AN ELEMENTARY SET FOR θ_n-CURVE PROJECTIONS

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Abstract. A finite set of nontrivial θ_n-curves is shown to be minimal among those which produce all projections of nontrivial θ_n-curves.

1. Introduction

A θ_n*-curve* is the image of a piecewise-linear embedding of the graph with two vertices and n edges between them into $\mathbf{R}^3$. A θ_1-curve is a simple arc, a θ_2-curve is called a *knot* and a θ_3-curve is called a θ*-curve*. A *projection* $\bar{\Theta}$ of a θ_n-curve Θ is the image of a projection map from Θ to a plane in general position with Θ in the sense that finitely many transverse double points away from the vertices are the only singular points of $\bar{\Theta}$. A projection together with over and under information of crossing points is called a *diagram*. Two θ_n-curves Θ and Θ' are said to be *equivalent*, if there exists a homeomorphism of $\mathbf{R}^3$ carrying Θ to Θ'. A θ_n-curve is said to be *trivial* if it is equivalent to a θ_n-curve contained in a plane. If $1 \leq m < n$, a θ_m*-subcurve* of a θ_n-curve Θ is a θ_m-curve consisting of m edges of Θ.

A set E of nontrivial θ_n-curves is called *elementary* if the set of the projections of the elements of E contains every projection of any nontrivial θ_n-curves and it is minimal among the sets with this property.

There are several results in this direction. Taniyama [T1, T2] showed that the set of only one trefoil knot is the only elementary set for knots and that the Hopf link and the split union of a trefoil knot and a trivial knot constitute an elementary set for 2-component links. It was shown in [HJO, KM] that $\{3_1^*, 3_1, 5_1\}$ is an elementary set for θ-curves. These three θ-curves are depicted in Figure 1. Taniyama and Yoshioka [TY] also constructed an elementary set for handcuff graphs having infinitely many elements.

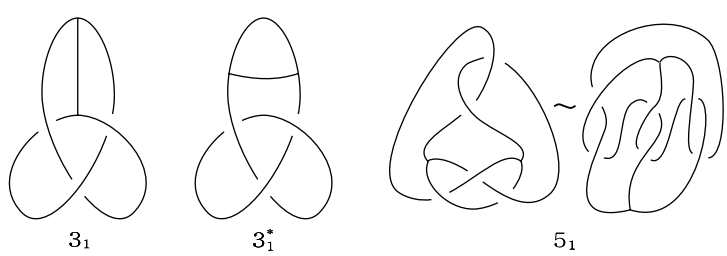

Figure 1. Elementary set for θ-curves

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2. Main result

For $n \geq 3$ and $1 \leq m \leq n$, we define θ_n-curves $E_m(n)$ as follows. We construct $E_1(n)$ and $E_2(n)$ by attaching a trivial $\theta_{(n-2)}$-curve to a trefoil knot as we attach an edge to obtain 3_1^* and 3_1, respectively. We construct $E_n(n)$ by drawing a diagram on a sphere so that it looks as in Figure 2 in each of the n regions divided by n great semicircles whose end points are at the north and south poles. Suzuki proved that $E_n(n)$'s are nontrivial and any θ_m-subcurves, for $1 \leq m < n$, are trivial [S]. Notice that $E_1(3)$, $E_2(3)$ and $E_3(3)$ are equivalent to 3_1^*, 3_1 and 5_1, respectively. For $3 \leq m < n$, we construct $E_m(n)$ by attaching a trivial $\theta_{(n-m)}$-curve to $E_m(m)$ inside the sphere used to construct it.

Figure 2. A portion of $E_n(n)$

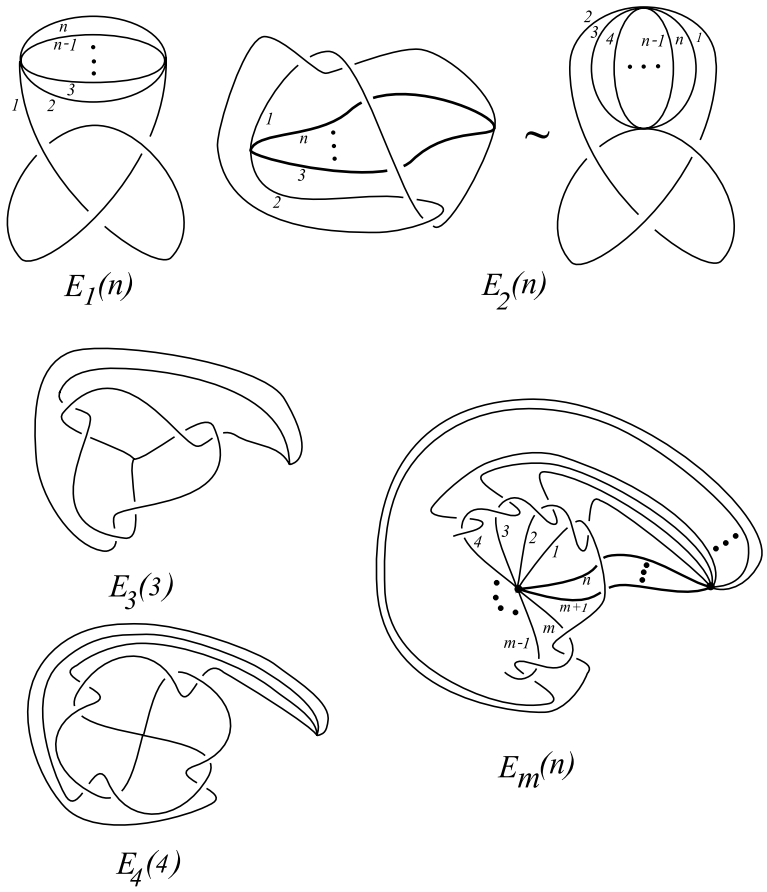

Figure 3. $E_m(n)$'s

Theorem 1. *For any $n \geq 3$, $\{E_1(n), E_2(n), E_3(n), \cdots, E_n(n)\}$ is an elementary set for θ_n-curves.*

Let $\bar{\Theta}$ be a projection of a θ_n-curve Θ. We may assume $\bar{\Theta}$ is contained in $\mathbf{R}^2 = \mathbf{R}^3_+ \cap \mathbf{R}^3_-$. For any subset Λ of $\mathbf{R}^3$, we use $\bar{\Lambda}$ to denote the image of Λ under the projection $\mathbf{R}^3 \to \mathbf{R}^2$ so that if $\Lambda \subset \Theta$ then $\bar{\Lambda} \subset \bar{\Theta}$. Let Λ be a closed connected subset of Θ. We use $\partial\Lambda$ to denote the boundary of Λ in Θ. See Figure 4(a) for an example in which the remaining part $\bar{\Theta} - \bar{\Lambda}$ is not drawn. By modifying the process of obtaining a descending diagram from a knot projection, we may find a homeomorphic image Λ' of Λ in $\mathbf{R}^3_+$ (resp. $\mathbf{R}^3_-$) satisfying the following conditions:

1. $\bar{\Lambda}' = \bar{\Lambda}$.
2. $\Lambda' \cap \mathbf{R}^2 = \partial\bar{\Lambda}$
3. There exists a non-singular disc Δ in $\mathbf{R}^3_+$ (resp. $\mathbf{R}^3_-$) containing Λ' such that $\Delta \cap \mathbf{R}^2 \subset \partial\Delta$.

The process of obtaining Λ' is called a *trivialization* of Λ into $\mathbf{R}^3_+$ (resp. $\mathbf{R}^3_-$).

Let Θ have edges $\{e_1, \ldots, e_n\}$ and vertices $\{v_1, v_2\}$. We say a projection $\bar{\Theta}$ of Θ is *in braided shape* if there exist parametrizations $e_i \colon t \mapsto e_i(t)$, $t \in [0,1]$, with $e_i(0) = v_1$, for $i = 1, 2, \ldots, n$, so that if $\bar{e}_i(t_i) = \bar{e}_j(t_j)$ then $t_i = t_j$. Note that a projection which has a self-crossing of an edge is not in braided shape. We say that $\bar{\Theta}$ is *m-braided*, $1 \leq m \leq n$, if the image of any θ_m-subcurve of Θ in $\bar{\Theta}$ is in braided shape. In particular, $\bar{\Theta}$ is 1-braided if and only if none of the edges of Θ has a self-crossing in $\bar{\Theta}$. Note that if $\bar{\Theta}$ is in braided shape then Θ is trivial, and if $\bar{\Theta}$ is m-braided, $m > 1$, then it is also $(m-1)$-braided.

A crossing point separating a projection into two disjoint parts is called a *nugatory* point. A projection is said to be *reduced*, if it has no nugatory points. If $\bar{\Theta}$ has nugatory points, we may isotope Θ to obtain Θ' whose projection $\bar{\Theta}'$ is reduced. See Figure 5. Therefore we may always assume that $\bar{\Theta}$ is reduced.

Theorem 1 easily follows from the three lemmas below.

Lemma 2. *If $\bar{\Theta}$ is not 1-braided, then it is a projection of $E_1(n)$ or $E_2(n)$.*

Lemma 3. *If $\bar{\Theta}$ is $(m-1)$-braided but not m-braided, $m = 2, \cdots, n$, then it is a projection of $E_m(n)$.*

Lemma 4. *No proper subset of $\{E_1(n), E_2(n), \cdots, E_n(n)\}$ is an elementary set for θ_n-curves.*

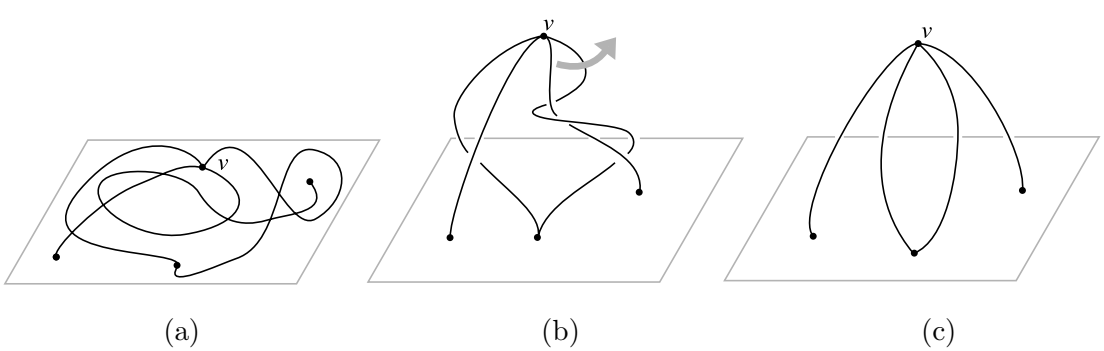

FIGURE 4. A trivialization which is followed by an isotopy

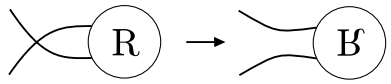

FIGURE 5. Removing a nugatory point

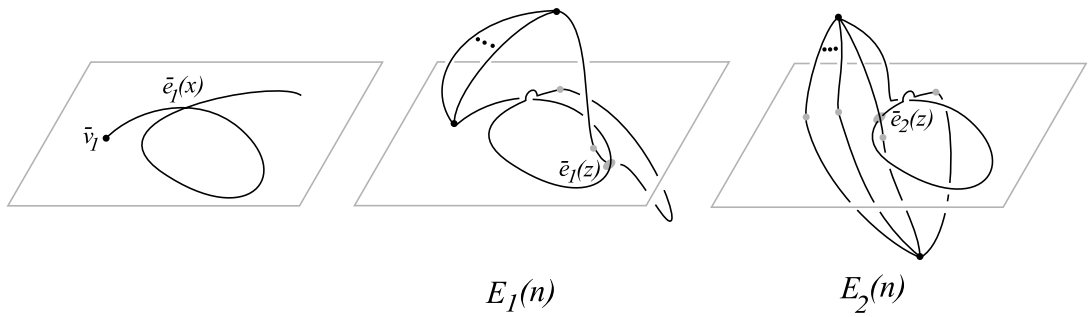

FIGURE 6. Realizing $E_1(n)$ and $E_2(n)$

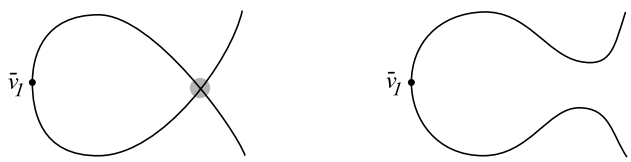

FIGURE 7. Eliminating a bigon with a vertex at $\bar{v}_1$

3. PROOF OF LEMMAS

Proof of Lemma 2. Assume that $\bar{\Theta}$ is not 1-braided. We parametrize all edges of Θ starting at a same vertex over the unit interval $[0,1]$. There is an edge, say e_1, whose projection has a self-crossing. Then $\bar{e}_1(x) = \bar{e}_1(y)$, for some x, y with $x < y$ such that $\bar{e}_1[x,y]$, the projection of the image of the closed interval $[x,y]$ under e_1, is a simple closed curve and $\bar{e}_1[0,x]$ is a simple arc which does not intersect $\bar{e}_1(x,y)$. See Figure 6. Since there is no nugatory crossing, $\bar{e}_1(x,y)$ contains another crossing point.

If this crossing is with e_1, again, we have w, z such that $x < w < y < z$ and $\bar{e}_1(w) = \bar{e}_1(z)$. For a sufficiently small $\epsilon > 0$, we trivialize the parts $\bar{e}_1[z+\epsilon, 1] \cup \bar{e}_2[0,1] \cup \cdots \cup \bar{e}_n[0,1]$ and $\bar{e}_1[y-\epsilon, y+\epsilon]$ into $\mathbf{R}^3_+$ and $\bar{e}_1[y+\epsilon, z+\epsilon]$ into $\mathbf{R}^3_-$. The resulting θ_n-curve is equivalent to $E_1(n)$.

If $\bar{e}_1(x,y)$ has a crossing point with an edge other edge than e_1, say e_2, we let $\bar{e}_2(z)$ be the first crossing point of $\bar{e}_2$ with $\bar{e}_1(x,y)$. For a sufficiently small $\epsilon > 0$, we trivialize the parts $\bar{e}_1[0, x-\epsilon] \cup \bar{e}_2[0, z+\epsilon] \cup \bar{e}_3[0,\epsilon] \cup \cdots \cup \bar{e}_n[0,\epsilon]$ and $\bar{e}_1[y-\epsilon, y+\epsilon]$ into $\mathbf{R}^3_+$ and $\bar{e}_1[y+\epsilon, 1] \cup \bar{e}_2[z+\epsilon, 1] \cup \bar{e}_3[\epsilon, 1] \cup \cdots \cup \bar{e}_n[\epsilon, 1]$ into $\mathbf{R}^3_-$. The resulting θ_n-curve is equivalent to $E_2(n)$. $\square$

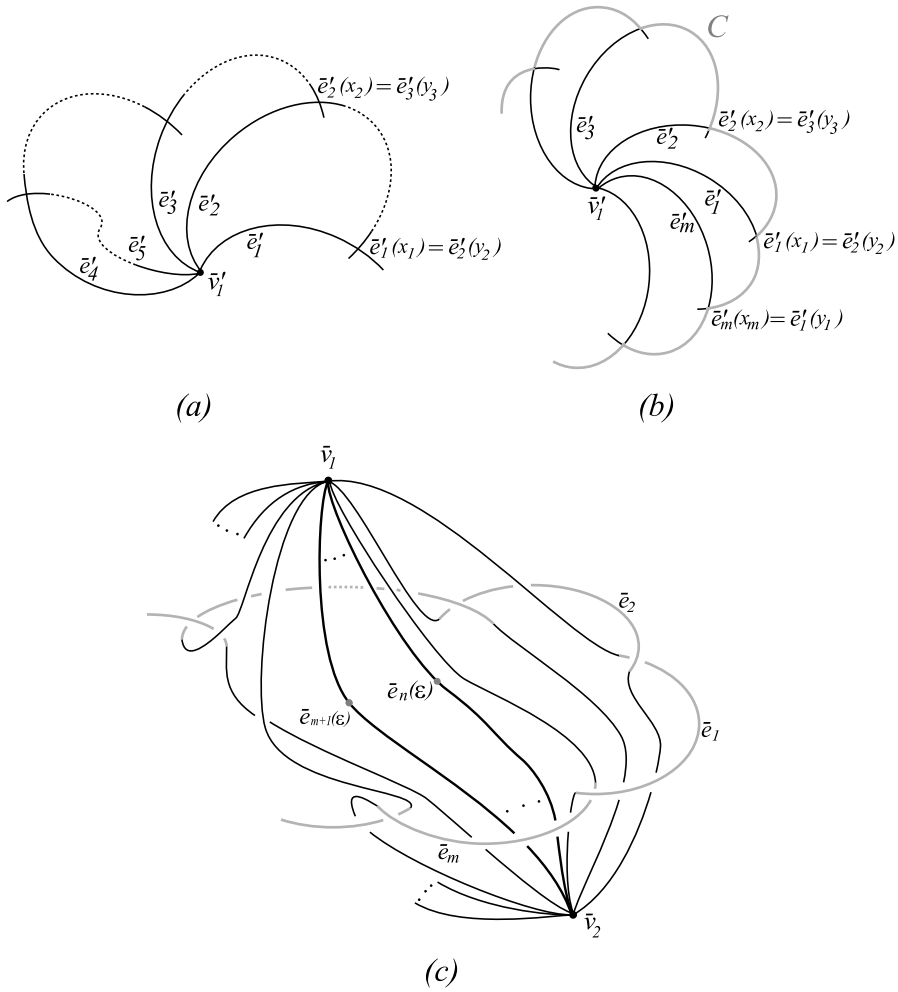

FIGURE 8. Finding a desired curve C

Proof of Lemma 3. Assume that $\bar{\Theta}_m$ is not m-braided for a subcurve Θ_m with m edges $\{e_1, \cdots, e_m\}$. Then $\bar{\Theta}_m$ is not in braided shape, but still $(m-1)$-braided. As drawn in Figure 7, we modify $\bar{\Theta}_m$ by eliminating all bigons with a vertex at $\bar{v}_1$, including those which may appear during this process. The resulting θ_m-curve projection, denoted by $\bar{\Theta}'_m$, is also $(m-1)$-braided but not in braided shape, and has no bigons with a vertex at $\bar{v}_1$. Now we concentrate only on $\bar{\Theta}'_m$. We parametrize every edge of $\bar{\Theta}'_m$ starting from $\bar{v}_1$ over the unit interval. Take an edge $\bar{e}'_1$ and let $\bar{e}'_1(x_1)$ be the crossing point where $\bar{e}'_1$ firstly meets the projection of the other edges, i.e. $\bar{e}'_1(0, x_1] \cap \{\bar{e}'_2 \cup \cdots \cup \bar{e}'_m\} = \bar{e}'_1(x_1)$. We may say $\bar{e}'_2$ crosses $\bar{e}'_1$ at $\bar{e}'_1(x_1) = \bar{e}'_2(y_2)$. Recall that each $\bar{e}'_i$ has no self-crossing. Since there is no bigon with a vertex at $\bar{v}_1$, the part $\bar{e}'_2(0, y_2)$ has crossing points and these can't come from $\bar{e}'_1$ because of $(m-1)$-braidedness. Again, we may say $\bar{e}'_2$ firstly meets $\bar{e}'_3$ at $\bar{e}'_2(x_2) = \bar{e}'_3(y_3)$. We continue the same process of renaming $\bar{e}'_i$'s for $i \leq m$. The $(m-1)$-braidedness of $\bar{\Theta}'_m$ guarantees that $\bar{e}'_i$, for $1 \leq i < j \leq m$, cannot meet $\bar{e}'_j(0, y_j)$. By the same reason $\bar{e}'_m(0, y_m)$ must have crossing points and these come from $\bar{e}'_1$ only. For, if $\bar{e}'_m(0, y_m)$ meets some other $\bar{e}'_i$ then the projection

of $m-i+1$ edges $\{e'_i, \cdots, e'_m\}$ is not in braided shape. It contradicts the $(m-1)$-braidedness of $\bar{\Theta}'_m$. Let $\bar{e}'_m(x_m) = \bar{e}'_1(y_1)$ be the first crossing point on $\bar{e}'_m$.

We claim that at each crossing point $\bar{e}'_i(x_i)$, $\bar{e}'_{i+1}$ doesn't cross $\bar{e}'_i$ from inside to outside of the simple closed curve $J_i = \bar{e}'_{i-1}[0, x_{i-1}] \cup \bar{e}'_i[0, y_i]$ as $\bar{e}'_5$ does in Figure 8(a). For, if so, the $(m-1)$-braidedness of $\bar{\Theta}'_m$ forces all of $\bar{e}'_j(0, y_j)$, $i+1 \le j \le m$, to be contained inside the curve J_i so that $\bar{e}_1$ has to cross $\bar{e}'_{i-1}(0, x_{i-1}) \cup \bar{e}'_i(0, y_i)$ to meet the first crossing point $\bar{e}'_m(x_m)$ of $\bar{e}'_m$. This contradicts the $(m-1)$-braidedness again. Furthermore the $(m-1)$-braidedness also guarantees that $C = \bar{e}'_1[x_1, y_1] \cup \cdots \cup \bar{e}'_m[x_m, y_m]$ is a simple closed curve separating the vertices $\bar{v}_1$ and $\bar{v}_2$. Finally, the part $\bar{e}'_1[0, y_1] \cup \cdots \cup \bar{e}'_m[0, y_m]$ of $\bar{\Theta}'_m$ is as depicted in Figure 8(b).

Now back to the original projection $\bar{\Theta}$. Two curves $\bar{\Theta}_m$ and $\bar{\Theta}'_m$ differ only on the small neighborhoods of the crossings of $\bar{\Theta}_m$ which are contained inside C and were eliminated to remove bigons with one vertex at $\bar{v}_1$. Therefore $\bar{\Theta}$ contains the curve C. We may relabel and parametrize the edges of $\bar{\Theta}$ so that $\bar{e}_1(x_1) = \bar{e}_2(y_2), \ldots, \bar{e}_{m-1}(x_{m-1}) = \bar{e}_m(y_m), \bar{e}_m(x_m) = \bar{e}_1(y_1)$, and $C = \bar{e}_1[x_1, y_1] \cup \cdots \cup \bar{e}_m[x_m, y_m]$. We choose a sufficiently small $\epsilon > 0$. By reparametrizing the edges $\bar{e}_{m+1}, \ldots, \bar{e}_n$, we may assume that $\bar{e}_{m+1}(\epsilon), \ldots, \bar{e}_n(\epsilon)$ are located inside C. At each crossing point $\bar{e}_i(y_i)$ on the curve C, we trivialize $\bar{e}_i[y_i - \epsilon, y_i + \epsilon]$ into $\mathbf{R}^3_+$. We also trivialize the parts $\bar{e}_1[0, x_1 - \epsilon] \cup \cdots \cup \bar{e}_m[0, x_m - \epsilon] \cup \bar{e}_{m+1}[0, \epsilon] \cup \cdots \cup \bar{e}_n[0, \epsilon]$ into $\mathbf{R}^3_+$ and $\bar{e}_1[y_1 + \epsilon, 1] \cup \cdots \cup \bar{e}_m[y_m + \epsilon, 1] \cup \bar{e}_{m+1}[\epsilon, 1] \cup \cdots \cup \bar{e}_n[\epsilon, 1]$ into $\mathbf{R}^3_-$. We may do it so that the resulting θ_n-curve is $E_m(n)$ as in Figure 8(c). □

Proof of Lemma 4. For each $m = 1, 2, \ldots, n$, let $\bar{E}_m(n)$ be the projection of $E_m(n)$ obtained from the diagrams in Figure 3. For $m = 2$, we take one on the right. Let $\mathcal{E}_m(n)$ be the set of all nontrivial θ_n-curves which project onto $\bar{E}_m(n)$.

Suppose $m \ge 3$. In $\bar{E}_m(n)$, for $i = 1, 2, \ldots, m$, every two adjacent edges e_i and e_{i+1} (e_1 if $i = m$) share a pair of crossings. If any of such pair is chosen to be non-alternating, the result is a trivial θ_n-curve. Furthermore, if every pair of such two crossings is alternating(i.e. it corresponds to a full twist), then $e_1 \cup \cdots \cup e_m$ is a nontrivial θ_m-curve. If all the pairs are of the same type, say a right-handed full twist, the nontriviality of this θ_m-curve is shown in [S]. If both a left-handed full twist and a right-handed full twist exist, we need Yokota's polynomial for θ_m-curves to prove the nontriviality [Y].

Each element of $\mathcal{E}_1(n)$ contains $n-1$ nontrivial θ_2-subcurves (i.e., knots). For $2 \le i \le m$, each element of $\mathcal{E}_i(n)$ contains a unique nontrivial θ_i-subcurve whose proper subcurves are all trivial. Therefore the sets $\mathcal{E}_i(n)$, $1 \le i \le n$, are mutually disjoint. Therefore any elementary set for θ_n-curves must have at least one element in common with each $\mathcal{E}_i(n)$, $1 \le i \le n$. In particular, any elementary set must have at least n elements. □

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