Personal notes on Delign-Beilinson Cohomology

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1. Definitions

Let X be a quasiprojective smooth variety over $\mathbb{C}$.

Recall 1.1. Poincaré ∂-lemma gives a resolution of $\mathbb{C}$:

$$\Omega^{\cdot}_X := 0 \to \Omega^0_X \to \Omega^1_X \to \Omega^2_X \to \cdots \to \Omega^{\dim_{\mathbb{C}} X}_X \to 0$$

where Ω^i_X is the sheaf of holomorphic differential forms of degree i on X. So, its de Rham cohomology $H^i(X, \mathbb{C})$ is also given by the hypercohomology $\mathbb{H}^i(X, \Omega^{\cdot}_X)$.

Recall 1.2. Let $F^n \Omega^{\cdot}_X$ to be the naive filtration of the complex $\Omega^{\cdot}_X$, i.e. $F^n \Omega^{\cdot}_X = \Omega^{\cdot \geq n}_X$. We have a inclusion $F^n \Omega^{\cdot}_X \to \Omega^{\cdot}_X$ which induces the filtration $F^n \mathbb{H}^i(X, \Omega^{\cdot}_X) = F^n H^i(X, \mathbb{C})$, which is by definition, the image of $\mathbb{H}^i(X, F^n \Omega^{\cdot}_X)$.

Recall 1.3. With above notations, with X projective, we also have the Hodge decomposition:

$$F^n H^i(X, \mathbb{C}) = \bigoplus_{\substack{p+q=k \\ p \geq n}} H^{p,q}(X)$$

where $H^{p,q}(X)$ is the space of classes of closed differential forms of type (p, q) on X. From now when, whenever convenient, we will assume that X is projective, hence compact and Kaehler.

Let A be a subring of $\mathbb{C}$, for example, $A = \mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. Let $A(n) = (2\pi i)^n A$ which is a subgroup of $\mathbb{C}$.

Definition 1.4 (The Delign Complex). *The following complex of sheaves of abelian groups*

$$A(n)_{\mathcal{D}} := A(n) \to \Omega^0_X \to \Omega^1_X \to \cdots \to \Omega^{n-1}_X \to 0$$

is called the Delign complex, where $A(n)$ is placed at degree 0, and Ω^p_X is place at degree $p+1$. We define $H^i_{\mathcal{D}}(X, A(n)) := \mathbb{H}^i(X, A(n)_{\mathcal{D}})$, which is called the Delign-Beilinson cohomology.

2. Basic Theorems and Examples

Recall 2.1 (Cone)**.** Let $V^{\cdot}$, $W^{\cdot}$ be two complexes in an abelian category, and let $f : V^{\cdot} \to W^{\cdot}$ be a chain map. The cone of f, denoted by C_f is by definition, a complex $C^i_f = V^{i+1} \oplus V^i$ equipped with a differential $d(v, w) = (-dv, f(v) + dw)$. We instantly check that

$$d^2(v, w) = (-d(-dv), f(-dv) + d(f(v) + dw)) = (d^2 v, -f(dv) + d(f(v) + d^2 w) = 0$$

because f was a chain map.

Recall 2.2 (Cylinder)**.** Let f, $V^{\cdot}$, $W^{\cdot}$ be as above. The cylinder of f, denoted by Cyl_f is by definition, a complex $Cyl^i_f = V^i \oplus V^{i+1} \oplus W^i$ equipped with a differential $d(v, v', w) = (dv - v', -dv', f(v') + dw)$. We also instantly check that

$$d^2(v, v', w) = (d(dv - v') + dv', d^2 v', f(-dv') + df(v') + d^2 w) = 0$$

because f was a chain map.

Recall 2.3. We have a short exact sequence of complexes:

$$0 \to W^{\cdot} \to C^{\cdot}_f \to V^{\cdot}[1] \to 0$$

so that we have the long exact sequence:

$$\to H^{k-1}(W^{\cdot}) \to H^{k-1}(C^{\cdot}_f) \to H^{k-1}(V^{\cdot}[1]) = H^k(V^{\cdot}) \overset{H^k(f)}{\to} H^k(W^{\cdot}) \to .$$

We consider $A(n)$ itself as a complex at the degree 0 place. Then, we have a chain map

$$\epsilon(n) : A(n) \to \Omega^{\cdot}_X.$$

Also, we have a natural inclusion

$$\sigma(n) : F^n\Omega^{\cdot}_X \to \Omega^{\cdot}_X.$$

So, we consider the chain map

$$f : A(n) \oplus F^n\Omega^{\cdot}_X \to \Omega^{\cdot}_X$$

with $f = \epsilon(n) - \sigma(n)$.

Theorem 2.4. *The inclusion*

$$i : A(n)_{\mathcal{D}} \to C^{\cdot}_f[-1]$$

is a quasi-isomorphism, whose quasi-inverse is the natural projection

$$j : C^{\cdot}_f[-1] \to A(n)_{\mathcal{D}}.$$

Proof. $j \circ i$ is the identity of $A(n)_{\mathcal{D}}$, so it induces the identity isomorphism on cohomology. So, it is enough to show that $i \circ j$ induces an isomorphism on cohomology. (later)

$\square$

Corollary 2.5. *Above $C^{\cdot}_f$ gives the same hypercohomology, i.e. the Delign-Beilinson cohomology. In particular, we obtain a long exact sequence for the derived functor of Γ for the cone short exact sequence, so that*

$$\to H^{k-1}(X, A(n)) \oplus F^n H^{k-1}(X, \mathbb{C}) \to H^{k-1}(X, \mathbb{C}) \to$$
$$H^k_{\mathcal{D}}(X, A(n)) \to H^k(X, A(n)) \oplus F^n H^k(X, \mathbb{C}) \to$$

is obtained.

Since $A(n)$ is a subgroup of $\mathbb{C}$, this inclusion induces a homomorphism $\alpha : H^{\cdot}(X, A(n)) \to H^{\cdot}(X, \mathbb{C})$, i.e. $\alpha = \epsilon(n)$. Hence, if we apply it to above Corollary, we obtain:

Corollary 2.6. *Above long exact sequence can be also written as:*

$$0 \to H^{k-1}(X, \mathbb{C})/(\alpha(H^{k-1}(X, A(n))) + F^n H^{k-1}(X, \mathbb{C})) \to H^k_{\mathcal{D}}(X, A(n))$$
$$\to H^k(X, A(n)) \cap \alpha^{-1}(F^n H^k(X, \mathbb{C})) \to 0.$$

Example 2.7 ($n = 0$)**.** *Let $A = \mathbb{Z}$. Then, $\mathbb{Z}(0)_{\mathcal{D}}$ is the constant sheaf $\mathbb{Z}$ at the degree 0, hence,*

$$H^k_{\mathcal{D}}(X, \mathbb{Z}(0)) = H^k(X, \mathbb{Z}).$$

Example 2.8 ($n = 1$)**.** *There is a quasi-isomorphism*

$$\exp : \mathbb{Z}(1)_{\mathcal{D}} \to \mathcal{O}^*_X[-1]$$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 2\pi i\mathbb{Z} \arrow[r] \arrow[d, "\exp"] & \mathcal{O}_X \arrow[r] \arrow[d, "\exp"] & 0 \arrow[d] \\
1 \arrow[r] & 1 \arrow[r] & \mathcal{O}^*_X \arrow[r] & 1
\end{tikzcd}$$

This is a quasi-isomorphism, because,

$$0 \to 2\pi i\mathbb{Z} \to \mathcal{O}_X \overset{\exp}{\to} \mathcal{O}^*_X \to 1$$

is exact, and above diagram sort of "bent down" the last two terms. So, we have

$$H^k_{\mathcal{D}}(X, \mathbb{Z}(1)) = H^k(X, \mathcal{O}^*_X[-1]) = H^{k-1}(X, \mathcal{O}^*_X).$$

In particular,

$$H^2_{\mathcal{D}}(X, \mathbb{Z}(1)) = H^1(X, \mathcal{O}^*_X) = \mathrm{Pic}(X).$$

Example 2.9 ($n = 2$)**.** *We use a similar technic we used for $n = 1$ case. We have a quasi-isomorphism of complexes:*

$$\mathbb{Z}(2)_{\mathcal{D}} \to \{\mathcal{O}_X^* \overset{d\log}{\to} \Omega_X^1\}[-1].$$

Explicitly,

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & (2\pi i)^2\mathbb{Z} \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d, "\exp(\frac{1}{2\pi i}-)"] & \Omega_X^1 \arrow[r] \arrow[d, "\frac{1}{2\pi i}"] & 0 \arrow[d] \\
0 \arrow[r] & 1 \arrow[r] & \mathcal{O}_X^* \arrow[r, "d\log"'] & \Omega_X^1 \arrow[r] & 0
\end{tikzcd}$$

Hence we have

$$H_{\mathcal{D}}^k(X, \mathbb{Z}(2)) \overset{\simeq}{\to} \mathbb{H}^{k-1}(X, \mathcal{O}_X^* \overset{d\log}{\to} \Omega_X^1).$$

Definition 2.10. *The p-th (Griffith) intermediate Jacobian of X was, by definition*

$$J^p(X) := H^{2p-1}(X, \mathbb{C})/(H^{2p-1}(X, \mathbb{Z}) + F^pH^{2p-1}(X, \mathbb{C})).$$

Also, the p-th set of Hodge classes was, by definition,

$$\mathrm{Hdg}^p(X) := H^{2p}(X, \mathbb{Z}) \cap \epsilon^{-1}H^{p,p}(X).$$

Look at Corollary 2.6 when $A = \mathbb{Z}$, $k = 2p$ and $n = p$. It reads,

$$0 \to H^{2p-1}(X, \mathbb{C})/(H^{2p-1}(X, \mathbb{Z}(p)) + F^pH^{2p-1}(X, \mathbb{C})) \to H_{\mathcal{D}}^{2p}(X, \mathbb{Z}(p)) \to$$
$$H^{2p}(X, \mathbb{Z}(p)) \cap \epsilon(p)^{-1}(F^pH^{2p}(X, \mathbb{C})) \to 0.$$

Since $\epsilon(p)$ is just a multiplication by $(2\pi i)^p$, we have

$$H^{2p-1}(X, \mathbb{C})/(H^{2p-1}(X, \mathbb{Z}(p)) + F^pH^{2p-1}(X, \mathbb{C})) \simeq J^p(X)$$

and similarly,

$$H^{2p}(X, \mathbb{Z}(p)) \cap \epsilon(p)^{-1}(F^pH^{2p}(X, \mathbb{C})) \simeq \mathrm{Hdg}^p(X).$$

Hence we have the following:

Proposition 2.11. *We have the following exact sequence:*

$$0 \to J^p(X) \to H_{\mathcal{D}}^{2p}(X, \mathbb{Z}(p)) \to \mathrm{Hdg}^p(X) \to 0.$$

Now, look at Corollary 2.6 with $A = \mathbb{R}$, $k = 2p$ and $n = p$, i.e. everything is same as above, except that $\mathbb{Z}$ was replaced by $\mathbb{R}$. Then, the first term of the short exact sequence vanishes (why?), so we obtain:

Proposition 2.12.

$$H_{\mathcal{D}}^{2p}(X, \mathbb{R}(p)) = H^{2p}(X, \mathbb{R}(p)) \cap H^{p,p}(X)$$

3. Product Structure

There is a multiplication of complexes

$$\mu : A(n)_{\mathcal{D}} \otimes A(m)_{\mathcal{D}} \to A(m+n)_{\mathcal{D}}$$

defined *mysteriously* as follows:

$$\mu(x \cdot y) = \begin{cases} xy & \deg x = 0 \\ x \wedge dy & \deg x > 0, \deg y > 0 \\ 0 & \text{otherwise} \end{cases}.$$

This is commutative and associative up to homotopy. This product induces a cup-product structure

$$\cup : H_{\mathcal{D}}^k(X, A(n)) \otimes_{\mathbb{Z}} H_{\mathcal{D}}^{k'}(X, A(m)) \to H_{\mathcal{D}}^{k+k'}(X, A(m+n)).$$

4. Cycle Class map

Recall 4.1 (Chow group)**.** Let S be a regular separated noetherian scheme of finite Krull dimension. For each $n \in \mathbb{N}$, let $S^{(n)}$ be the set of points of codimension n in S, and let $\mathcal{Z}^n(S)$ be the free abelian group generated by $S^{(n)}$. If $y \in S^{(n-1)}$ and $f \in k(y)^*$ is a nonzero rational function field over the integral scheme $Y = \bar{\{y\}}$, we let $\operatorname{div}(f) \in \mathcal{Z}^n(S)$ be the divisor of f. The *Chow group* $CH^n(S)$ is by definition the quotient of $\mathcal{Z}^n(S)$ be the subgroup generated by all elements $\operatorname{div}(f)$ for $f \in k(y)^*$ and $y \in S^{(n-1)}$.

Recall 4.2. When X is a complex manifold, we had a cycle class map

$$cl : \mathcal{Z}^n(X) \to H^{2n}(X, \mathbb{Z}) \simeq H^{2n}(X, \mathbb{Z}(n))$$

where $\mathbb{Z} \simeq \mathbb{Z}(n)$ as abelian groups. For the construction of this map, see Chapter 11 of Vol I of [5] and Chapter 9 of Vol II of [5].

Recall 4.3 (Differential Character)**.** Let X be a differentiable manifold. Let $C_l^{diff}(X)$ be the group of differentiable singular chians of dimension l, and let Z_l^{diff} be the subgroup of closed chains for the differential ∂.

$\Xi^l_{diff}(X)$ is the subgroup of $\operatorname{Hom}(Z_l^{diff}, \mathbb{R}/\mathbb{Z})$ consisting of $\xi : Z_l^{diff}\mathbb{R}/\mathbb{Z}$, called *differential characters*, such that there exists a real differential form (obviously uniquely determined by χ) $\omega \in A^{l+1}(X)$ satisfying

$$\chi(\partial\phi) = \int_{\Delta_{l+1}} \phi^*\omega \mod \mathbb{Z}, \quad \forall \phi \in C^{diff}_{l+1}(X).$$

It is easy to see that (see [5]) above form ω is closed, and the associated de Rham class $[\omega]$ is an integral class, i.e. lies in the image of the natural map

$$H^{l+1}(X, \mathbb{Z}) \to H^{l+1}(X, \mathbb{R}).$$

Now let X be a complex manifold and $\mu \in A^{l-1}(X)$ be a real form. Let $\phi = \sum_i n_i\phi_i$ with $\phi_i : \Delta_l \to X$ be a closed chain. Then,

$$\overline{\int_{\Delta_l} \phi^*(i\partial\mu)} = \int_{\Delta_l} \phi^*(-i\bar{\partial}\mu) = \int_{\Delta_l} \phi^*(-id\mu + i\partial\mu) \overset{\text{Stokes}}{=} \int_{\Delta_l} \phi^*(i\partial\mu),$$

so that $\int_{\Delta_l} \phi^*(i\partial\mu) := \sum_i n_i \int_{\Delta_l} \phi_i^*(i\partial\mu) \in \mathbb{R}$.

Thus, we can associate to μ the differential character $\phi \to \int_{\Delta_l} \chi^*(i\partial\mu) \mod \mathbb{Z}$ which we write $\int i\partial\mu$. Then we have the following result from [5]:

Theorem 4.4. *Let X be compact Kähler. Let*

$$\Xi^{2p-1}_{diff}(X)^{p,p} \subset \Xi^{2p-1}_{diff}(X)$$

be the subgroup consisting of the differential characters whose associated form ω is of type (p, p). Then,

$$H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)) \simeq K^{2p-1}_{diff}(X) := \Xi^{2p-1}_{diff}(X)^{p,p}/\sim$$

where $\sim$ is given by the subgroup generated by all $\int i\partial\mu$ for $\mu \in A^{p-1,p-1}_{\mathbb{R}}(X)$.

The proof uses Theorem 2.4. The point is that elements in some Delign-Beilinson cohomology groups can be represented by differential characters. Using this idea, from [5] we can define the following cycle class map $cl_{\mathcal{D}}$:

$$cl_{\mathcal{D}} : CH^p(X) \to H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))$$

and for a cycle Z, we let $[Z]_{\mathcal{D}} = cl_{\mathcal{D}}(Z)$. This lifts the class $[Z] \in \mathrm{Hdg}^p(X) = H^{2p}(X, \mathbb{Z}(p)) \cap H^{p,p}(X)$ to $[Z]_{\mathcal{D}}$ for the short exact sequence of Proposition 2.11:

$$0 \to J^p(X) \to H^{2p}_{calD}(X, \mathbb{Z}(p)) \to \mathrm{Hdg}^p(X) \to 0.$$

Recall 4.5. Let $CH^p_{hom}(X) = \ker(cl : CH^p(X) \to H^{2p}(X, \mathbb{Z}(p)))$, i.e. cycles Z homologous to 0. Let $B^p(X)$ be the cokernel of cl.

Theorem 4.6. *We have the following commutative diagram[7]:*

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & CH^p_{hom}(X) \arrow[r] \arrow[d, "\phi_p = cl_{\mathcal{D}}|_{CH^p_{hom}(X)}"] & CH^p(X) \arrow[r] \arrow[d, "cl_{\mathcal{D}}"] & B^p(X) \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & J^p(X) \arrow[r] & H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)) \arrow[r] & \mathrm{Hdg}^p(X) \arrow[r] & 0
\end{tikzcd}$$

Here, the map ϕ_p coincides with the map

$$\Phi^p_X : CH^p_{hom}(X) \to J^p(X)$$

which is the Abel-Jacobi map.

Remark. The Delign class map $cl_{\mathcal{D}}$ respects the cup product for the Delign-Beilinson cohomology, and functorial. For the proof, see [6].

5. Applications

6. References

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