SPECTRAL RADIUS OF A STAR WITH ONE LONG ARM

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ABSTRACT. A tree is said to be starlike if exactly one vertex has degree greater than two. In this paper, we will study the spectral properties of $S(n, k \cdot 1)$, that is, the starlike tree with k branches of length 1 and one branch of length n. The largest eigenvalue λ_1 of $S(n, k \cdot 1)$ satisfies $\sqrt{k+1} \le \lambda_1 < k/\sqrt{k-1}$. Moreover, the largest eigenvalue of $S(n, k \cdot 1)$ is equal to the largest eigenvalue of $S(k \cdot (n+1))$, which is the starlike tree that has k branches of length $n-1$. Using the spectral radii of $S(n, k \cdot 1)$ we can show that there is a sequence of Salem numbers that converges to each integer > 1.

1. INTRODUCTION

A tree which has exactly one vertex of degree greater than two is said to be *starlike*. Spectral properties of starlike trees are recently studied in [LG01, LG02, BS98].

Let P_n be the path with n vertices. We denote $S(n_1, n_2, \cdots, n_k)$ a starlike tree in which removing the central vertex v_1 leaves disjoint paths such that

$$S(n_1, n_2, \cdots, n_k) - v_1 = P_{n_1} \cup P_{n_2} \cup \cdots \cup P_{n_k}.$$

We say that the starlike tree $S(n_1, n_2, \cdots, n_k)$ has k branches and the lengths of branches are $n_1, n_2, \cdots, n_k$. It will be assumed that $n_1 \ge n_2 \ge \cdots \ge n_k$.

For a simple graph G of order n, the *spectrum* of G is the set of eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ of its adjacency matrix A. The characteristic polynomial $\det(\lambda I - A)$ of A is called the characteristic polynomial of G, denoted $\phi(G, \lambda)$ or simply $\phi(G)$. It is known that if G is a graph and v is any vertex, then

$$\phi(G) = \lambda \phi(G - v) - \sum_u \phi(G - v - u) - 2 \sum_C \phi(G - C),$$

where the first summation is over vertices u adjacent to the vertex v and the second summation is over all cycles C embracing the vertex v. Applying to the starlike trees we obtain

$$\phi(S(n_1, n_2, \cdots, n_k)) = \lambda \prod_{i=1}^{k} \phi(P_{n_i}) - \sum_{i=1}^{k} \left[\phi(P_{n_i - 1}) \prod_{j \in I_i} \phi(P_{n_j}) \right], \tag{1}$$

where $I_i = \{1, 2, \cdots, k\} \setminus \{i\}$.

Using Equation (1), Lepović and Gutman [LG01] determine the bounds for the largest eigenvalues of starlike trees.

Key words and phrases. starlike trees, largest eigenvalue, Salem number.

Theorem 1.1. [LG01, Theorem 2] *If λ_1 is the largest eigenvalue of the starlike tree $S(n_1, n_2, \cdots, n_k)$, then*

$$\sqrt{k} \leq \lambda_1 < \frac{k}{\sqrt{k-1}}$$

for any positive integers $n_1 \geq n_2 \geq \cdots \geq n_k \geq 1$.

The lower bound $\sqrt{k}$ for λ_1 is realized by the star on $k+1$ vertices. The upper bound can be achieved asymptotically by the starlike trees with $n_1 = n_2 = \cdots = n_k = n$. In such case, we will denote this starlike tree by $S(k \cdot n)$ instead of $S(n_1, n_2, \cdots, n_k)$.

In this paper, we will discuss the spectral properties of a star with one long arm; let $S(n, k \cdot 1)$ be the starlike tree with k branches of length 1 and one branch with length n. Note that $S(n, k \cdot 1)$ is a tree on $n+k+1$ vertices.

Two nonisomorphic graphs with the same spectrum are called *cospectral*. It is known that no two starlike trees are cospectral [LG02]. However, the spectral radius does not distinguish starlike trees. We will show that there are infinitely many pairs of nonisomorphic starlike trees that have the same spectral radius.

Theorem A. *For any positive integer $k \geq 3$, starlike trees $S(n, k \cdot 1)$ and $S(k \cdot (n+1))$ have the same largest eigenvalue.*

By Theorem 1.1 the largest eigenvalue λ_1 of $S(n, k \cdot 1)$ satisfies

$$\sqrt{k+1} \leq \lambda_1 < \frac{k+1}{\sqrt{k}}.$$

As a consequence of Theorem A, we have a sharper upper bound for the starlike tree $S(n, k \cdot 1)$.

Theorem B. *If λ_1 is the largest eigenvalue of $S(n, k \cdot 1)$, then*

$$\sqrt{k+1} \leq \lambda_1 < \frac{k}{\sqrt{k-1}}$$

for any positive integers $n \geq 1$ and $k \geq 3$.

There are two special algebraic integers related to the largest eigenvalue of starlike trees. A *Salem number* is an algebraic integer $\alpha > 1$, all of whose other conjugates have modulus ≤ 1, with at least one conjugate of modulus 1. A *Pisot number* is an algebraic integer $\beta > 1$, all of whose other conjugates have modulus < 1. With Theorem B and the work of McKee–Rowlinson–Smyth [MRS99], we have the following corollary.

Corollary 3.2. *For $n \geq 2$ and $k \geq 3$ let λ_1 be the largest eigenvalue of the starlike tree $S(n, k \cdot 1)$. Then the number $t > 1$ defined by*

$$\sqrt{t} + \frac{1}{\sqrt{t}} = \lambda_1, \tag{2}$$

is a Salem number.

Using the spectral properties of the starlike tree $S(n, k \cdot 1)$, the author studied the stretch factors of pseudo-Anosov mapping classes of closed orientable surfaces. In particular, the number t defined by (2) is the stretch factor of a pseudo-Anosov mapping class from Thurston's construction whose configuration graph is $S(n, k \cdot 1)$. For more about this topic, see [Shi].

2. Bounds for the largest eigenvalue

In this section we will prove main theorems of this paper. Lepović and Gutman [LG01] show that the number $t > 1$, defined by $\sqrt{t} + 1/\sqrt{t} = \lambda_1$, where λ_1 is the largest eigenvalue of $S(k \cdot (n+1))$, is the root of the polynomial equation

$$t^{n+3} - (k-1)t^{n+2} + (k-1)t - 1 = 0. \tag{3}$$

To prove Theorem A we will show that when λ_1 is the largest eigenvalue of $S(n, k \cdot 1)$, the number t given by $\sqrt{t} + 1/\sqrt{t} = \lambda_1$, is again the root of the polynomial (3).

Proof of Theorem A. Equation (1) reduces to

$$\begin{aligned} \phi\left(S(n, k \cdot 1)\right) &= \lambda \phi(P_n)\phi(P_1)^k - \left(k\, \phi(P_n)\phi(P_1)^{k-1} + \phi(P_{n-1})\phi(P_1)^k\right) \\ &= \lambda^{k+1}\phi(P_n) - k\lambda^{k-1}\phi(P_n) - \lambda^k \phi(P_{n-1}) \\ &= \lambda^{k-1}\left(\lambda^2\phi(P_n) - k\phi(P_n) - \lambda\phi(P_{n-1})\right). \end{aligned}$$

Therefore the largest eigenvalue of $S(n, k \cdot 1)$ is the root of

$$\lambda^2\phi(P_n) - k\phi(P_n) - \lambda\phi(P_{n-1}) = 0. \tag{4}$$

By substituting $\lambda = 2\cos\theta$, we get $\phi(P_n) = \sin(n+1)\theta / \sin\theta$ (see [CDS95, p.73]) and Equation (4) becomes

$$(4\cos^2\theta - k)\frac{\sin(n+1)\theta}{\sin\theta} - 2\cos\theta\frac{\sin n\theta}{\sin\theta} = 0. \tag{5}$$

By setting $t^{1/2} = e^{i\theta}$, we have

$$\lambda = 2\cos\theta = t^{1/2} + t^{-1/2}$$

and

$$\sin n\theta = \frac{t^{n/2} - t^{-n/2}}{2i}.$$

By substituting and simplifying, Equation (5) becomes

$$t^{n+3} - (k-1)t^{n+2} + (k-1)t - 1 = 0. \tag{6}$$

If t^* is a root of Equation (6), then the number λ^*, defined by $\lambda^* = \sqrt{t^*} + 1/\sqrt{t^*}$, is a root of Equation (4). Since Equation (6) is identical with Equation (3) we can conclude that the largest eigenvalue of $S(n, k \cdot 1)$ is equal to the largest eigenvalue of $S(k \cdot (n+1))$. □

Theorem B follows directly from Theorem A and the work of Lepović and Gutman.

Proof of Theorem B. A star with $k+2$ vertices is a subgraph of $S(n, k\cdot 1)$ and its largest eigenvalue is $\sqrt{k+1}$. By the interlacing theorem we have $\sqrt{k+1} \leq \lambda_1$.

On the other hand, Lepović and Gutman also show that Equation (6) has a zero in the interval $(k-2, k-1)$ and it follows that $\lambda_1 < k/\sqrt{k-1}$ (See [LG01, prrof of Theorem 2]). Therefore we have

$$\sqrt{k+1} \leq \lambda_1 < \frac{k}{\sqrt{k-1}}.$$

□

Remark. In the paper of Lepović and Gutman, they study the properties of the polynomial

$$t^{n+2} - (k-1)t^{n+1} + (k-1)t - 1$$

and one can easily see that all results are also true for Equation (6).

3. Algebraic integers associated with starlike trees

It is known that a starlike tree has at most one eigenvalue > 2. We say that a starlike tree is *hyperbolic* if it has exactly one eigenvalue greater than 2. It happens that all starlike trees are hyperbolic except $S(n-3,1,1)$, for $n \geq 4$, $S(5,2,1), S(4,2,1), S(3,3,1), S(3,2,1), S(2,2,2), S(2,2,1)$, and $S(1,1,1,1)$ [LG01, Theroem 1]. Hence for $n \geq 2$ and $k \geq 3$, $S(n, k\cdot 1)$ is hyperbolic.

Let λ_1 be the largest eigenvalue of $S(n_1, n_2, \cdots, n_k)$. If the starlike tree is hyperbolic, then the number $t > 1$, defined by $\sqrt{t} + 1/\sqrt{t} = \lambda_1$, is associated with the dynamical complexity of an automorphism of an orientable surface (for more about this topic, see [Lei04] or [Shi]). In particular, t is a special algebraic integer, characterized by the following theorem.

Theorem 3.1. [MRS99, Corollary 9] *Let S be a starlike tree whose largest eigenvalue λ_1 is not an integer, and suppose that S is hyperbolic. Then $t > 1$, defined by $\sqrt{t} + 1/\sqrt{t} = \lambda_1$, is a Salem number. If λ_1 is an integer then t is a quadratic Pisot number.*

Now we have the following result.

Corollary 3.2. *For $n \geq 2$ and $k \geq 3$ let λ_1 be the largest eigenvalue of the starlike tree $S(n, k\cdot 1)$. Then the number $t > 1$ defined by*

$$\sqrt{t} + \frac{1}{\sqrt{t}} = \lambda_1,$$

is a Salem number.

Proof. By Theorem B we have

$$k + 1 < \lambda_1^2 < \frac{k^2}{k-1} = k + 1 + \frac{1}{k-1}$$

and hence λ_1 is not an integer. By Theorem 3.1, t is a Salem number. □

Let $Q_n(t)$ be the polynomial in Equation (6) and let $\rho\left(Q_n(t)\right)$ be the largest real root of $Q_n(t)$. Let m be any fixed positive integer. It is shown that for sufficiently large n, $Q_n(t)$ has a root in the interval $(k-1-\frac{1}{10^m}, k-1)$ (see the proof of Corollary 2.1. in [LG01]). This implies that

$$\lim_{n\to\infty} \rho\left(Q_n(t)\right) = k - 1.$$

Since the largest root of $Q_n(t)$ is a Salem number for each n, there is a sequence of Salem numbers that converges to each integer greater than 1.

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