a $f : F \to G$ is smooth if and only if the above condition is true for small extensions $B \to A$ in (Art/Λ).

Proposition 3.2. *All functors of Artin rings are supposed to be satisfying the condition H_0). Under this assumption:*

1. *Let $f : R \to S$ be a morphism in $(CLov/\Lambda)$. Then, f is formally smooth if and only if the morphism of functors $h_f : h_{S/\Lambda} \to h_{R/\Lambda}$ is smooth.*
2. *If $f : F \to G$ is smooth, then f is surjective for each $A \in (Art/\Lambda)$. In particular, $df : t_F \to t_G$ is surjective.*
3. *If $f : F \to G$ is smooth, then F is smooth if and only if G is smooth.*
4. *If $f : F \to G$ is smooth, then the induced morphism $\widehat{f} : \widehat{F} \to \widehat{G}$ is smooth.*
5. *If $F \to G$ and $G \to H$ are smooth, then the composition $F \to H$ is smooth.*
6. *If $F \to G$ and $H \to G$ are morphisms, and $F \to G$ is smooth, then $F \times_G H \to H$ is smooth.*
7. *If F and G are smooth, then $F \times G$ is smooth.*

Proof. (1) Let $B \to A$ be a surjection in (Art/k), and consider the induced diagram

$$\begin{tikzcd}
h_S(B) \arrow[r] \arrow[d] & h_R(B) \arrow[d] \\
h_S(A) \arrow[r] & h_R(A).
\end{tikzcd}$$

To give an element in $h_S(A) \times_{h_R(A)} h_R(B)$ is equivalent to give a commutative diagram of homomorphism of Λ-algebras

$$\begin{tikzcd}
R \arrow[r] \arrow[d, "f"'] & B \arrow[d] \\
S \arrow[r] & A.
\end{tikzcd}$$

The formal smoothness of f is equivalent to the existence of a homomorphism $S \to B$ so that the resulting diagram

$$\begin{tikzcd}
R \arrow[r] \arrow[d, "f"'] & B \arrow[d] \\
S \arrow[r] \arrow[ur] & A
\end{tikzcd}$$

is commutative. But, this is equivalent to the surjectivity of

$$h_S(B) \to h_S(A) \times_{h_R(A)} h_R(B).$$

(2) For any surjection $B \to k$, we have a corresponding surjection

$$F(B) \xrightarrow{q} F(k) \times_{G(k)} G(B).$$

But, since $F(k) \simeq G(k)$ is a singleton, this surjection q gives a surjection $F(B) \to G(B)$.

(3) Let $\mu : B \to A$ be s surjection, and consider the diagram

$$\begin{tikzcd}
F(B) \arrow[r] & F(A) \times_{G(A)} G(B) \arrow[r] \arrow[d] & G(B) \arrow[d, "G(\mu)"] \\
 & F(A) \arrow[r, "f(A)"'] & G(A).
\end{tikzcd}$$

Suppose that G is smooth. They by (2), $G(\mu)$ is surjective so that $F(A) \times_{G(A)} G(B) \to F(A)$ is surjective. Thus, $F(B) \to F(A)$ is surjective, proving that F is smooth.

Conversely, let $\xi \in G(A)$. Since $f(A)$ is surjective, there is $\eta \in F(A)$ that maps to ξ. Since $F(B) \to F(A)$ is surjective, there is $\chi \in F(B)$ that maps to η. Hence $F(B)(\chi) \in G(B)$ is sent to ξ. Hence G is smooth.

(4) Let $\widehat{v} = \{v_n\} \in \widehat{G}(R)$. Since f is smooth, the map

$$F(R/m_R^2) \to F(k) \times_{G(k)} G(R/m_R^2) = G(R/m_R^2)$$

is surjective. Therefore, there is $w_1 \in F(R/m_R^2)$ that maps to v_1. Suppose that for $i = 1, \cdots, n-1$, there are $w_i \in F(R/m_R^{i+1})$ that maps to v_i and $w_i \mapsto w_{i-1}$. Then, the surjectivity of

$$F(R/m_R^{n+1}) \to F(R/m_R^n) \times_{G(R/m_R^n)} G(R/m_R^{n+1})$$

shows that there is $w_n \in F(R/m_R^{n+1})$ that maps to (w_{n-1}, v_n). It gives $\widehat{w} \in \widehat{F}(R)$ that mapsto $\widehat{v}$.

The rest are easy.

□

4. Versality of formal pairs

Definition 4.1. Let F be a functor of artin rings such that $F(k)$ is a singleton. Let $(R, \widehat{u})$ be a formal pair for F.

(1) $\widehat{u} \in \widehat{F}(R)$ is called *versal* if the induced morphism $h_{R/\Lambda} \to F$ is smooth.
(2) $\widehat{u}$ is called *semiuniversal* if it is versal, and the differential $d\widehat{u} : t_{R/\Lambda} \to t_F$ is bijective.
(3) $\widehat{u}$ is called *universal* if the induced morphism $h_{R/\Lambda} \to F$ is an isomorphism.

Certainly, universal ⇒ semiuniversal ⇒ versal.