Proper $\mathrm{SL}(2,\mathbb{R})$-actions on pseudo-Riemannian Symmetric Spaces

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Introduction

Classical Geometries

geometry	curvature	space form
Spherical geometry	1	$\Gamma\backslash\mathbb{S}^n = \Gamma\backslash\mathrm{SO}(n+1)/\mathrm{SO}(n)$
Euclidean geometry	0	$\Gamma\backslash\mathbb{E}^n = \Gamma\backslash\mathrm{E}(n)/\mathrm{O}(n)$
Hyperbolic geometry	-1	$\Gamma\backslash\mathbb{H}^n = \Gamma\backslash\mathrm{SO}(1,n)_0/\mathrm{SO}(n)$

In general, for a connected Lie group G and a closed subgroup H, we say that a smooth manifold M is Clifford-Klein space form of G/H if it admits a complete $(G/H, G)$-structure.

Introduction

It is not even obvious whether a given homogeneous manifold G/H admits nontrivial Clifford-Klein form.

Question

Is there a nontrivial Clifford-Klein form of a homogeneous manifold G/H? Is there a nontrivial discrete subgroup of G that acts properly discontinuously and freely on G/H?

- Calabi and Markus showed that only a finite group can act properly discontinuously on the Lorentz space $\mathrm{SO}(n,1)_0/\mathrm{SO}(n-1,1)_0$, $n \geq 3$.
- Kobayashi found the equivalent condition that $\pi_1(\Gamma\backslash G/H)$ is always finite, when G/H is a reductive type homogeneous manifold.

Introduction

In this talk, we answer the following question:

Question

Under what conditions, does $\mathrm{SO}(p, q)/\mathrm{SO}(p-i, q-j) \times \mathrm{SO}(i, j)$ *admit proper discontinuous action of a surface group?*

Remark

$\mathrm{SO}(p, q)/\mathrm{SO}(p-i, q-j) \times \mathrm{SO}(i, j)$ *is interesting pseudo-Riemannian manifold because it generalizes many geometric spaces*

- $S^{p,q} = \mathrm{SO}(p+1, q)/\mathrm{SO}(p, q)$, *the pseudo-Riemannian manifold of signature* (p, q) *with constant sectional curvature* 1.
- $H^{p,q} = \mathrm{SO}(p, q+1)/\mathrm{SO}(p, q)$, *the pseudo-Riemannian manifold of signature* (p, q) *with constatn sectional curvature* -1.

Introduction

Question

Under what conditions, does $\mathrm{SO}(p,q)/\mathrm{SO}(p-i,q-j)\times \mathrm{SO}(i,j)$ *admit proper discontinuous action of a surface group?*

Our approach is:

1. Surface group Γ can be embedded as a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$.
2. Γ acts properly discontinuously on G/H if $\mathrm{SL}(2,\mathbb{R})$ does.
3. Find the conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$ in G. If two subgroups are conjugate then one acts properly on G/H if and only if the other does.
4. Show that each conjugacy class contains "good representative" and apply the Kobayashi theory.

Introduction

The main theorem of this talk is (terms will be defined)

Theorem

Let $\mathfrak{g} = \mathfrak{so}(p, q)$, $p \geq q > 0$. Let $\mathfrak{l}$ be an $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$ of type $(d_1, \cdots, d_e; d_{e+1}, \cdots, d_{e+l})$. Let $\mathfrak{h} = \mathfrak{so}(p-i, q-j) \oplus \mathfrak{so}(i, j)$. Let G, H and L be the corresponding Lie groups for $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$ respectively. Then, $(\mathfrak{g}, \mathfrak{h}, \mathfrak{l})$ satisfies the Kobayashi's criterion (i.e., L acts on G/H properly) if and only if

$$\min\{i, j\} + \min\{p-i, q-j\} < \lceil d_1 \rceil + \cdots + \lceil d_{e+l} \rceil.$$

Preliminaries

Definition

- Let G be a Lie group, X a smooth manifold where G acts smoothly. We say that G acts on X properly when $\{g \in G \mid gK \cap K \neq \emptyset\}$ is compact in G for all compact set $K \subset X$.
- We say that G acts on X properly discontinuously if, in addition, G is discrete.

Remark

Let Γ be a discrete subgroup of a Lie group L. Then Γ acts properly discontinuously on a space X if L acts properly on X.

Theorem

Every closed surface Σ of genus > 1 admits a complete hyperbolic structure. The associated holonomy representation $\pi_1(\Sigma) \to \mathrm{PSL}(2, \mathbb{R})$ can be lifted to $\mathrm{SL}(2, \mathbb{R})$ with discrete image.

Conjugacy Classes of $\mathrm{SL}(2,\mathbb{R})$ in $\mathrm{SO}(p,q)$

Following theorem connects Lie group representations and Lie algebra representations.

Theorem

There is a 1-1 correspondence between Lie algebra representations $\mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{gl}(n,\mathbb{R})$ and Lie group representations $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$. Moreover, the correspondence preserves irreducibility.

Therefore, studying conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$ in $\mathrm{SO}(p,q)$ amounts to studying conjugacy classes of $\mathfrak{sl}(2,\mathbb{R})$ in $\mathfrak{so}(p,q)$.

Conjugacy Classes of $\mathrm{SL}(2,\mathbb{R})$ in $\mathrm{SO}(p,q)$

The following theorem is well known for $\mathfrak{sl}(2,\mathbb{C})$ case. But we can get the same result for $\mathfrak{sl}(2,\mathbb{R})$.

Theorem

Let $\pi : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{gl}(V)$ be an n-dimensional irreducible representation on a $\mathbb{R}$-vector space V. Then there is a basis $\{v_0, \cdots, v_{n-1}\}$ for V satisfying following properties

- $\pi(H)(v_k) = (d-k)v_k$
- $\pi(X)(v_k) = \sqrt{\frac{1}{2}k(2d-k+1)}v_{k-1}$
- $\pi(Y)(v_k) = \sqrt{\frac{1}{2}(k+1)(2d-k)}v_{k+1}$.

for $k = 0, 1, \cdots, n-1$. Here, $n = 2d+1$ and $v_{-1} = v_n = 0$ and

$$H = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Conjugacy Classes of $\mathrm{SL}(2,\mathbb{R})$ in $\mathrm{SO}(p,q)$

Definition

- Let $\mathfrak{g}$ be a linear Lie algebra. By an $\mathfrak{sl}_2$-subalgebra in $\mathfrak{g}$, we mean any subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.
- $\mathfrak{sl}_2$-subalgebra $\mathfrak{l}$ is of type $(d_1,\cdots,d_e;d_{e+1},\cdots,d_{e+l})$ if irreducible components of the inclusion $\mathfrak{l}\hookrightarrow\mathfrak{g}$ (regarded as an Lie algebra representation) have dimension $2d_1+1,\cdots,2d_{e+l}+1$. We assume that $d_1,\cdots,d_e$ are half-integers and $d_{e+1},\cdots,d_{e+l}$ are integers.

Remark

- $\mathfrak{sl}(2,\mathbb{R})$ *is semisimple. Thus, Weyl theorem guarantees that the representation* $\mathfrak{l}\to\mathfrak{g}$ *is completely reducible.*
- *Types are well defined up to ordering.*

Conjugacy Classes of $\mathrm{SL}(2,\mathbb{R})$ in $\mathrm{SO}(p,q)$

Proposition

Let $\mathfrak{l}$ be an $\mathfrak{sl}_2$-subalgebra in $\mathfrak{so}(p,q)$ of type $(d_1,\cdots,d_e;d_{e+1},\cdots,d_{e+l})$.

- *$d_1,\cdots,d_e$ are nonnegative half-integers and $d_{e+1},\cdots,d_{e+l}$ are nonnegative integers. At least one of d_i is nonzero.*
- *$n_E := 2(d_1+\cdots+d_e)+e$ is divisible by 4.*
- *There is an nonnegative integer $r \le l$ such that*
  $p - n_E/2 = d_{e+1}+\cdots+d_{e+l}+r,$
  $q - n_E/2 = d_{e+1}+\cdots+d_{e+l}+l-r.$
- *$d_{e+1}+\cdots+d_{e+l} \le \min\{p,q\} - n_E/2.$*

Proof.

We interpret $\mathfrak{so}(p,q)$ in terms of invariant nondegenerate symmetric bilinear form. Using standard representation of $\mathfrak{sl}(2,\mathbb{R})$, one can show that such form exists if and only if $d_1,\cdots,d_e$ must occur in pairs. □

Conjugacy Classes of $\mathrm{SL}(2,\mathbb{R})$ in $\mathrm{SO}(p,q)$

Proposition

There is an one-to-one correspondence between conjugacy classes of $\mathfrak{sl}_2$-subalgebras in $\mathfrak{so}(p,q)$ and $e+l$ tuples $(d_1,\cdots,d_e;d_{e+1},\cdots,d_{e+l})$ such that

- $d_1,\cdots,d_e$ *are half integers and* $d_{e+1},\cdots,d_{e+l}$ *are integers with* $d_1\le d_2\le\cdots\le d_e$, $d_{e+1}\le\cdots d_{e+l}$.
- e *is even and* $d_1=d_2$, $d_3=d_4$, $\cdots$, $d_{e-1}=d_e$.
- $p-n_E/2=d_{e+1}+\cdots+d_{e+l}+r$,
  $q-n_E/2=d_{e+1}+\cdots+d_{e+l}+l-r$ *for some nonnegative* $s\le l$.

Proof.

Forward correspondence is due to the previous proposition. Converse correspondence follows by direct construction. □

Proper $\mathrm{SL}(2, \mathbb{R})$ actions

Let G be a semisimple Lie group with finitely many connected component. Let H, L be reductive subgroups of G. Denote their Lie algebra by $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$ repectively.

- Let θ be any Cartan involution of $\mathfrak{g}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{g}$ corresponding to θ.
- Let $\bigstar = \mathfrak{h}$ or $\mathfrak{l}$. There is a Cartan involution $\theta_\bigstar : \mathfrak{g} \to \mathfrak{g}$ whose restriction on $\bigstar$ is a Cartan involution of $\bigstar$.
- Let $\mathfrak{a}_\bigstar$ be a maximal split abelian subspace of $\bigstar$. Since all Cartan involutions are conjugate, there are $g_\bigstar$ such that $\mathrm{Ad}(g_\bigstar)\mathfrak{a}_\bigstar \subset \mathfrak{a}$.

Proper $\mathrm{SL}(2,\mathbb{R})$ actions

In our case,

- θ is the minus of transpose, i.e., $\theta(X) = -X^T$.
- a maximal split abelian subspace $\mathfrak{a}$ (with respect to θ) of $\mathfrak{g} = \mathfrak{so}(p,q)$, $p \geq q > 0$, $p + q \geq 3$ is of the form

$$\left(\begin{array}{c|c|c} \multicolumn{2}{c|}{\multirow{2}{*}{$0_{p\times p}$}} & \begin{pmatrix} 0 & 0 & \cdots & a_1 \\ 0 & \cdots & a_2 & 0 \\ 0 & \iddots & 0 & 0 \\ a_q & 0 & \cdots & 0 \end{pmatrix} \\ \cline{3-3} \multicolumn{2}{c|}{} & 0_{(p-q)\times q} \\ \hline \begin{pmatrix} 0 & 0 & \cdots & a_q \\ 0 & \cdots & a_{q-1} & 0 \\ 0 & \iddots & 0 & 0 \\ a_1 & 0 & \cdots & 0 \end{pmatrix} & 0_{q\times(p-q)} & 0_{q\times q} \end{array}\right)$$

Proper $\mathrm{SL}(2,\mathbb{R})$ actions

Here, $\mathfrak{a}$ is spanned by basis E_i, $i = 1, 2, \cdots, q$ where E_i is of the form

$$\left(\begin{array}{c|c|c} \multicolumn{2}{c|}{0_{p\times p}} & \begin{array}{c}\begin{pmatrix} 0 & 0 & \cdots & a_1 \\ 0 & \cdots & a_2 & 0 \\ 0 & \iddots & 0 & 0 \\ a_q & 0 & \cdots & 0 \end{pmatrix} \\ \hline 0_{(p-q)\times q}\end{array} \\ \hline \begin{pmatrix} 0 & 0 & \cdots & a_q \\ 0 & \cdots & a_{q-1} & 0 \\ 0 & \iddots & 0 & 0 \\ a_1 & 0 & \cdots & 0 \end{pmatrix} & 0_{q\times(p-q)} & 0_{q\times q} \end{array}\right)$$

with only $a_i = 1$ and all other a_j's equal 0.

Proper $\mathrm{SL}(2,\mathbb{R})$ actions

Lemma

Let L_1, L_2 and H be two reductive subgroup of $G = \mathrm{SO}(p,q)$. Suppose that L_1 and L_2 are conjugate via $\mathrm{SO}(p,q)$. Then L_1 acts on G/H properly if and only if L_2 does. If L_1, L_2 acts properly discontinuously and freely, then $L_1\backslash G/H$ is isometric to $L_2\backslash G/H$.

Theorem

Let $p \geq q > 0$, $n := p+q \geq 3$ and let $\mathfrak{g} = \mathfrak{so}(p,q)$. Let $\mathfrak{l}$ be a $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$ of type $(d_1, \cdots, d_e; d_{e_1}, \cdots, d_{e+l})$. Then there is $\mathfrak{sl}_2$-subalgebra $\mathfrak{l}'$ and $g \in \mathrm{SO}(p,q)_0$ such that $\mathfrak{l}' = \mathrm{Ad}(g).\mathfrak{l} = g\mathfrak{l}g^{-1}$ and that $\mathfrak{a}_{\mathfrak{l}'}$ is an 1-dimensional subspace of $\mathfrak{a}$ spanned by $\sum_{i=1}^{\lceil d_1\rceil+\cdots+\lceil d_{e+l}\rceil} c_iE_i$ where c_i are nonzero numbers. Here, $\lceil\cdot\rceil$ is the ceiling function.

Note that $\mathfrak{l}'$ in the theorem is what we called "good representative" of conjugacy class.

Proper $\mathrm{SL}(2,\mathbb{R})$ actions

Now we use Kobayashi's theorem:

Theorem (1989, Kobayashi)

Let G be a real reductive Lie group. Let L, H be reductive subgroups of G. Fix a maximal split abelian subspaces $\mathfrak{a}_{\mathfrak{l}}$, $\mathfrak{a}_{\mathfrak{h}}$ and $\mathfrak{a}$ of L, H and G. Assume that $\mathfrak{a}_{\mathfrak{l}}, \mathfrak{a}_{\mathfrak{h}} \subset \mathfrak{a}$. Then, L acts on G/H properly if and only if

$$\mathfrak{a}_{\mathfrak{h}} \cap W(\mathfrak{g}, \mathfrak{a}) \cdot \mathfrak{a}_{\mathfrak{l}} = \{0\}.$$

where $W(\mathfrak{g}, \mathfrak{a})$ is the Weyl group.

We will investigate when $\mathfrak{a}_{\mathfrak{h}} \cap W(\mathfrak{g}, \mathfrak{l}).\mathfrak{a}_{\mathfrak{l}} = \{0\}$ happen.

Proper $\mathrm{SL}(2,\mathbb{R})$ actions

We state and prove the main theorem:

Theorem

Let $\mathfrak{g} = \mathfrak{so}(p,q)$, $p \geq q > 0$. Let $\mathfrak{l}$ be an $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$ of type $(d_1, \cdots, d_e; d_{e+1}, \cdots, d_{e+l})$. Let $\mathfrak{h} = \mathfrak{so}(p-i, q-j) \oplus \mathfrak{so}(i,j)$. Let G, H and L be the corresponding Lie groups for $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$ respectively. Then, $(\mathfrak{g}, \mathfrak{h}, \mathfrak{l})$ satisfies the Kobayashi's criterion (i.e., L acts on G/H properly) if and only if

$$\min\{i,j\} + \min\{p-i, q-j\} < \lceil d_1 \rceil + \cdots + \lceil d_{e+l} \rceil.$$

Proof.

Because each $w \in W(\mathfrak{g}, \mathfrak{a})$ just permutes the coordinates (up to sign), the number of nonzero coordinates of $x \in \mathfrak{a}$ is not changed under the Weyl group action. Since the number of nonzero coordinates of $x \in \mathfrak{a}_{\mathfrak{l}}$ is $\lceil d_1 \rceil + \cdots + \lceil d_{e+l} \rceil$, the theorem follows. □

Application to the Surface Group Action

As an application, we can reprove the following fact

Corollary (2013, Okuda)

Let $\mathfrak{h} = \mathfrak{so}(i,j) \oplus \mathfrak{so}(p-i, q-j)$ *be a subalgebra of* $\mathfrak{g} = \mathfrak{so}(p,q)$ *with* $p \geq q > 0$. *Let* G, H *be the corresponding Lie groups.*

- *If* $p \neq q$ *or* $p = q$ *is even then* G/H *admits proper* $\mathrm{SL}(2,\mathbb{R})$ *action if and only if*

$$\min\{i,j\} + \min\{p-i, q-j\} < \min\{p,q\} = q.$$

- *If* $p = q$ *is odd, then* G/H *admits proper* $\mathrm{SL}(2,\mathbb{R})$ *action if and only if*

$$\min\{i,j\} + \min\{p-i, q-j\} < \min\{p,q\} = q \quad \textit{and} \quad |i-j| \neq 1.$$

In fact, Okuda gives full list of semisimple pairs $(\mathfrak{g}, \mathfrak{h})$ that can admit proper $\mathrm{SL}(2,\mathbb{R})$ action. But his proof was much abstract than our proof.

Application to the Surface Group Action

Another application is

Corollary (1981, Kulkarni)

Let $p + q \geq 2$. The fundamental group π of closed surface of genus $g > 1$ acts properly discontinuously and freely on $S^{p,q} = \mathrm{SO}(p+1, q)_0 / \mathrm{SO}(p, q)_0$ when $p < q - 1$ or $p = q - 1$ is odd. In particular, the homogeneous space $S^{p,q}$ admits Clifford-Klein space form which has fundamental group isomorphic to π when $p < q - 1$ or $p = q - 1$ is odd.

Proof.

Direct application of previous corollary. To treat freeness of action, use Selberg's lemma. □

We shall show that the converse is also true.

Application to the Surface Group Action

To prove the converse statement, we must use more intrinsic method. Let $\mathfrak{g}$ as before and fix a positive system Π of restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$.

- Denote Σ^+ the positive roots with respect to Π.
- Let $\mathfrak{a}_+ = \{A \in \mathfrak{a} \,|\, \alpha(A) \geq 0 \text{ for any } \alpha \in \Sigma^+\}$.
- Fix $w_0 \in W(\mathfrak{g}, \mathfrak{a})$ having the longest length. Let

$$\mathfrak{b} = \{A \in \mathfrak{a} \,|\, -w_0.A = A\}, \quad \textit{and} \quad \mathfrak{b}_+ = \mathfrak{b} \cap \mathfrak{a}_+.$$

Theorem (1996, Benoist)

There is a Γ not virtually abelian discrete subgroup of G which acts properly discontinuously on G/H if and only if $\mathfrak{b}_+ \not\subset w.\mathfrak{a}_\mathfrak{h}$ for any $w \in W(\mathfrak{g}, \mathfrak{a})$.

Application to the Surface Group Action

Direct computation shows that

Proposition

$\mathfrak{b}_+ \not\subset w.\mathfrak{a}_{\mathfrak{h}}$ *for any* $w \in W(\mathfrak{g}, \mathfrak{a})$ *if and only if* $p \neq q$ *or* $p = q$ *is even.*

Because surface groups are not virtually abelian,

Corollary

Let $p + q \geq 2$*. The fundamental group* π *of closed surface of genus* $g > 1$ *acts properly discontinuously and freely on* $S^{p,q} = \mathrm{SO}(p+1, q)_0/\mathrm{SO}(p, q)_0$ *when* $p < q - 1$ *or* $p = q - 1$ *is odd. In particular, the homogeneous space* $S^{p,q}$ *admits Clifford-Klein space form which has fundamental group isomorphic to* π *if and only if* $p < q - 1$ *or* $p = q - 1$ *is odd.*