Equicovering Subgraphs of Graphs and Hypergraphs

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Outline

1. Definitions
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4. The $t$-EVP of Hypergraphs
5. Open Questions
1 Definitions

2 The 2-EUP and 2-EVP of Graphs

3 The $t$-EVP of Graphs

4 The $t$-EVP of Hypergraphs

5 Open Questions
A hypergraph $H$ has the \textit{t-Equal Union Property} (t-EUP) if there are $t$ distinct subhypergraphs $H_1, \ldots, H_t$ of $H$ such that

1. $E(H_i) \cap E(H_j) = \emptyset$ for $1 \leq i < j \leq t$
2. $\bigcup_{e \in E(H_i)} e = \bigcup_{e \in E(H_j)} e$ for $1 \leq i < j \leq t$
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The hypergraph on the left has 2-EUP.
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![Diagram showing Has 2-EUP](image_url)
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**Theorem (Lindström (1972))**

*If a hypergraph $H$ has more than $(t - 1)n$ edges, then $H$ has the t-EUP.*
t-Equal Valence Property

A hypergraph $H$ has the **$t$-Equal Valence Property** ($t$-EVP) if there are $t$ distinct subhypergraphs $H_1, \ldots, H_t$ of $H$ such that

1. $E(H_i) \cap E(H_j) = \emptyset$ for $1 \leq i < j \leq t$
2. $d_{H_i}(v) = d_{H_j}(v)$ for $v \in V(H)$ and $1 \leq i < j \leq t$

Note that the $t$-EVP is a stronger property than $t$-EUP.
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![Diagram](image-url)
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Note that the $t$-EVP is a stronger property than $t$-EUP.
Theorem (Lindström (1972))

If a hypergraph $H$ has more than $(t - 1)n$ edges, then $H$ has the $t$-EUP.
Theorem (Lindström (1972))

If a hypergraph \( H \) has more than \( (t - 1)n \) edges, then \( H \) has the \( t \)-EUP.

Maximum number of edges \( \mathbb{U}(n, t) \) in a hypergraph without the \( t \)-EUP.
Maximum number of edges \( \mathbb{V}(n, t) \) in a hypergraph without the \( t \)-EVP.
Equicovering Subgraphs of Graphs and Hypergraphs

Notation

**Theorem (Lindström (1972))**

\[ \mathcal{U}(n, t) \leq (t - 1)n \]

Maximum number of edges \( \mathcal{U}(n, t) \) in a hypergraph without the \( t \)-EUP. Maximum number of edges \( \mathcal{V}(n, t) \) in a hypergraph without the \( t \)-EVP.
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<table>
<thead>
<tr>
<th>( n )-vertex graphs</th>
<th>( t )-EUP</th>
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</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{U}_2(n, t) )</td>
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Note that \( A(n, t) \geq A_k(n, t) \) for \( A \in \{ \mathcal{U}, \mathcal{V} \} \).
1 Definitions

2 The 2-EUP and 2-EVP of Graphs

3 The $t$-EVP of Graphs

4 The $t$-EVP of Hypergraphs

5 Open Questions
Characterization of Graphs with the 2-EUP

**Theorem**

*A graph $G$ has the 2-EUP if and only if $G$ has an even cycle or has two odd cycles in the same component.*

**Corollary**

$$\mathbb{U}_2(n, 2) = n$$

*Equality holds only for either connected graphs with only one odd cycle or the disjoint union of odd cycles.*
Characterization of Graphs with the 2-EUP: the Proof

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A graph $G$ has the 2-EVP if and only if $G$ has an even circuit.
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A graph is an *odd-cycle-forest* if it can be obtained in this fashion:
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A graph $G$ does not have the 2-EVP if any only if $G$ is an odd-cycle-forest.
Characterization of Graphs with the 2-EVP

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Theorem
A graph $G$ does not have the 2-EVP if any only if $G$ is an odd-cycle-forest.

Corollary
\[ V_2(n, 2) = \left\lfloor \frac{4}{3} n \right\rfloor - 1 \]

Equality holds only for odd-cycle-trees obtained by replacing all vertices in a tree by triangles.
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5 Open Questions
The $t$-EVP of Graphs: Upper Bound

Theorem

For $t \in \mathbb{N}$,

$$V_2(n, t) \leq 4(t-1)n$$

Proof.

Let $G$ have $4(t-1)n + 1$ edges.

$\Rightarrow$ a bipartite $H \subseteq G$ has $2(t-1)n + 1$ edges

$\Rightarrow$ a subgraph $H' \subseteq H$ has $(q-1)n + 1$ edges for prime $t \leq q < 2t$

$\Rightarrow$ a $q$-divisible subgraph $Q \subseteq H'$

A graph $Q$ is a $q$-divisible graph if the degree of each vertex in $Q$ is a multiple of an integer $q$.

Lemma

If $Q$ is a $q$-divisible bipartite graph, then $Q$ has the $q$-EVP.
The $t$-EVP of Graphs: Upper Bound

**Theorem**

*For* $t \in \mathbb{N}$,

$$V_2(n, t) \leq 4(t - 1)n$$
The $t$-EVP of Graphs: Upper Bound

**Theorem**

For $t \in \mathbb{N}$,

$$\mathbb{V}_2(n, t) \leq 4(t - 1)n$$

**Proof.** Let $G$ have $4(t - 1)n + 1$ edges.
The $t$-EVP of Graphs: Upper Bound

**Theorem**

For $t \in \mathbb{N}$, \[ \nabla_2(n, t) \leq 4(t - 1)n \]

**Proof.** Let $G$ have $4(t - 1)n + 1$ edges.

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For $t \in \mathbb{N}$,

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$\Rightarrow$ a $q$-divisible subgraph $Q \subseteq H'$

A graph $Q$ is a **$q$-divisible graph** if the degree of each vertex in $Q$ is a multiple of an integer $q$.

**Lemma**

If $Q$ is a $q$-divisible bipartite graph, then $Q$ has the $q$-EVP.
Every \((t-1)\)-degenerate graph does not have the \(t\)-EVP. This gives
\[
\forall_2(n, t) \geq (t - 1)n
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Every \((t - 1)\)-degenerate graph does not have the \(t\)-EVP. This gives

\[
\mathcal{V}_2(n, t) \geq (t - 1)n
\]

Construct \(G_t^a\) in the following manner:

Let \(W_t = \overline{K_{t-2}} \lor C_{t+1}\). For \(a\) copies of \(W_t\), do:
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Construct \(G^a_t\) in the following manner:

Let \(W_t = \overline{K_{t-2}} \lor C_{t+1}\). For \(a\) copies of \(W_t\), do:
Every \((t - 1)\)-degenerate graph does not have the \(t\)-EVP. This gives

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The $t$-EVP of Graphs: Lower Bound

Lemma

$G^a_t$ does not have the $t$-EVP.
The $t$-EVP of Graphs: Lower Bound

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$G_t^a$ does not have the $t$-EVP.
The $t$-EVP of Graphs: Lower Bound

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Let $H_{t,k}^a$ be a $k$-uniform hypergraph such that $V(H_{t,k}^a) = V(G_t^a) \cup S$ and $E(H_{t,k}^a) = \{e \cup S : e \in E(G_t^a)\}$. Then $H_{t,k}^a$ does not have the $t$-EVP.
The $t$-EVP of Graphs: Lower Bound

**Lemma**

$G^a_t$ does not have the $t$-EVP.

Let $H^a_{t,k}$ be a $k$-uniform hypergraph such that $V(H^a_{t,k}) = V(G^a_t) \cup S$ and $E(H^a_{t,k}) = \{e \cup S : e \in E(G^a_t)\}$. Then $H^a_{t,k}$ does not have the $t$-EVP.

**Theorem**

For some polynomial $f_k(t)$ with degree at most 2,

$$\forall_k(n, t) \geq \left(t - 1 + \frac{1}{2(t-1)}\right)n - f_k(t)$$
1. Definitions

2. The 2-EUP and 2-EVP of Graphs

3. The $t$-EVP of Graphs

4. The $t$-EVP of Hypergraphs

5. Open Questions
The 2-EVP of Hypergraphs: Proof

**Theorem**

For an $n$-vertex $k$-uniform hypergraph with $k \geq 3$,

$$\nabla_k(n, 2) < (\log_2 k + (1 + \varepsilon_k) \log_2 \log_2 k)n$$

for some $\varepsilon_k > 0$, where $\varepsilon_k \to 0$ as $k \to \infty$. 
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**Proof.** $H$ with $m = (\log k + (1 + \varepsilon_k) \log \log k)n$ edges has $2^m$ subgraphs. Let $(d_1, \ldots, d_n)$ be the degree list of $H$. Since $\sum_{i=1}^{n} d_i = km$, there exists at most $\prod_{i=1}^{n}(d_i + 1) \leq \left(\frac{km}{n} + 1\right)^n$ degree lists for subhypergraphs.
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**Theorem**

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**Proof.** Let $H$ with $m = (\log k + (1 + \varepsilon_k) \log \log k)n$ edges has $2^m$ subgraphs. Let $(d_1, \ldots, d_n)$ be the degree list of $H$. Since $\sum_{i=1}^n d_i = km$, there exists at most $\prod_{i=1}^n (d_i + 1) \leq \left(\frac{km}{n} + 1\right)^n$ degree lists for subhypergraphs.

If $H$ does not have 2-EVP $\Rightarrow$ subhypergraphs must have different lists.

$$2^m \leq \left(\frac{km}{n} + 1\right)^n \Rightarrow (k(\log_2 k)^{1+\varepsilon_k})^n \leq (k \log_2 k + (1+\varepsilon_k)k \log_2 \log_2 k+1)^n$$

With appropriate $\varepsilon_k$, contradiction. Hence, $H$ has the 2-EVP. $\Box$
The 2-EVP of Hypergraphs

**Theorem**

For an $n$-vertex $k$-uniform hypergraph with $k \geq 3$,

$$\nabla_k(n, 2) < (\log_2 k + (1 + \varepsilon_k) \log_2 \log_2 k)n$$

for some $\varepsilon_k > 0$, where $\varepsilon_k \to 0$ as $k \to \infty$.

$\varepsilon_k$ must be greater than a root of

$$(\log_2 k)^{1+\varepsilon_k} - \log_2 k - (1 + \varepsilon_k) \log_2 \log_2 k - \frac{1}{k} = 0$$

As $k$ increases, $\varepsilon_k \to 0$, but at a very slow rate.

When $k = 10^{42}$ the constant $\varepsilon_k$ still needs to be larger than 0.01.

**Corollary**

$$\nabla_3(n, 2) < 3.5377n$$
The $t$-EVP of Hypergraphs

By refining the previous argument using known results about $r$-$\Delta$-systems, we prove the following theorems.

**Theorem**

Let $t \in \mathbb{N}_{\geq 3}$ and $\varepsilon > 0$. There exists $N = N(t, \varepsilon)$ such that for $n \geq N$,

$$ V(n, t) < (4 + \varepsilon)n^2 \left( \frac{\log n}{\log \log \log n} \right)^2 $$

**Theorem**

Let $t \in \mathbb{N}_{\geq 3}, k \in \mathbb{N}_{\geq 2}$, and $\varepsilon > 0$. There exists $N = N(t, k, \varepsilon)$ such that for $n \geq N$,

$$ V_k(n, t) < (1 + \varepsilon)n^2 \left( \frac{\log n}{\log \log \log n} \right)^2 $$
Equicovering Subgraphs of Graphs and Hypergraphs

Open Questions

1. Definitions

2. The 2-EUP and 2-EVP of Graphs

3. The $t$-EVP of Graphs

4. The $t$-EVP of Hypergraphs

5. Open Questions
We know $\mathcal{U}_2(n, 2) = n$ and $\mathcal{V}_2(n, 2) = \lfloor \frac{4}{3} n \rfloor - 1$. 

Equality holds only for adding an edge to an odd-cycle-tree obtained by replacing all vertices in a tree by triangles.

What about other exact values for graphs? In particular, $\mathcal{V}_2(n, 3)$?

We know $\left( t - 1 + \frac{1}{2} (t - 1) \right) n - f_k(t) \leq \mathcal{V}_2(n, t) \leq 4(t - 1) n$ for some function $f_k(t)$ of degree at most two. Close the gap?

Better lower bounds?
We know $U_2(n, 2) = n$ and $V_2(n, 2) = \lfloor \frac{4}{3} n \rfloor - 1$.

**Theorem**

$$V_2'(n, 3) = \left\lfloor \frac{4}{3} n \right\rfloor$$

*Equality holds only for adding an edge to an odd-cycle-tree obtained by replacing all vertices in a tree by triangles.*
We know \( \mathcal{U}_2(n, 2) = n \) and \( \mathcal{V}_2(n, 2) = \lfloor \frac{4}{3} n \rfloor - 1 \).

**Theorem**

\[
\mathcal{V}'_2(n, 3) = \left\lfloor \frac{4}{3} n \right\rfloor
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1. We know $U_2(n, 2) = n$ and $V_2(n, 2) = \left\lfloor \frac{4}{3} n \right\rfloor - 1$.

**Theorem**

$$V'_2(n, 3) = \left\lfloor \frac{4}{3} n \right\rfloor$$

*Equality holds only for adding an edge to an odd-cycle-tree obtained by replacing all vertices in a tree by triangles.*

What about other exact values for graphs? In particular, $V_2(n, 3)$?

2. We know $\left(t - 1 + \frac{1}{2(t-1)}\right) n - f_k(t) \leq V_2(n, t) \leq 4(t - 1)n$ for some function $f_k(t)$ of degree at most two. Close the gap?
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What about other exact values for graphs? In particular, $\mathcal{V}_2(n, 3)$?

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3. Better lower bounds?
Thank you!