Small asymptotic translation lengths of pseudo-Anosov maps on the curve complex

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Abstract

Let M be a hyperbolic fibered 3-manifold whose first Betti number is greater than 1 and let S be a fiber with pseudo-Anosov monodromy ψ. We show that there exists a sequence (R_n, ψ_n) of fibers and monodromies contained in the fibered cone of (S, ψ) such that the asymptotic translation length of ψ_n on the curve complex $\mathcal{C}(R_n)$ behaves asymptotically like $1/|\chi(R_n)|^2$. As applications, we can reprove the previous result by Gadre–Tsai that the minimal asymptotic translation length of a closed surface of genus g asymptotically behaves like $1/g^2$. We also show that this holds for the cases of hyperelliptic mapping class group and hyperelliptic handlebody group.

Keywords: pseudo-Anosov, curve complex, asymptotic translation length, fibered 3-manifold, hyperelliptic mapping class group, handlebody group

Mathematics Subject Classification (2010). 57M99, 37E30

1 Introduction

Let $S_{g,n}$ be an orientable surface of genus g with n punctures. We will simply denote it by S. The *mapping class group* of S, denoted $\mathrm{Mod}(S)$, is the group of isotopy classes of orientation-preserving homeomorphisms of S. By the Nielsen–Thurston classification theorem, each element of $\mathrm{Mod}(S)$ is either periodic, reducible, or pseudo-Anosov.

For a non-sporadic surface S, that is, a surface with $3g-3+n \geq 2$, the *curve complex* $\mathcal{C}(S)$ is defined to be a simplicial complex whose vertex set $\mathcal{C}^0(S)$ is the set of homotopy classes of essential simple closed curves in S, and whose k-simplices are formed by $k+1$ distinct vertices whose representatives can be chosen to be pairwise disjoint. We will restrict our attention to the 1-skeleton $\mathcal{C}^1(S)$ of $\mathcal{C}(S)$

with path metric $d_{\mathcal{C}}$ by assigning each edge length 1. Then $\mathrm{Mod}(S)$ acts on $\mathcal{C}(S)$ by isometry and the *asymptotic translation length* of $f \in \mathrm{Mod}(S)$ on $\mathcal{C}^1(S)$ is defined by

$$\ell_{\mathcal{C}}(f) = \liminf_{j \to \infty} \frac{d_{\mathcal{C}}(\alpha, f^j(\alpha))}{j},$$

where α is an essential simple closed curve in S. It follows from the definition that $\ell_{\mathcal{C}}(f)$ is independent of the choice of α and that $\ell_{\mathcal{C}}(f^k) = k\ell_{\mathcal{C}}(f)$ for $k \in \mathbb{N}$.

Masur and Minsky [MM99] showed that $f \in \mathrm{Mod}(S)$ is pseudo-Anosov if and only if $\ell_{\mathcal{C}}(f) > 0$, and Bowditch [Bow08] proved that there exists a constant $m > 0$ depending only on the surface S such that for each pseudo-Anosov mapping class f in $\mathrm{Mod}(S)$, f^k has an invariant geodesic axis on $\mathcal{C}(S)$ for some $k \le m$. In other words, $\ell_{\mathcal{C}}(f)$ is a positive rational number with bounded denominator.

For any subgroup $H < \mathrm{Mod}(S)$, let us denote by $L_{\mathcal{C}}(H)$ the minimum of $\ell_{\mathcal{C}}(f)$ over all pseudo-Anosov elements $f \in H$. Then $L_{\mathcal{C}}(H) \ge L_{\mathcal{C}}(\mathrm{Mod}(S))$. We also write $F \asymp G$ if there exists a universal constant $C > 0$ so that $1/C \le F/G \le C$. For the closed surface S_g of genus g, Gadre and Tsai [GT11] showed that

$$L_{\mathcal{C}}(\mathrm{Mod}(S_g)) \asymp \frac{1}{g^2}.$$

In fact, using the invariant train tracks constructed by Bestvina and Handel [BH95] and the nesting lemma by Masur and Minsky [MM99], they obtained in [GT11] the lower bound of the asymptotic translation lengths in terms of the Euler characteristic $\chi(S_{g,n})$ of $S_{g,n}$. That is,

$$\ell_{\mathcal{C}}(f) \ge \frac{1}{18\chi(S_{g,n})^2 + 30|\chi(S_{g,n})| - 10n}$$

for any pseudo-Anosov element $f \in \mathrm{Mod}(S_{g,n})$. To obtain the upper bound, they use an explicit family of pseudo-Anosov mapping classes. This family was first considered by Penner [Pen91] to find small stretch factors of pseudo-Anosov maps;

$$L_{\mathcal{C}}(\mathrm{Mod}(S_g)) \le \frac{4}{g^2 + g - 4}.$$

In this paper, we describe a way of generating a sequence of pseudo-Anosov mapping classes $\psi_n \in \mathrm{Mod}(S_n)$ with small asymptotic translation lengths on the curve complex. We say that a sequence $\{\psi_n\}$ has a *small* asymptotic translation length if $\ell_{\mathcal{C}}(\psi_n) \asymp 1/|\chi(S_n)|^2$, where $\chi(S_n)$ is the Euler characteristic of the corresponding surface S_n such that $|\chi(S_n)| \to \infty$ as $n \to \infty$.

Let M be a hyperbolic fibered 3-manifold with the first Betti number $b_1(M) \ge 2$ and let $S \subset M$ be a fiber with pseudo-Anosov monodromy ψ. Then the assumption $b_1(M) \ge 2$ implies that there is a primitive cohomology class $\xi_0 \in H^1(S; \mathbb{Z})$ fixed by ψ, that is, $\xi_0 \circ \psi_* = \xi_0$, where $\psi_* : H_1(S; \mathbb{Z}) \to H_1(S; \mathbb{Z})$. Let $p : \widetilde{S} \to S$ be a $\mathbb{Z}$-covering map corresponding to ξ_0 whose deck transformation group is generated by $h : \widetilde{S} \to \widetilde{S}$ and let $\widetilde{\psi}$ be a lift of ψ to $\widetilde{S}$. Then we have the following main theorem.

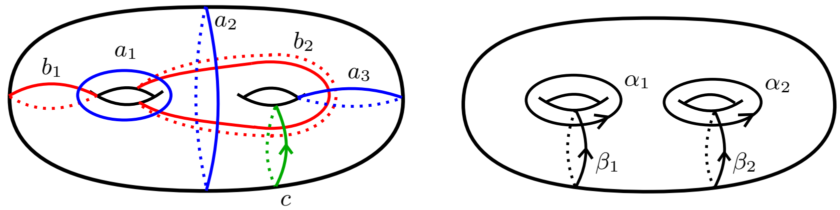

Figure 1: Simple closed curves and the standard basis for $H_1(S_2;\mathbb{Z})$.

Theorem A. *For all sufficiently large n, $R_n = \widetilde{S}/\langle h^n\widetilde{\psi}\rangle$ is a fiber of M with $|\chi(R_n)| \asymp n$ whose pseudo-Anosov monodromy ψ_n satisfies*

$$\ell_{\mathcal{C}}(\psi_n) \asymp \frac{1}{|\chi(R_n)|^2}.$$

The above family of fibers in a fibered 3-manifold was first considered by McMullen and he proved the following theorem providing short geodesics on the moduli space when S is a closed surface.

Theorem 1.1 (McMullen, Theorem 10.2 in [McM00])**.** *For all n sufficiently large,*

$$R_n = \widetilde{S}/\langle h^n\widetilde{\psi}\rangle$$

is a closed surface of genus $g_n \asymp n$, and $h^{-1} : \widetilde{S} \to \widetilde{S}$ descends to a pseudo-Anosov mapping class $\psi_n \in \mathrm{Mod}(R_n)$ with

$$\log \lambda(\psi_n) \asymp \frac{1}{g_n},$$

where $\lambda(\psi_n)$ is the stretch factor of ψ_n.

Although McMullen dealt with closed hyperbolic 3-manifolds in Theorem 1.1, we can adopt the same proof for the general case of fibers of cusped hyperbolic 3-manifolds. In such case, we have to say $\log \lambda(\psi_n) \asymp 1/|\chi(R_n)|$ and $|\chi(R_n)| \asymp n$.

As a consequence of Theorem A, we can determine the behavior of minimal asymptotic translation lengths of a few subgroups of mapping class groups. First of all, the fact that $L_{\mathcal{C}}(\mathrm{Mod}(S_g)) \asymp 1/g^2$ also follows from Theorem A by considering the genus 2 surface S_2 and any mapping class fixing a nontrivial cohomology class. For instance, consider the mapping class $\psi = T_{a_1}T_{a_2}T_{a_3}T_{b_1}^{-1}T_{b_2}^{-1}$ of the closed surface of genus 2 as in Figure 1, where T_γ is the left-handed Dehn twist about a simple closed curve γ. (We apply elements of the mapping class group from right to left.) Then ψ is pseudo-Anosov because it is coming from Penner's construction (see, for instance, [FM12, Theorem 14.4]). The action of ψ on the first homology $H_1(S_2;\mathbb{Z})$ with respect to the basis $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ as in Figure 1 is given by the matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

	$\mathrm{Mod}(S_{0,n})$	$\mathrm{Mod}(S_{1,2n})$	For any fixed $g \geq 2$, $\mathrm{Mod}(S_{g,n})$
$L_{\mathcal{C}}(\cdot)$	$\asymp \dfrac{1}{n^2}$ [Val14]	$\asymp \dfrac{1}{n^2}$ [GT11]	$\asymp \dfrac{1}{n}$ [Val14]

Table 1: Minimal asymptotic translation lengths.

Therefore there is a 1-dimensional subspace of $H_1(S_2;\mathbb{Z})$ and its dual $\xi_0 \in H^1(S_2;\mathbb{Z})$ is given by the algebraic intersection number with an oriented simple closed curve c. Hence, ξ_0 is a cohomology class fixed by ψ.

Valdivia [Val14] showed that fixing $g \geq 2$ as $n \to \infty$,

$$L_{\mathcal{C}}(\mathrm{Mod}(S_{g,n})) \asymp \frac{1}{n},$$

and for the remaining cases of $S_{0,n}$ and $S_{1,2n}$ with even number of punctures as $n \to \infty$, see Table 1. We will determine the minimal asymptotic translation lengths of a few other types of surfaces including the surface of genus 1 with odd number of punctures. Let D_n be the closed disk D with n-punctures and let $\mathrm{Mod}(D_n)$ be the mapping class group of D_n fixing the boundary ∂D of the disk D pointwise. As an application of Theorem A, we have the following results.

Theorem B. *We have*

(1) $L_{\mathcal{C}}(\mathrm{Mod}(D_n)) \asymp \dfrac{1}{n^2}$, *and*

(2) $L_{\mathcal{C}}(\mathrm{Mod}(S_{1,n})) \asymp \dfrac{1}{n^2}$.

Furthermore, we improve the upper bound for the minimal asymptotic translation length for S_g. The *hyperelliptic mapping class group* $\mathcal{H}(S_g)$ of S_g is the subgroup of $\mathrm{Mod}(S_g)$ consisting of elements with representative homeomorphisms that commute with some fixed hyperelliptic involution.

Theorem C. *For closed surfaces* S_g *with* $g \geq 3$*,*

$$L_{\mathcal{C}}(\mathcal{H}(S_g)) \leq \frac{1}{g^2 - 2g - 1},$$

and as a direct consequence, we have

$$L_{\mathcal{C}}(\mathrm{Mod}(S_g)) \leq \frac{1}{g^2 - 2g - 1}.$$

We remark that for $g \geq 4$, this is a sharper upper bound than Gadre–Tsai's.

As another application, we determine the asymptotes of minimal asymptotic translation lengths for handlebody groups and hyperelliptic handlebody groups. Let $\mathbb{H}_g$ be the handlebody of genus g, that is, a 3-manifold bounded by a closed orientable surface S_g of genus g. The *handlebody group* $\mathrm{Mod}(\mathbb{H}_g)$ is the subgroup of $\mathrm{Mod}(S_g)$ consisting of elements whose representative homeomorphisms of S_g can be extended to homeomorphisms of $\mathbb{H}_g$. Then the *hyperelliptic handlebody group* is defined by

$$\mathcal{H}(\mathbb{H}_g) = \mathrm{Mod}(\mathbb{H}_g) \cap \mathcal{H}(S_g).$$

Theorem D. *We have*

$$L_{\mathcal{C}}(\mathcal{H}(\mathbb{H}_g)) \asymp \frac{1}{g^2}.$$

The following is an immediate corollary of the previous Theorem D and the lower bound by Gadre–Tsai.

Corollary E. *We have*

$$L_{\mathcal{C}}(\mathcal{H}(S_g)) \asymp \frac{1}{g^2} \ \ \textit{and} \ \ L_{\mathcal{C}}(\mathrm{Mod}(\mathbb{H}_g)) \asymp \frac{1}{g^2}.$$

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2 Proof of Theorem A

In this section, we begin with the following simple observation.

Lemma 2.1. *Let $f \in \mathrm{Mod}(S)$ be a pseudo-Anosov mapping class and let α be any essential simple closed curve in S. If $d_{\mathcal{C}}(\alpha, f^m(\alpha)) = 1$ for some $m \in \mathbb{N}$, then*

$$\ell_{\mathcal{C}}(f) \leq \frac{1}{m}.$$

Proof. By the triangle inequality, we have

$$\begin{aligned} \ell_{\mathcal{C}}(f^m) &= \liminf_{j \to \infty} \frac{d_{\mathcal{C}}(\alpha, f^{jm}(\alpha))}{j} \\ &\leq \liminf_{j \to \infty} \frac{\sum_{i=1}^{j} d_{\mathcal{C}}(f^{(i-1)m}(\alpha), f^{im}(\alpha))}{j} \\ &= \liminf_{j \to \infty} \frac{j \cdot d_{\mathcal{C}}(\alpha, f^m(\alpha))}{j} = 1 \end{aligned}$$

Since $\ell_{\mathcal{C}}(f^m) = m\,\ell_{\mathcal{C}}(f)$, this completes the proof. □

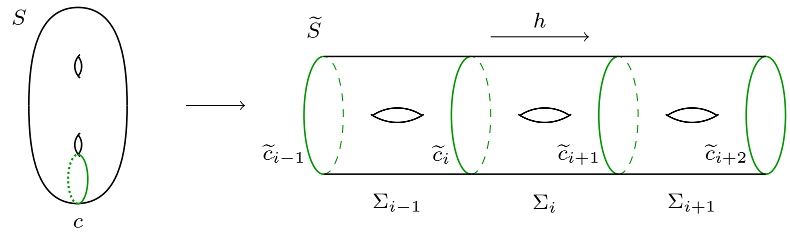

Figure 2: $\mathbb{Z}$-cover corresponding to ξ_0.

Now we prove our main theorem.

Proof of Theorem A. Since the lower bound was established by Gadre–Tsai, it is enough to show that there exists some constant C such that

$$\ell_{\mathcal{C}}(\psi_n) \leq \frac{C}{|\chi(R_n)|^2}.$$

The proof consists of the following three steps. In step 1, we establish the structure of the $\mathbb{Z}$-cover $\widetilde{S}$ of S as the union of $\mathbb{Z}$-copies of $S \setminus \{c\}$, where $[c]$ is the homology class dual to the primitive cohomology ξ_0 fixed by ψ. In step 2, using the decomposition of $\widetilde{S}$ in step 1, we will find an integer r so that $\psi_n^r(\overline{\alpha})$ and $\overline{\alpha}$ are disjoint in the quotient surface R_n, where $\overline{\alpha}$ is either a simple closed curve or a simple proper arc and ψ_n is a pseudo-Anosov monodromy on R_n. In step 3, using Lemma 2.1, we deduce that the asymptotic translation length of ψ_n is less than or equal to $1/r$ and we show that we can choose r to be quadratic in n. This finishes the proof.

Step 1. (*The decomposition of* $\widetilde{S}$) Let $[c]$ be a homology class in $H_1(S;\mathbb{Z})$ which is dual to the primitive cohomology $\xi_0 \in H^1(S;\mathbb{Z})$. Since ξ_0 is primitive, $[c]$ is also a primitive element. If S is a closed surface, one can find a representative c that is an oriented simple closed curve and if S is a surface with punctures, c can be chosen to be a simple proper arc or union of disjoint simple proper arcs (see, for instance, Proposition 6.2 in [FM12]). Let $\widetilde{S}$ be the surface obtained by cutting S along c and concatenating $\mathbb{Z}$-copies of $S \setminus \{c\}$ together (see Figure 2 in the case of closed surfaces and see Figure 8 in the case of punctured surfaces). Then the natural projection map $p : \widetilde{S} \to S$ is a covering map corresponding to ξ_0 because the kernel of the composition $\pi_1(S) \to H_1(S;\mathbb{Z}) \xrightarrow{\xi_0} \mathbb{Z}$ of the Hurewicz map and ξ_0 is equal to $p_*(\pi(\widetilde{S}))$. Let Σ_i be the copies of $S \setminus \{c\}$ on $\widetilde{S}$ such that the generator $h : \widetilde{S} \to \widetilde{S}$ for the deck transformation group is given by $h(\Sigma_i) = \Sigma_{i+1}$ for all i (See Figure 2).

Step 2. (*Finding a positive integer r such that $\psi_n^r(\overline{\alpha})$ and $\overline{\alpha}$ are disjoint*) Choose a lift $\widetilde{\psi}$ and take a constant $k = k(\widetilde{\psi})$ such that

$$\widetilde{\psi}(\Sigma_0) \subset \Sigma_{-k} \cup \ldots \cup \Sigma_{k-1} \cup \Sigma_k.$$

(For instance, in Figure 3, $k = 1$). Note then we have $h^n\widetilde{\psi}(\Sigma_0) \subset \Sigma_{n-k} \cup \ldots \cup \Sigma_{n+k}$.

Suppose n is large enough so that $n - k > 1$. (More precise condition on n will be determined later.) Then $h^n\widetilde{\psi}(\Sigma_0)$ and Σ_0 are disjoint and no orbit of Σ_0 under the cyclic group $\langle h^n\widetilde{\psi}\rangle$ intersect with $\Sigma_1 \cup \Sigma_2 \cup \ldots \cup \Sigma_{n-k-1}$. Let α be a simple closed curve or a simple proper arc contained in Σ_0 and let $\overline{\alpha}$ be the $\langle h^n\widetilde{\psi}\rangle$-orbit of α in $\widetilde{S}$. (One can choose α to be one of parallel copies of c, and α in Figure 4 is not the case.) It follows that if a simple closed curve or a simple proper arc β lies in $\Sigma_1 \cup \ldots \cup \Sigma_{n-k-1}$, i.e., disjoint from both α and $h^n\widetilde{\psi}(\alpha)$, then $\overline{\beta}$ and $\overline{\alpha}$ are disjoint in $R_n = \widetilde{S}/\langle h^n\widetilde{\psi}\rangle$. Since h^{-1} induces a pseudo-Anosov map on R_n, let us find a positive integer r as large as possible such that one of the representative of $\overline{h^{-r}(\alpha)}$ is contained in $\Sigma_1 \cup \ldots \cup \Sigma_{n-k-1}$ (see Figure 4). Then this representative is disjoint from both α and $h^n\widetilde{\psi}(\alpha)$, and because of the previous argument, $\overline{h^{-r}(\alpha)}$ and $\overline{\alpha}$ are disjoint in R_n. By the fact that h^{-1} descends to a psuedo-Anosov ψ_n in R_n together with Lemma 2.1, this allows us to obtain the upper bound for the asymptotic translation length of ψ_n. To find such r, first note that since α is in Σ_0, we have $\widetilde{\psi}(\alpha) \subset \Sigma_{-k} \cup \ldots \cup \Sigma_k$ and $\widetilde{\psi}^m(\alpha) \subset \Sigma_{-mk} \cup \ldots \cup \Sigma_{mk}$ for any $m \in \mathbb{N}$. Since the generator h of the deck transformation group translates Σ_i's, after applying h^{mk+1}, we have $h^{mk+1}\widetilde{\psi}^m(\alpha) \subset \Sigma_1 \cup \ldots \cup \Sigma_{2mk+1}$. In order that $h^{mk+1}\widetilde{\psi}^m(\alpha)$ lies in $\Sigma_1 \cup \ldots \cup \Sigma_{n-k-1}$, we require that $2km + 1 \leq n - k - 1$. Let us choose the biggest such m, that is

$$m = \lfloor \frac{n-k-2}{2k} \rfloor$$

(note that the precise assumption on n is $(n - k - 2)/2k \geq 1$ because we want m to be positive). Since $\overline{\widetilde{\psi}(\alpha)} = \overline{h^{-n}(\alpha)}$ and hence $\overline{h^{mk+1}\widetilde{\psi}^m(\alpha)} = \overline{h^{-(n-k)m+1}(\alpha)}$, the desired integer for $\overline{h^{-r}(\alpha)}$ and $\overline{\alpha}$ being disjoint is $r = (n - k)m - 1$.

Step 3. (*Small asymptotic translation length* $\ell_{\mathcal{C}}(\psi_n)$) We first remark that arc and curve complex $\mathcal{AC}(S)$ and curve complex $\mathcal{C}(S)$ are 2-bilipschitz (see, for instance, [MM00, Lemma 2.2] or [HPW15]). This implies that the asymptotic translation lengths $\ell_{\mathcal{AC}}(f)$ and $\ell_{\mathcal{C}}(f)$ of a pseudo-Anosov mapping class f on the 1-skeletons $\mathcal{AC}^1(S)$ and $\mathcal{C}^1(S)$, respectively, have the same asymptotic behavior, that is,

$$\ell_{\mathcal{AC}}(f) \asymp \ell_{\mathcal{C}}(f).$$

So without loss of generality, we may assume that α is a simple closed curve and compute the asymptotic translation length on the curve complex $\mathcal{C}(R_n)$. In the case when α is a simple proper arc, we think of computing the asymptotic translation length of the arc and curve complex $\mathcal{AC}(R_n)$ and it gives us the same asymptotic behavior on the curve complex.

By the previous step, $\psi_n^{(n-k)m-1}(\overline{\alpha})$ and $\overline{\alpha}$ are disjoint in R_n. Then by Lemma 2.1, we have

$$\ell_{\mathcal{C}}(\psi_n) \leq \frac{1}{(n-k)m-1},$$

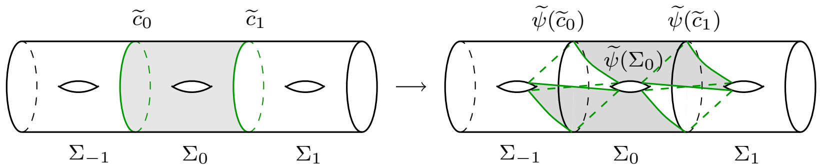

Figure 3: A lift of ψ, where ψ is given as in Figure 1. Curves $\widetilde{c}_0$ and $\widetilde{c}_1$, which are the lifts of c, determines a fundamental region Σ_0. Then the images $\widetilde{\psi}(\widetilde{c}_0)$ and $\widetilde{\psi}(\widetilde{c}_1)$, lifts of $\psi(c)$, bound the image $\widetilde{\psi}(\Sigma_0)$, which lies in $\Sigma_{-1} \cup \Sigma_0 \cup \Sigma_1$.

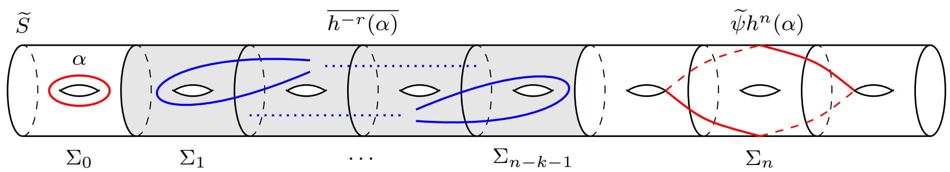

Figure 4: The curves $\overline{h^{-r}(\alpha)}$ and $\overline{\alpha}$ are disjoint in $R_n = \widetilde{S}/\langle\widetilde{\psi}h^n\rangle$.

and by the fact that $\chi(R_n)$ is a linear function in n, we have

$$\ell_{\mathcal{C}}(\psi_n) \leq \frac{C}{|\chi(R_n)|^2}$$

for some $C > 0$. This completes the proof. □

3 Backgrounds for Theorems B, C, and D.

This section includes some backgrounds and basic facts for the proofs of the rest of theorems. Consider a pseudo-Anosov mapping class $\psi \in \mathrm{Mod}(S)$. Let $\Psi : S \to S$ be any representative of ψ. The *mapping torus* M_ψ is defined by

$$M_\psi = S \times [0,1]/\sim,$$

where $\sim$ identifies $(x,1)$ with $(\Psi(x),0)$ for each $x \in S$. Then the manifold M_ψ fibering over the circle S^1 is hyperbolic. Suppose that there is a primitive cohomology class $\xi_0 \in H^1(S;\mathbb{Z})$ fixed by ψ. This implies that $b_1(M_\psi) \geq 2$. Then Theorem A says that for n sufficiently large, $R_n = \widetilde{S}/\langle h^n\widetilde{\psi}\rangle$ is a fiber of M_ψ with $\chi(R_n) \asymp n$ such that the pseudo-Anosov monodromy ψ_n defined on the fiber R_n satisfies $\ell_{\mathcal{C}}(\psi_n) \asymp 1/|\chi(R_n)|^2$.

3.1 Fibered 3-manifolds from braids

Let B_n be the the braid group with n strands. In this paper braids are depicted vertically. We define the product $\beta\beta'$ of $\beta, \beta' \in B_n$ in the usual way, namely, we

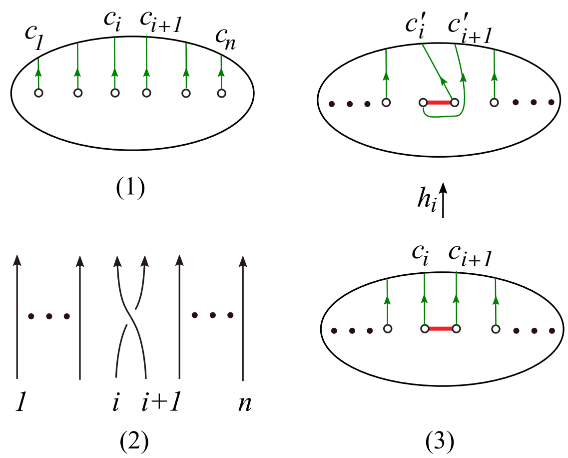

Figure 5: (1) Arcs c_i in the n-punctured disk D_n. (2) Generators σ_i. (3) Half twist h_i. ($c'_i = h_i(c_i)$ and $c'_{i+1} = h_i(c_{i+1})$.)

stack β on β' and concatenate the bottom ith end point of β with the top ith end point of β' for each $i = 1, \cdots, n$. Then we obtain n strands. The product $\beta\beta'$ is the resulting n-braid after rescaling.

We briefly review a relation between B_n and $\mathrm{Mod}(D_n)$. To do this we assign an orientation for each n-braid from the bottom endpoints to the top endpoints (see Figure 5(2)). We take a natural basis $t_i \in H_1(D_n; \mathbb{Z})$, where a representative of t_i is a small oriented loop in D_n centered at the ith puncture of D_n for $i = 1, \cdots, n$. Let c_i be a simple proper arc in D_n which connects the ith puncture of D_n to the boundary ∂D as in Figure 5(1). Then there is an isomorphism

$$\Gamma : B_n \to \mathrm{Mod}(D_n)$$

which sends the generator σ_i of B_n to the left-handed half twist h_i (see Figure 5(2)(3)). The orientation of braids as we described above induces the motion of n punctures in the disk, which defines the above map Γ.

We have a natural homomorphism

$$\mathfrak{c} : \mathrm{Mod}(D_n) \to \mathrm{Mod}(S_{0,n+1})$$

collapsing the boundary ∂D of the disk to the $(n+1)$th puncture of $S_{0,n+1}$. By definition, $\mathfrak{c}(\mathrm{Mod}(D_n))$ is isomorphic to the subgroup of $\mathrm{Mod}(S_{0,n+1})$ which fixes this puncture. We sometimes identify $f \in \mathrm{Mod}(D_n)$ with $\mathfrak{c}(f) \in \mathrm{Mod}(S_{0,n+1})$. We simply denote by β, both mapping classes $\Gamma(\beta) \in \mathrm{Mod}(D_n)$ and $\mathfrak{c}(\Gamma(\beta)) \in \mathrm{Mod}(S_{0,n+1})$.

The closure $\mathrm{cl}(\beta)$ of $\beta \in B_n$ is a knot or link in the 3-sphere S^3. Let $\mathcal{A}$ be a braid axis of β which is an unknot in S^3. Then $\mathrm{cl}(\beta)$ runs around the unknot $\mathcal{A}$ in a monotone manner. We set $\mathrm{br}(\beta) = \mathrm{cl}(\beta) \cup \mathcal{A}$ which is a link in S^3 whose number of the components is greater than or equal to 2, and let us set $M_\beta = S^3 \setminus \mathrm{br}(\beta)$. The 3-manifold M_β is homeomorphic to the interior of the mapping torus of the

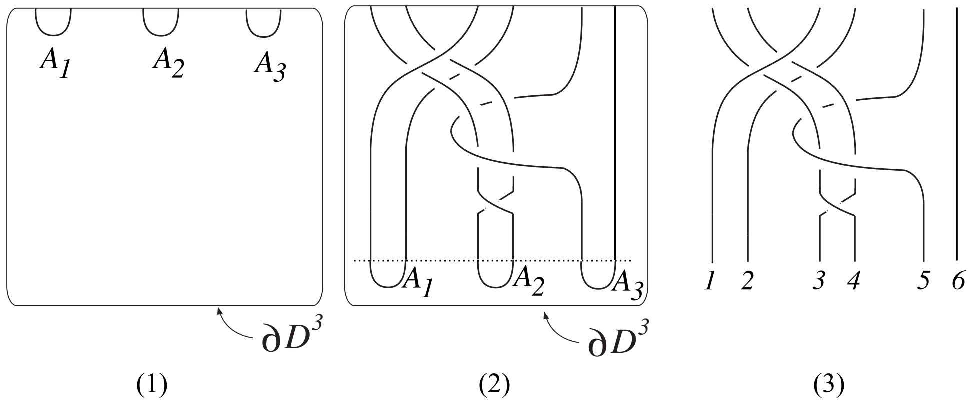

Figure 6: (1) $\mathbf{A}$ in the case $n = 3$. (2) ${}^{w}\mathbf{A}$. (3) $w \in SW_6 < SB_6$.

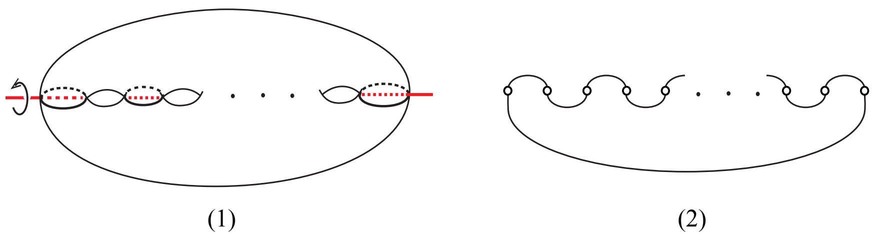

Figure 7: (1) $\mathcal{I} : S_g \to S_g$. (2) Sphere $S_g/\mathcal{I}$ with $2g+2$ marked points. Small circles in the figure indicate marked points.

monodromy $\beta \in \mathrm{Mod}(D_n)$, and $b_1(M_\beta) \geq 2$. A spanning disk by the unknot $\mathcal{A}$ has n punctures in M_β, and such a disk with punctures is a fiber of M_β with monodromy β.

3.2 Subgroups of mapping class groups

Let SB_m be the *spherical* m-braid group. We now introduce the subgroup SW_{2n} of SB_{2n}. Let $A_1, A_2, \cdots, A_n$ be n disjoint unknotted arcs properly embedded in the 3-ball D^3 so that $\mathbf{A} = A_1 \cup \cdots \cup A_n$ is unlinked as in Figure 6(1). The boundary $\partial\mathbf{A}$ is the set of $2n$ points in the 2-sphere ∂D^3.

For $b \in SB_{2n}$, we stack b on $\mathbf{A}$, and concatenate the bottom endpoints of b with the endpoints of $\mathbf{A}$. As a result we obtain n disjoint (knotted) arcs ${}^{b}\mathbf{A}$ properly embedded in D^3 (see Figure 6(2)). The *wicket group* SW_{2n} is the subgroup of SB_{2n} generated by braids b's such that ${}^{b}\mathbf{A}$ is isotopic to $\mathbf{A}$ relative to $\partial\mathbf{A}$. It is easy to see that the braid $w \in SB_6$ as shown in Figure 6(3) is an element of SW_6.

There is a spherical version of the isomorphism $\Gamma : B_n \to \mathrm{Mod}(D_n)$, namely we have a surjective homomorphism $SB_m \to \mathrm{Mod}(S_{0,m})$ which sends the generator σ_i of SB_m to the left-handed half twist between the ith and $(i+1)$st punctures (cf. Figure 5(2)(3)). We also denote this homomorphism by

$$\Gamma : SB_m \to \mathrm{Mod}(S_{0,m}).$$

Its kernel is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ generated by a full twist $\Delta^2 \in SB_m$, where Δ is a half twist. When $m = 2n$ the image $\Gamma(SW_{2n})$ of SW_{2n} under the map Γ is a subgroup of $\mathrm{Mod}(S_{0,2n})$ which is so-called *Hilden group*, denoted by SH_{2n}, and

$$SH_{2g+2} \simeq SW_{2g+2}/\langle \Delta^2 \rangle$$

holds (see [HK17]).

For the proof of Theorem D, we recall a connection between the wicket group and the hyperellitic handlebody group. We first state a theorem by Birman and Hilden which relates $\mathcal{H}(S_g)$ to $\mathrm{Mod}(S_{0,2g+2})$. Each homeomorphism on S_g which commutes with some fixed hyperelliptic involution $\mathcal{I} : S_g \to S_g$ (Figure 7(1)) preserves the set of fixed points of $\mathcal{I}$ consisting of $2g + 2$ points. Such a homeomorphism induces a homeomorphism on a sphere $S_g/\mathcal{I}$ which preserves these fixed points (Figure 7(2)). Thus we have a map

$$q : \mathcal{H}(S_g) \to \mathrm{Mod}(S_{0,2g+2})$$

by choosing a representative of each mapping class of $\mathcal{H}(S_g)$ which commutes with $\mathcal{I}$. It is shown in [BH71] that the map q is well-defined and it is a surjective homomorphism whose kernel is generated by $\iota = [\mathcal{I}] \in \mathcal{H}(S_g)$. In particular we have

$$\mathcal{H}(S_g)/\langle \iota \rangle \simeq \mathrm{Mod}(S_{0,2g+2}) \simeq SB_{2g+2}/\langle \Delta^2 \rangle.$$

On the other hand, it is proved in [HK17] that there is a surjective homomorphism

$$Q : \mathcal{H}(\mathbb{H}_g) \to SH_{2g+2}$$

whose kernel is generated by ι. The map Q is given by the restriction

$$q|_{\mathcal{H}(\mathbb{H}_g)} : \mathcal{H}(\mathbb{H}_g) \to SH_{2g+2} < \mathrm{Mod}(S_{0,2g+2}).$$

Putting all things together, we have

$$\mathcal{H}(\mathbb{H}_g)/\langle \iota \rangle \simeq SH_{2g+2} \simeq SW_{2g+2}/\langle \Delta^2 \rangle.$$

Thus an element $f \in SH_{2g+2}$ can be described by a braid $v \in SW_{2g+2}$, i.e., $f = \Gamma(v)$. Moreover a lift $\widehat{f}$ of f under the map $q|_{\mathcal{H}(\mathbb{H}_g)} = Q$ is an element of $\mathcal{H}(\mathbb{H}_g)$. We simply denote by v, the element $\Gamma(v)$ in the Hilden group SH_{2g+2}.

The following lemma is used in the proofs of the rest of theorems (other than Theorem B(2)).

Lemma 3.1. *Let $f \in \mathrm{Mod}(S_{0,2g+2})$ for $g \geq 2$ and let $\widehat{f} \in \mathcal{H}(S_g)$ be a lift of f under the map $q : \mathcal{H}(S_g) \to \mathrm{Mod}(S_{0,2g+2})$. We take any $\alpha \in \mathcal{AC}^0(S_{0,2g+2})$, i.e., α is a homotopy class of an arc or simple closed curve in $S_{0,2g+2}$. Suppose that $d_{\mathcal{AC}}(\alpha, f^m(\alpha)) = 1$ for some $m \in \mathbb{N}$, where $d_{\mathcal{AC}}$ is the path metric on $\mathcal{AC}(S_{0,2g+2})$. Then*

$$\ell_{\mathcal{C}}(\widehat{f}) \leq \frac{1}{m}.$$

It is well-known and not hard to see that if $f \in \mathrm{Mod}(S_{0,2g+2})$ is pseudo-Anosov, then $\widehat{f} \in \mathcal{H}(S_g)$ is also pseudo-Anosov.

Proof of Lemma 3.1. By abuse of the notation, a representative of $\alpha \in \mathcal{AC}^0(S_{0,2g+2})$ is denoted by the same α. Let $\widehat{\alpha} \subset S_g$ be the preimage $q^{-1}(\alpha)$ of a simple arc or simple closed curve α in $S_{0,2g+2}$ under the map q. If α is a simple arc, then $\widehat{\alpha}$ is a non-separating simple closed curve in S_g which means $\widehat{\alpha}$ is essential. Hence $\widehat{\alpha} \in \mathcal{C}^0(S_g)$. The assumption implies that $d_{\mathcal{C}}(\widehat{\alpha}, (\widehat{f})^m(\widehat{\alpha})) = 1$. The claim follows from Lemma 2.1.

If α is a simple closed curve, then α cuts $S_{0,2g+2}$ into two components $S_{(1)}$ and $S_{(2)}$ which are disks with punctures $n_1 \geq 2$ and $n_2 \geq 2$ respectively. Since $n_1 + n_2 = 2g + 2$, both n_1 and n_2 have the same parity.

We first consider the case where n_1 and n_2 are odd. Then $n_1, n_2 \geq 3$. Observe that $\widehat{\alpha}$ is a single simple closed curve. Since $\widehat{\alpha}$ cuts S_g into the essential surfaces $q^{-1}(S_{(1)})$ and $q^{-1}(S_{(2)})$ with positive genera, $\widehat{\alpha}$ is a separating and essential simple closed curve. We have $d_{\mathcal{C}}(\widehat{\alpha}, (\widehat{f})^m(\widehat{\alpha})) = 1$ by the assumption. Thus $\ell_{\mathcal{C}}(\widehat{f}) \leq \frac{1}{m}$ holds.

Let us consider the remaining case where both n_1 and n_2 are even with $n_1, n_2 \geq 2$. Observe that $\widehat{\alpha}$ has two components $\widehat{\alpha}_{(1)}$ and $\widehat{\alpha}_{(2)}$ which are non-separating simple closed curves. Hence $\widehat{\alpha}_{(i)} \in \mathcal{C}^0(S_g)$ for $i = 1, 2$. We have $d_{\mathcal{C}}(\widehat{\alpha}_{(i)}, (\widehat{f})^m(\widehat{\alpha}_{(i)})) = 1$ by the assumption, and hence $\ell_{\mathcal{C}}(\widehat{f}) \leq \frac{1}{m}$ holds. We complete the proof. □

4 Proof of Theorem B

This section is devoted to the proof of Theorem B. In the proof of Theorem B(2), we reprove the previous result $\mathrm{Mod}(S_{1,2n}) \asymp \frac{1}{n^2}$ by Gadre–Tsai.

Proof of Theorem B(1). We separate the proof into two cases, depending on the parity of the number of punctures of D_n. We first deal with the case where n is even. We consider the pseudo-Anosov braid $\beta = \sigma_1^{-2}\sigma_2 \in B_3$ (Figure 8(1)) and the fibered hyperbolic 3-manifold M_β. We take a fiber $S = D_3$ with monodromy $\psi = \beta$ of M_β. Let $\xi_0 \in H^1(S; \mathbb{Z})$ be the primitive cohomology class which is dual to the homology class of the proper arc $c = c_1$ in S (see Figure 5(1) for c_1).

From Figure 8(1), one sees that the induced h $\psi_* = \beta_* : H_1(D_3; \mathbb{Z}) \to H_1(D_3; \mathbb{Z})$ maps $t_1, t_2, t_3 \in H_1(D_3; \mathbb{Z})$ to t_1, t_3, t_2 respectively, where the set of t_i's is a natural basis of $H_1(D_n; \mathbb{Z})$ (see Section 3.1). This tells us that ξ_0 is fixed by ψ. Figure 8(2) illustrates the $\mathbb{Z}$-cover $\widetilde{S}$ corresponding to ξ_0. We consider the *canonical* lift $\widetilde{\psi} : \widetilde{S} \to \widetilde{S}$ of ψ which means that $\widetilde{\psi}$ fixes the preimage $p^{-1}(\partial D)$ of the (outer) boundary of the 3-punctured disk pointwise. (In Figure 9(1)(2), the set $p^{-1}(\partial D) \cap \Sigma_i$ is thickened.) We set $\widetilde{c}_{(i)} = \Sigma_{i-1} \cap \Sigma_i$ which is a connected component of the preimage $p^{-1}(c)$ of c (see Figure 8(2)). In other words, $\widetilde{c}_{(i)}$ and $\widetilde{c}_{(i+1)}$ bound the copy Σ_i. To see the image $\widetilde{\psi}(\Sigma_i)$ of Σ_i under $\widetilde{\psi}$, we consider $\widetilde{\psi}(\widetilde{c}_{(i)})$ and $\widetilde{\psi}(\widetilde{c}_{(i+1)})$ which are determined by the proper arc $\psi(c) = \beta(c)$ (see

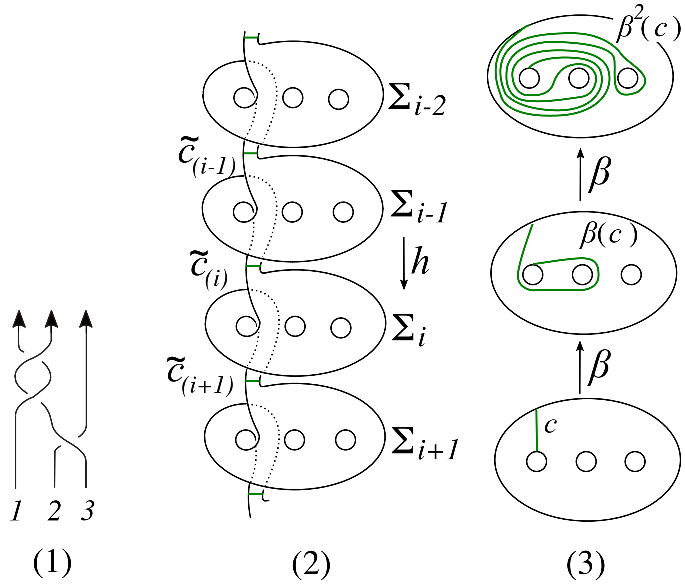

Figure 8: (1) $\beta = \sigma_1^{-2}\sigma_2 \in B_3$. (2) $\mathbb{Z}$-cover $\widetilde{S}$ over $S = D_3$ corresponding to the dual to $c = c_1$. (3) c, $\beta(c)$ and $\beta^2(c)$.

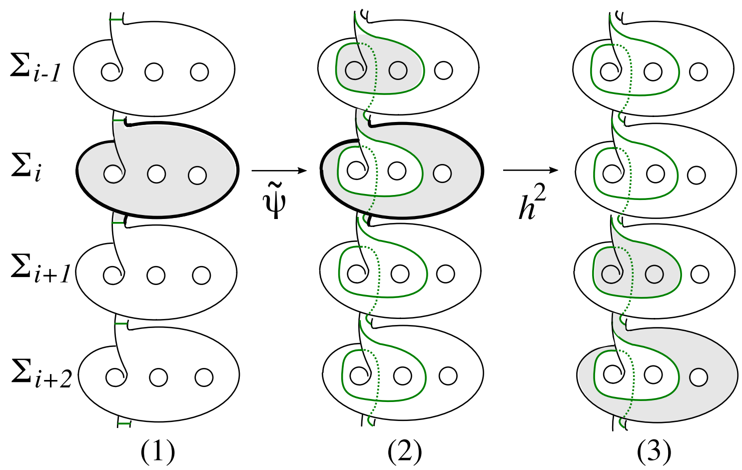

Figure 9: Illustration of $h^2\widetilde{\psi} : \widetilde{S} \to \widetilde{S}$. Shaded regions in (1)(2) and (3) are Σ_i, $\widetilde{\psi}(\Sigma_i)$ and $h^2\widetilde{\psi}(\Sigma_i)$ respectively.

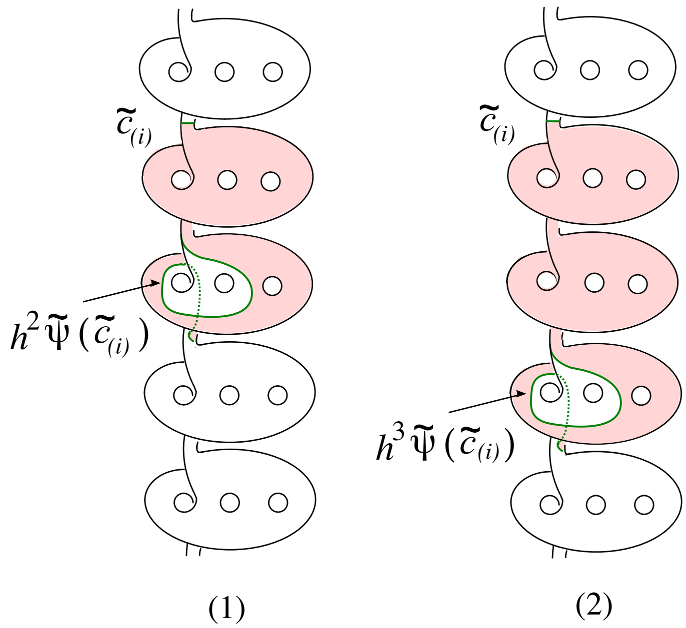

Figure 10: (1) Shaded region descends to $R_2 \simeq S_{0,5}$. See also Figure 9. (2) Shaded region descends to $R_3 \simeq S_{0,7}$. (Note that $[\widetilde{c}_{(i)}] = [h^n \widetilde{\psi}(\widetilde{c}_{(i)})]$ in R_n.)

Figure 8(3)). Observe (from Figure 9(1) and (2)) that

$$\widetilde{\psi}(\Sigma_i) \subset \Sigma_{i-1} \cup \Sigma_i \text{ and } \widetilde{\psi}^{-1}(\Sigma_i) \subset \Sigma_i \cup \Sigma_{i+1}.$$

Hence for each $n \geq 0$

$$h^n \widetilde{\psi}(\Sigma_i) \subset \Sigma_{i-1+n} \cup \Sigma_{i+n} \text{ and } (h^n \widetilde{\psi})^{-1}(\Sigma_i) = h^{-n} \widetilde{\psi}^{-1}(\Sigma_i) \subset \Sigma_{i-n} \cup \Sigma_{i-n+1}.$$

For $\ell > 0$, we have

$$\begin{array}{rcl} (h^n \widetilde{\psi})^{\ell}(\Sigma_i) & \subset & \Sigma_{i-\ell+\ell n} \cup \cdots \cup \Sigma_{i-1+\ell n} \cup \Sigma_{i+\ell n}, \\ (h^n \widetilde{\psi})^{-\ell}(\Sigma_i) & \subset & \Sigma_{i-\ell n} \cup \Sigma_{i-\ell n+1} \cup \cdots \cup \Sigma_{i-\ell n+\ell}. \end{array}$$

Notice that if we fix $n \geq 2$, then $(h^n \widetilde{\psi})^{\pm \ell}(\Sigma_i) \cap \Sigma_i = \emptyset$ for each $\ell > 0$, and hence $R_n = \widetilde{S}/\langle h^n \widetilde{\psi} \rangle$ is a surface. In fact R_n is a disk with $2n$ punctures, and hence we can think of R_n as a sphere with $2n+1$ punctures (see Figures 9 and 10). Note that one of the punctures of R_n, say p_∞, comes from the preimage of the boundary of the disk under the projection $p : \widetilde{S} \to S = D_3$. By Theorem 1.1, we know h^{-1} descends to the monodromy ψ_n, and we see that ψ_n maps p_∞ to itself. Thus $\psi_n \in \mathrm{Mod}(D_{2n})$. By Theorem A, we have $\ell_{\mathcal{C}}(\psi_n) \leq C/n^2$ for some constant C, and hence $L_{\mathcal{C}}(\mathrm{Mod}(D_{2n})) \leq C/n^2$.

For the case where the number of the punctures of D_n is odd, we turn to the pseudo-Anosov braid $\phi = \beta^2 \in B_3$. The hyperbolic fibered 3-manifold M_ϕ has a

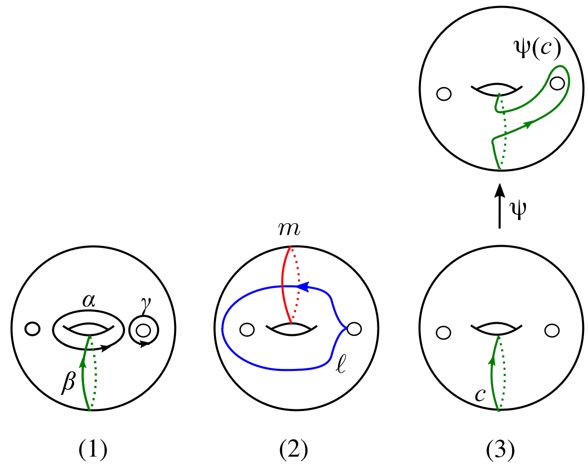

Figure 11: Two small circles indicate punctures of $S_{1,2}$. (1) A basis $\alpha, \beta, \gamma \in H_1(S_{1,2}; \mathbb{Z})$. (2) m, ℓ in $S_{1,2}$. (3) Image of c under $\psi = T_m^{-1} f_\ell$.

fiber $S = D_3$ with monodromy ϕ. The dual to $c = c_1$ is the primitive cohomology class fixed by ϕ. Consider the $\mathbb{Z}$-cover $\widetilde{S}$ corresponding to this cohomology class. Let $\widetilde{\phi} = (\widetilde{\psi})^2 : \widetilde{S} \to \widetilde{S}$ be the canonical lift of ϕ as before. By using the proper arc $\phi(c) = \beta^2(c)$ (see Figure 8(3)), we see where each copy Σ_i maps on $\widetilde{S}$ under $\widetilde{\phi}$. We use the same argument as above replacing $\widetilde{\psi}$ with $\widetilde{\phi} = (\widetilde{\psi})^2$, and construct a surface $\widetilde{S}/\langle h^n \widetilde{\phi} \rangle$ concretely. Then we find that this surface is a sphere with $2n+2$ punctures which is a fiber of M_ϕ for n large. Also we see that ϕ_n fixes one of the punctures of the fiber (which comes form the preimage of the boundary of the disk). Thus $\phi_n \in \mathrm{Mod}(D_{2n+1})$. By Theorem A, we have $\ell_{\mathcal{C}}(\phi_n) \leq C'/n^2$ for some constant $C' > 0$. This tells us that $L_{\mathcal{C}}(\mathrm{Mod}(D_{2n+1})) \leq C'/n^2$. This completes the proof. □

Proof of Theorem B(2). We first consider the case where the number of punctures is odd. Let L_W be the Whitehead link in S^3. The complement $S^3 \setminus L_W$ is a fibered hyperbolic 3-manifold with a fiber $S_{1,2}$. Consider its pseudo-Anosov monodromy ψ defined on the fiber $S_{1,2}$ (see [KR, Appendix B] for more details), and we use a basis $\alpha, \beta, \gamma \in H_1(S_{1,2}; \mathbb{Z})$ as in Figure 11(1). Let m be a simple closed curve in $S_{1,2}$, and let ℓ be an oriented loop based at one of the punctures of $S_{1,2}$ as in Figure 11(2). Let c be a representative of the generator $\beta \in H_1(S_{1,2}; \mathbb{Z})$ as in Figure 11(3). We set $\psi = T_m^{-1} f_\ell \in \mathrm{Mod}(S_{1,2})$ where f_ℓ is the mapping class which represents the point-pushing map along ℓ (see Figure 11(3)). Then ψ is the monodromy of a fibration on $S^3 \setminus L_W$, i.e., M_ψ is homeomorphic to $S^3 \setminus L_W$. In particular ψ is pseudo-Ansosov since $S^3 \setminus L_W$ is hyperbolic. Observe that the induced map $\psi_* : H_1(S_{1,2}; \mathbb{Z}) \to H_1(S_{1,2}; \mathbb{Z})$ sends a, β and γ to $\alpha - \beta - \gamma$, $\beta + \gamma$ and γ respectively. Then the cohomology class $\xi_0 \in H^1(S_{1,2}; \mathbb{Z})$ which is dual to c, is primitive and fixed by ψ. We consider the $\mathbb{Z}$-cover $\widetilde{S}$ over $S = S_{1,2}$ corresponding to ξ_0, and we take a lift $\widetilde{\psi} : \widetilde{S} \to \widetilde{S}$ such that $\widetilde{\psi}(\Sigma_i) \subset \Sigma_{i-1} \cup \Sigma_i$ (see Figures 12

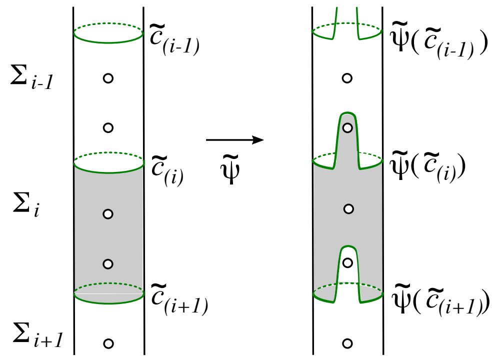

Figure 12: A lift $\widetilde{\psi} : \widetilde{S} \to \widetilde{S}$ of ψ with $\widetilde{\psi}(\Sigma_i) \subset \Sigma_{i-1} \cup \Sigma_i$. The regions of Σ_i and $\widetilde{\psi}(\Sigma_i)$ are shaded.

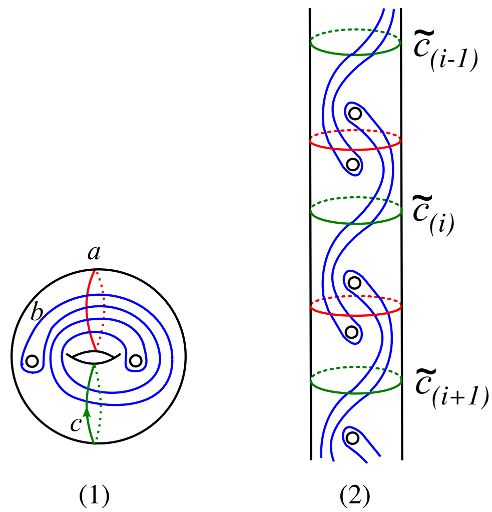

Figure 13: (1) Simple closed curves a, b in $S_{1,2}$. (2) $\mathbb{Z}$-cover $\widetilde{S}$ over $S = S_{1,2}$ corresponding to the dual of c.

and 11(3)). By the same argument as in the proof of Theorem B(1), we verify that R_n is a torus with $2n+1$ punctures if $n \geq 2$. By Theorem A, we conclude that $L_{\mathcal{C}}(\mathrm{Mod}(S_{1,2n+1})) \leq C/n^2$ for some constant $C > 0$.

We turn to the case where the number of punctures is even. Let a and b be simple closed curves in $S_{1,2}$ as in Figure 13(1), and let c be as before, i.e., $\beta = [c]$. Consider $\psi = T_b^{-1} T_a \in \mathrm{Mod}(S_{1,2})$ which is pseudo-Anosov by Penner's construction. The induced map ψ_* maps a basis a, β and γ of $H_1(S_{1,2}; \mathbb{Z})$ to $\alpha + \beta + \gamma$, β, and γ, respectively. Thus ψ fixes a primitive cohomology class $\xi_0 \in H^1(S_{1,2}; \mathbb{Z})$ which is dual to c. Consider the $\mathbb{Z}$-cover $\widetilde{S}$ over S corresponding to ξ_0 (Figure 13(2)) and pick a lift of $\widetilde{\psi} : \widetilde{S} \to \widetilde{S}$ of ψ. We can apply Theorem A

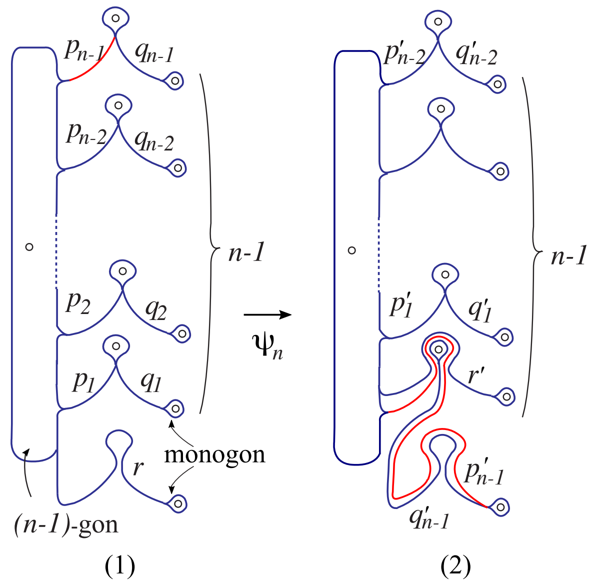

Figure 14: Small circles indicate punctures of $S_{0,2n+1}$. (1) Train track τ_n. (2) $\psi_n(\tau_n)$, where $e' = \psi_n(e)$. (The puncture p_∞ is not drawn here.)

for the fiber $(S_{1,2}, \psi)$ of the mapping torus M_ψ together with $\xi_0 \in H^1(S_{1,2}; \mathbb{Z})$ fixed by ψ. Theorem 1.1 says that for all n sufficiently large, R_n is a fiber of M_ψ. In this case R_n is a torus with $2n + n_0$ punctures, where n_0 is an even number which depends on the choice of the lift $\widetilde{\psi}$. By Theorem A we conclude that $L_{\mathcal{C}}(\mathrm{Mod}(S_{1,2n})) < C'/n^2$ for some constant $C' > 0$. This completes the proof. $\square$

5 Proof of Theorem C

This section includes the proof of Theorem C.

In the proof of Theorem B(1), we used the hyperbolic fibered 3-manifold $M_\beta = M_{\sigma_1^{-2}\sigma_2}$, the so-called *magic manifold* and its double cover M_{β^2}, depending on the parity of the number of punctures in the disk. Here we only use M_β and a sequence (R_n, ψ_n) of the fibers $R_n = D_{2n}$ of M_β with the monodromy ψ_n for $n \geq 2$ as in the proof of Theorem B(1). *Train tracks* play an important role in the proof. Terminology related to train tracks can be found in [BH95] or [FM12] for example.

We think of R_n as a sphere with $2n+1$ punctures. An invariant train track τ_n and a train track representative $\mathfrak{p}_n : \tau_n \to \tau_n$ of $\psi_n : S_{0,2n+1} \to S_{0,2n+1}$ are studied in [Kin15, Example 4.6]. Figure 14 shows the train track $\tau_n \subset S_{0,2n+1}$ and its image $\psi_n(\tau_n)$. Each of the monogon components of $S_{0,2n+1} \setminus \tau_n$ (bounded by loop edges of τ_n) contains a puncture of $S_{0,2n+1}$, the $(n-1)$-gon of $S_{0,2n+1} \setminus \tau_n$ contains another puncture, and the other connected component of $S_{0,2n+1} \setminus \tau_n$

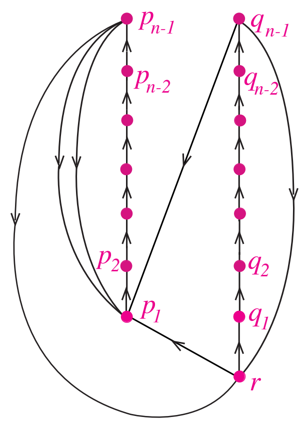

Figure 15: Directed graph Γ_n.

contains the other puncture p_∞ in the proof of Theorem B(1). Recall that ψ_n maps p_∞ to itself. Figure 15 gives the directed graph Γ_n of $\mathfrak{p}_n : \tau_n \to \tau_n$ for $n \geq 3$. The set of vertices of Γ_n equals the set of non-loop edges r, p_1, q_1, $\cdots$, p_{n-1}, q_{n-1} of τ_n as shown in Figure 14. The edges of Γ_n tell the locations of $\mathfrak{p}_n(e)$, $\mathfrak{p}_n^2(e)$, $\mathfrak{p}_n^3(e), \cdots$ in $S_{0,2n+1}$ for each non-loop edge e of τ_n. More precisely, j edges of Γ_n running from the vertex e to the vertex e' mean that $\mathfrak{p}_n(e)$ passes through the edge e' of τ_n j times. One can construct Γ_n viewing $\psi_n(\tau_n)$ and τ_n. The "vertical" consecutive edges of Γ_n in Figure 15 reveal the dynamics of $\psi_n : S_{0,2n+1} \to S_{0,2n+1}$ which is just like a translation on a "big" subsurface of $S_{0,2n+1}$.

We first prove the following.

Proposition 5.1. *For $n \geq 4$, we have*

$$L_{\mathcal{C}}(\mathrm{Mod}(D_{2n-1})) \leq \frac{1}{n^2 - 4n + 2} \quad \textit{and} \quad L_{\mathcal{C}}(\mathrm{Mod}(D_{2n})) \leq \frac{1}{n^2 - 4n + 2}.$$

Proof. We first prove the latter upper bound. We assume $n \geq 4$. Let $\mathcal{N}(\tau_n) \subset S_{0,2n+1}$ be a *fibered neighborhood* of τ_n (see [PP87, page 360] for the definition) equipped with a retraction $\mathcal{N}(\tau_n) \searrow \tau$. For a connected subset $\tau' \subset \tau_n$, we define a *fibered neighborhood* $\mathcal{N}(\tau')$ of τ' as follows.

$$\mathcal{N}(\tau') = \mathcal{N}(\tau_n) \cap U(\tau'),$$

where $U(\tau')$ is a small neighborhood of τ' in the 2-sphere S^2. We take n points $v_0, v_1, v_2, \cdots, v_{n-1} \subset \tau_n$, each of which lies on an edge of the $(n-1)$-gon, see Figure 16(1). For $1 \leq i < j \leq n-1$, let $\tau(i,j)$ be the connected component of $\tau_n \setminus \{v_{i-1}, v_j\}$ containing $p_i, q_i, p_{i+1}, q_{i+1}, \cdots, p_j, q_j$ (see Figure 16(2)). We consider its fibered neighborhood $\mathcal{N}(\tau(i,j))$, and we set

$$\mathcal{N}(p_i q_i p_{i+1} q_{i+1} \cdots p_j q_j) = \mathcal{N}(\tau(i,j)).$$

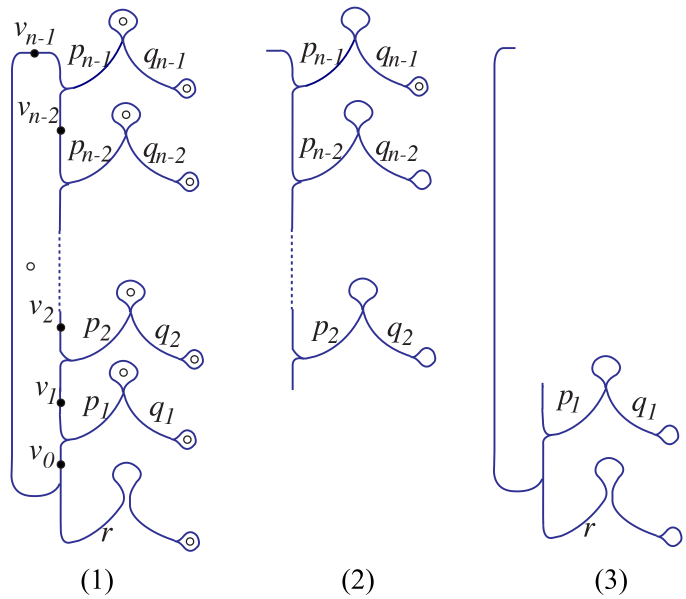

Figure 16: (1) Points $v_0, v_1, \cdots, v_{n-1}$. (2) $\tau(2, n-1) \subset \mathcal{N}(p_2q_2 \cdots p_{n-1}q_{n-1})$. (3) $\tau(1) \subset \mathcal{N}(rp_1q_1)$.

For $1 \leq j \leq n-2$, let $\tau(j)$ be the connected component of $\tau_n \setminus \{v_j, v_{n-1}\}$ containing $r, p_1, q_1, \cdots, p_j, q_j$ (see Figure 16(3)). Let

$$\mathcal{N}(rp_1q_1 \cdots p_jq_j) = \mathcal{N}(\tau(j)).$$

The notation $\mathcal{N}(rp_1q_1 \cdots p_jq_j)$ tells a property that it contains $r, p_1, q_1, \cdots, p_j, q_j$. The same thing holds for $\mathcal{N}(p_iq_ip_{i+1}q_{i+1} \cdots p_jq_j)$.

We take an essential arc c connecting the two punctures as in Figure 17(1). Then c is carried by τ_n. Notice that if $i \geq 2$, then $\mathcal{N}(p_iq_ip_{i+1}q_{i+1} \cdots p_{n-1}q_{n-1})$ is disjoint from c. Since $c \subset \mathcal{N}(rp_1q_1)$, we have

$$\begin{aligned} \psi_n(c) &\subset \mathcal{N}(p_1q_1p_2q_2), \\ \psi_n^2(c) &\subset \mathcal{N}(p_2q_2p_3q_3), \\ &\vdots \\ \psi_n^{1+(n-3)}(c) = \psi_n^{n-2}(c) &\subset \mathcal{N}(p_{n-2}q_{n-2}p_{n-1}q_{n-1}) \end{aligned}$$

(see Figures 15 and 17). Observe that $\psi_n^2(\psi_n^{n-2}(c)) = \psi_n^n(c) \subset \mathcal{N}(rp_1q_1p_2q_2)$. We have

$$\begin{aligned} \psi_n^{n+1}(c) &\subset \mathcal{N}(p_1q_1p_2q_2p_3q_3), \\ &\vdots \\ \psi_n^{(n+1)+(n-4)}(c) = \psi_n^{2n-3}(c) &\subset \mathcal{N}(p_{n-3}q_{n-3}p_{n-2}q_{n-2}p_{n-1}q_{n-1}). \end{aligned}$$

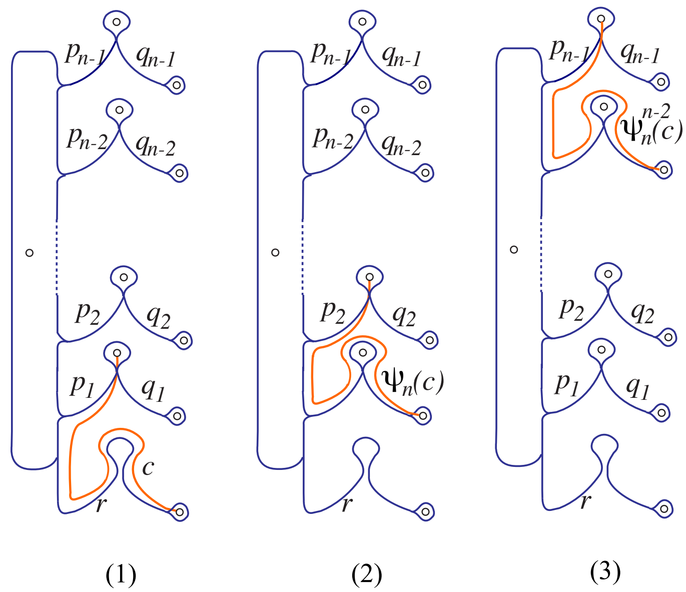

Figure 17: (1) $c \subset \mathcal{N}(rp_1q_1)$. (2) $\psi_n(c) \subset \mathcal{N}(p_1q_1p_2q_2)$. (3) $\psi_n^{n-2}(c) \subset \mathcal{N}(p_{n-2}q_{n-2}p_{n-1}q_{n-1})$.

In the same manner, for $2 \leq k \leq n-2$, we have

$$\psi_n^{(k-1)n-k}(c) \subset \mathcal{N}(p_{n-k}q_{n-k} \cdots p_{n-1}q_{n-1}).$$

When $k = n-2$,

$$\psi_n^{(n-3)n-(n-2)}(c) = \psi_n^{n^2-4n+2}(c) \subset \mathcal{N}(p_2q_2 \cdots p_{n-1}q_{n-1}).$$

Hence clearly we have

$$d_{\mathcal{AC}}(c, \psi_n^{n^2-4n+2}(c)) = 1.$$

If we consider a regular neighborhood of c in S^2, then we obtain an essential simple closed curve α in $S_{0,2n+1}$ as the boundary of the neighborhood in question. Notice that α is also carried by τ_n and $\alpha \subset \mathcal{N}(rp_1q_1)$. The above argument shows that $\psi_n^{n^2-4n+2}(\alpha) \subset \mathcal{N}(p_2q_2 \cdots p_{n-1}q_{n-1})$ and α is disjoint from $\psi_n^{n^2-4n+2}(\alpha)$. Recall that ψ_n is defined on $R_n = D_{2n}$. This together with Lemma 2.1 implies that

$$L_{\mathcal{C}}(\mathrm{Mod}(D_{2n})) \leq \frac{1}{n^2-4n+2}.$$

To show the former upper bound of $L_{\mathcal{C}}(\mathrm{Mod}(D_{2n-1})$ in the claim, we fill the puncture in the $(n-1)$-gon of $S_{0,2n+1} \setminus \tau_n$. The assumption $n-1 \geq 3$ ensures that τ_n extends to a train track $\overline{\tau}_n$ in $S_{0,2n}$ and $\psi_n : S_{0,2n+1} \to S_{0,2n+1}$ extends to $\overline{\psi}_n : S_{0,2n} \to S_{0,2n}$ which is still pseudo-Anosov. In particular $\overline{\psi}_n$ maps the puncture p_∞ to itself. We can think of $\overline{\psi}_n : S_{0,2n} \to S_{0,2n}$ as an element of

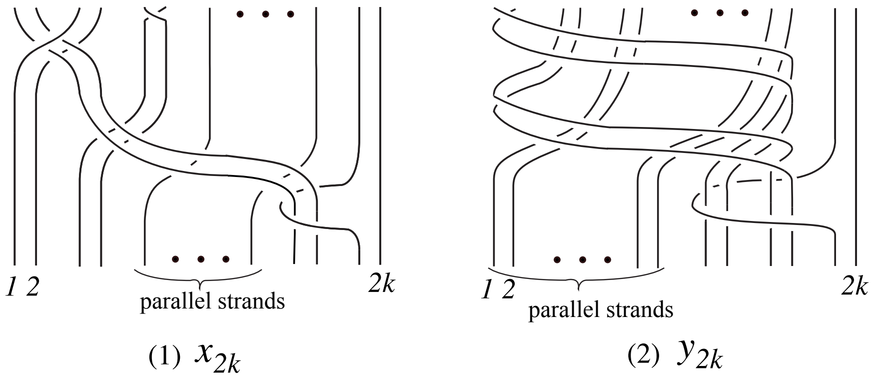

Figure 18: (1) $x_{2k} \in SW_{2k}$. (2) $y_{2k} \in SW_{2k}$.

$\mathrm{Mod}(D_{2n-1})$. The train track representative $\mathfrak{p}_n : \tau_n \to \tau_n$ also extends to a train track representative $\overline{\mathfrak{p}}_n : \overline{\tau}_n \to \overline{\tau}_n$ of $\overline{\psi}_n : S_{0,2n} \to S_{0,2n}$. All non-loop edges of $\overline{\tau}_n$ are coming from those of τ_n, and hence the directed graph $\overline{\Gamma}_n$ for $\overline{\mathfrak{p}}_n : \overline{\tau}_n \to \overline{\tau}_n$ is the same as Γ_n for $\mathfrak{p}_n; \tau_n \to \tau_n$. For the arc $\overline{c}$ and the simple closed curve $\overline{\alpha}$ in $S_{0,2n}$ coming from c and α in $S_{0,2n+1}$, respectively, the above argument tells us that

$$d_{\mathcal{AC}}(\overline{c},\ \overline{\psi}_n^{n^2-4n+2}(\overline{c})) = 1 \ \text{ and } \ d_{\mathcal{C}}(\overline{\alpha},\ \overline{\psi}_n^{n^2-4n+2}(\overline{\alpha})) = 1. \qquad (5.1)$$

The latter equality in (5.1) with Lemma 2.1 gives the desired upper bound. □

We are now ready to prove Theorem C.

Proof of Theorem C. By Lemma 3.1 together with either of the equalities for $\overline{\psi}_n : S_{0,2n} \to S_{0,2n}$ in (5.1), we have $L_{\mathcal{C}}(\mathcal{H}(S_{n-1})) \leq \frac{1}{n^2-4n+2}$ for $n \geq 4$. Thus for $g \geq 3$,

$$L_{\mathcal{C}}(\mathcal{H}(S_g)) \leq \frac{1}{(g+1)^2 - 4(g+1) + 2} = \frac{1}{g^2 - 2g - 1}.$$

□

6 Proof of Theorem D

In this section, we finally prove Theorem D.

Proof of Theorem D. The proof is separated into two cases, depending on the parity of the genera. First of all we introduce spherical braids $x_{2k}, y_{2k} \in SB_{2k}$ for $k \geq 5$ as shown in Figure 18. It is straightforward to see that they are elements of SW_{2k}. We define $w_{2k} \in SW_{2k}$ for each $k \geq 5$ as follows.

$$\begin{aligned} w_{4n+8} &= x_{4n+8}(y_{4n+8})^n && \text{if } \ 2k = 4n+8 \ \text{ for some } n \geq 1, \\ w_{4n+10} &= (x_{4n+10})^2(y_{4n+10})^n && \text{if } \ 2k = 4n+10 \ \text{ for some } n \geq 0. \end{aligned}$$

Consider an element in the Hilden group SH_{2k} corresponding to w_{2k} (see Section 3.2) and its mapping torus $M_{w_{2k}}$. In [HK17] it is shown that when $2k = 4n+8$ for $n \geq 1$, $M_{w_{2k}}$ is homeomorphic to the mapping torus M_w of the element in SH_6 corresponding to the pseudo-Anosov braid $w \in SW_6$ (see Figure 6(3)). In other words, M_w is hyperbolic and it has a fiber $S_{0,2k}$ with pseudo-Anosov monodromy w_{2k} when $2k = 4n+8$. We claim that a sequence of fibers $(S_{0,4n+10}, w_{4n+10})$ of M_w comes from a fibered 3-manifold as in Theorem A. More precisely, if we remove the 6th strand of w, then we obtain a spherical braid with 5 strands. Regarding such a braid as the one on the disk, we have a 5-braid, say $\psi \in B_5$. Clearly M_ψ is homeomorphic to M_w. We consider a fiber $S = D_5$ with monodromy ψ of the mapping torus $M_\psi \simeq M_w$. Since ψ_* maps the generator t_5 to itself (see the 5th strand of the braid w in Figure 6(3)), the cohomology class $\xi_0 \in H^1(S; \mathbb{Z})$ which is dual to the proper arc $c = c_5$ is fixed by ψ. Let $\widetilde{S}$ be the $\mathbb{Z}$-cover of S corresponding to ξ_0. We consider the canonical lift $\widetilde{\psi} : \widetilde{S} \to \widetilde{S}$ of ψ. Then $R_n = \widetilde{S}/\langle h^n \widetilde{\psi} \rangle$ is a fiber of M_ψ with monodromy ψ_n for n large. In this case, R_n is a sphere with $4n+8$ punctures, and we find that the monodromy ψ_n is given by the braid $w_{4n+8} \in SW_{4n+8}$ from the argument in [HK17, Section 3]. By the proof of Theorem A, there exist $\alpha \in \mathcal{AC}(R_n)^0$ and $m \asymp n^2$ such that $d_{\mathcal{AC}}(\alpha, (\psi_n)^m(\alpha)) = 1$. Notice that a lift $\widehat{\psi} = \widehat{\psi_n}$ of ψ_n under the map q is an element of $\mathcal{H}(\mathbb{H}_{2n+3})$ (see Section 3.2). By Lemma 3.1, $\ell_{\mathcal{C}}(\widehat{\psi}) \leq 1/m$, which implies $\ell_{\mathcal{C}}(\widehat{\psi}) \leq C/n^2$ for some constant $C > 0$. Thus we have $L_{\mathcal{C}}(\mathcal{H}(\mathbb{H}_{2n+3})) \leq C/n^2$ in the case of the odd genus.

To obtain the upper bound $L_{\mathcal{C}}(\mathcal{H}(\mathbb{H}_{2n+4})) \leq C'/n^2$ for some $C' > 0$ in the case of the even genus, we take the second power $\psi^2 \in B_5$ of the above ψ and we set $\phi = \psi^2$. We consider a fiber $S = D_5$ with monodromy ϕ in the mapping torus M_ϕ. Note that ϕ fixes the same $\xi_0 \in H^1(S; \mathbb{Z})$. Let $\widetilde{S}$ be the $\mathbb{Z}$-cover over S as before and let $\widetilde{\phi} = (\widetilde{\psi})^2 : \widetilde{S} \to \widetilde{S}$ which is the canonical lift of ϕ. Now we apply Theorem A for the fiber (S, ϕ) of M_ϕ together with ξ_0. One sees that for n large, $\widetilde{S}/\langle h^n \widetilde{\phi} \rangle$ is a fiber of M_ϕ which is the sphere with $4n+10$ punctures. The same argument as in [HK17, Section 3] tells us that the monodromy of the fiber $\widetilde{S}/\langle h^n \widetilde{\phi} \rangle$ is described by the braid $w_{4n+10} \in SW_{4n+10}$. As in the case of the odd genus, we obtain the desired upper bound of $L_{\mathcal{C}}(\mathcal{H}(\mathbb{H}_{2n+4}))$. This completes the proof. □

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