MINIMAL ASYMPTOTIC TRANSLATION LENGTHS OF TORELLI GROUPS AND PURE BRAID GROUPS ON THE CURVE GRAPH

HYUNGRYUL BAIK AND HYUNSHIK SHIN

ABSTRACT. In this paper, we show that the minimal asymptotic translation length of the Torelli group $\mathcal{I}_g$ of the surface S_g of genus g on the curve graph asymptotically behaves like $1/g$, contrary to the mapping class group $\mathrm{Mod}(S_g)$, which behaves like $1/g^2$. We also show that the minimal asymptotic translation length of the pure braid group PB_n on the curve graph asymptotically behaves like $1/n$, contrary to the braid group B_n, which behaves like $1/n^2$.

1. INTRODUCTION

Let $S = S_{g,n}$ be an orientable surface of genus g with n punctures. The *mapping class group* of S, denoted by $\mathrm{Mod}(S)$, is the group of isotopy classes of orientation-preserving homeomorphisms of S. An element of $\mathrm{Mod}(S)$ is called a mapping class. The curve graph $\mathcal{C}(S)$ of S is one of the important spaces upon which $\mathrm{Mod}(S)$ is acting. The *curve graph* is a simplicial graph whose vertices are isotopy classes of essential simple closed curves in S and two vertices are joined by an edge if they are realized by a pair of disjoint curves. Assigning each edge length 1 induces a path metric $d_{\mathcal{C}}(\cdot, \cdot)$ on $\mathcal{C}(S)$. Then $\mathrm{Mod}(S)$ acts on $\mathcal{C}(S)$ by isometries. For an element $f \in \mathrm{Mod}(S)$, we define the *asymptotic translation length* of f by

$$\ell_{\mathcal{C}}(f) = \liminf_{j \to \infty} \frac{d_{\mathcal{C}}(\alpha, f^j(\alpha))}{j},$$

where α is an element in $\mathcal{C}(S)$. Note that $\ell_C(f)$ is independent of the choice of α, and $\ell_C(f^m) = m\ell_C(f)$ for each $m \in \mathbb{N}$.

In this paper, We always assume that the complexity of S is $\xi(S) = 3g - 3 + n \geq 2$. Masur and Minsky [MM99] showed that $C(S)$ is Gromov-hyperbolic and show that $\ell_{\mathcal{C}}(f) > 0$ if and only if f is a pseudo-Anosov mapping class. Bowditch [Bow08] showed that there exists some positive integer m, depending only on S, such that for any pseudo-Anosov $f \in \mathrm{Mod}(S)$, f^m acts by translation along a geodesic axis in $\mathcal{C}(S)$. As a consequence, $\ell_{\mathcal{C}}(f)$ is a rational number with bounded denominator. There has been work of many authors on estimating asymptotic translation lengths on curve graphs. For instance, see [FLM08, GT11, GHKL13, Val14, AT15, KS17, Val17, BSW18] and references therein.

One can think of the behavior of the minimal asymptotic translation length on a closed surface S_g of genus g. For any subgroup $H < \mathrm{Mod}(S)$, let us define

$$L_{\mathcal{C}}(H) = \min\{\ell_{\mathcal{C}}(f) | f \in H, \text{pseudo-Anosov}\}.$$

Gadre and Tsai [GT11] proved that

$$L_{\mathcal{C}}(\mathrm{Mod}(S_g)) \asymp \frac{1}{g^2},$$

where $F(g) \asymp G(g)$ implies that there exist positive constants C and D such that $CG(g) \leq F(g) \leq DG(g)$. The second author and Kin [KS17] showed that the minimal asymptotic translation lengths of hyperelliptic mapping class group, handlebody group, and hyperelliptic handlebody group of S_g also behave like $1/g^2$.

The Torelli group $\mathcal{I}_g$ is another important subgroup of $\mathrm{Mod}(S_g)$. Farb–Leininger–Margalit [FLM08] proved that $L_{\mathcal{C}}(\mathcal{I}_g) \to 0$ as $g \to \infty$ but the exact asymptote is not known. It turns out that the behavior of $L_{\mathcal{C}}(\mathcal{I}_g)$ is different from that of $L_{\mathcal{C}}(\mathrm{Mod}(S_g))$.

Theorem 1.1. *For $g \geq 2$, we have*

$$L_{\mathcal{C}}(\mathcal{I}_g) \asymp \frac{1}{g}.$$

The lower bound for Theorem 1.1 is a direct consequence of the result by Tsai [Tsa09, Lemma 3.1] together with Bestvina–Handel algorithm [BH95]. For the upper bound of $L_{\mathcal{C}}(\mathcal{I}_g)$, we use an explicit sequence of pseudo-Anosov $f_g \in \mathcal{I}_g$ such that $\ell_{\mathcal{C}}(f_g) \leq D/g$ for some constant $D > 0$. This sequence is arising from Penner's construction [Pen88] and this technique can be applied to all other surfaces.

We also investigate the minimal asymptotic translation lengths of the pure braid group PB_n. The braid group B_n can be regarded as the mapping class group of the n-punctured disk D_n fixing boundary pointwise. Then the pure braid group PB_n is analogous to the Torelli group $\mathcal{I}_g$ in the sense that the action of PB_n on the first homology $H_1(D_n)$ is trivial. (For an introduction to braid groups and pure braid groups, we refer the reader to [FM12, Chapter 9] or [BB05].) Our next theorem in this paper is as follows.

Theorem 1.2. *For a pure braid group PB_n, we have*

$$L_{\mathcal{C}}(\mathrm{PB}_n) \asymp \frac{1}{n}.$$

On the contrary, it is known that $L_{\mathcal{C}}(\mathrm{B}_n) \asymp 1/n^2$ (see [KS17, Theorem B]). To obtain the lower bound, we cannot use the Lemma 3.1 of [Tsa09] as in the Torelli group. Instead, we use the calculations in [GT11] and the nesting lemma by Masur–Minsky [MM99]. The upper bound of $L_{\mathcal{C}}(\mathrm{PB}_n)$ is again obtained by an explicit sequence of pure braids $f_n \in \mathrm{PB}_n$ from Penner's construction.

We also discuss the minimal translation length for the pure mapping class group $\mathrm{PMod}(S)$. In fact, our method in the proof of Theorem 1.2 can be easily adapted in this case, and one can obtain another proof of the following theorem of Valdivia.

Theorem 1.3 (c.f., [Val14])**.** *For any fixed $g \geq 0$,*

$$L_{\mathcal{C}}(\mathrm{PMod}(S_{g,n})) \asymp \frac{1}{n}.$$

The original theorem by Valdivia in [Val14] was proved for $g \geq 2$ using a technique which is similar to one we used to prove Theorem 1.1. Our proof also works for $g = 0$ or 1 as well as $g \geq 2$, and hence we can state this theorem for all $g \geq 0$.

2. Background

2.1. **Train tracks and Bestvina–Handel algorithm.** In the late 1970's, Thurston introduced a powerful tool to study measured geodesic laminations on a surface, so called train tracks. For more discussion about train tracks, see [PH92] or [BH95].

A *train track* τ is a smooth 1-complex embedded in a surface S where there is a well-defined tangent line at each vertex, and that there are edges tangent in each direction. Vertices and edges of τ are also called switches and branches, respectively. We require that each vertex of τ is at least tri-valent. We assign each edge of τ a nonnegative number, called a *weight*, so that it satisfies switch conditions. That is, the sum of weights on incoming edges is equal to the sum of weights on outgoing edges. The set μ of weights on τ is called a *measure*.

Every pseudo-Anosov $f \in \mathrm{Mod}(S)$ has a weighted train track τ such that $f(\tau)$ collapses to τ. We call such τ an invariant train track for f. In [BH95], Bestvina and Handel gave an algorithm to find an invariant train track τ for a given pseudo-Anosov f. The branches of τ consist of two types, *real* branches and *infinitesimal* branches. There are at most $9|\chi(S)|$ real branches when S is closed, and at most $3|\chi(S)|$ real branches when S is punctured. Furthermore, there are at most $24|\chi(S)| - 8n$ infinitesimal branches (see Section 4 of [GT11]).

Let $\mathcal{R}$ be the set of real branches of τ. In [BH95], Bestvina and Handel showed that the transition matrix M of f on τ is of the form

$$M = \begin{pmatrix} A & B \\ O & M_{\mathcal{R}} \end{pmatrix},$$

where A is a permutation matrix on infinitesimal branches, and $M_{\mathcal{R}}$ is the transition matrix on real branches where $M_{\mathcal{R}}$ is *primitive*, that is, for some $m \in \mathbb{N}$, $M_{\mathcal{R}}^m$ is a positive matrix .

2.2. Nesting lemma and lower bound of asymptotic translation length.

Let τ be a train track, and let $P(\tau)$ be the polyhedron of measures supported on τ, that is, a cone satisfying switch conditions in $\mathbb{R}_{\geq 0}^B$, where B is the number of branches of τ. We say that a measure μ on τ is positive if it has positive weights on every branch. Let $int(P(\tau))$ be the set of positive measures supported on τ. We say a train track τ is *recurrent* if there is a positive measure $\mu \in int(P(\tau))$. A train track is *transversely recurrent* if given a branch of τ, there is a simple closed curve on S that crosses the branch, intersects τ transversely, and the union of τ and the simple closed curve has no complementary bigons. A train track is said to be *birecurrent* if it is both recurrent and transversely recurrent.

We say that a train track τ fills the surface S if the complementary region $S \setminus \tau$ is a union of ideal polygons containing at most one puncture. If τ fills S, then a train track σ is called a *diagonal extension* of τ if τ is sub-track of σ and each branch of $\sigma \setminus \tau$ has its endpoints terminating in the cusps of complementary regions of τ. Let

$$PE(\tau) = \bigcup_{\sigma \in E(\tau)} P(\sigma),$$

and let $int(PE(\tau))$ be the set of measures in $PE(\tau)$ that are positive on each branch of some diagonal extension of τ. Abusing the notation, $int(PE(\tau))$ and $PE(\tau)$ also denote the set of curves which defines measures in $int(PE(\tau))$ and $PE(\tau)$, respectively. Masur and Minsky showed the following lemma in [MM99].

Lemma 2.1 (Nesting Lemma). *Let τ be a birecurrent train track that fills the surface S. Then*

$$\mathcal{N}_1(int(PE(\tau))) \subset PE(\tau),$$

where $\mathcal{N}_1(X)$ is the 1-neighborhood of X in the curve complex.

From this lemma, Gadre and Tsai established the way to obtain lower bound of asymptotic translation length on the curve graph of a given pseudo-Anosov mapping class. Following the proof of Lemma 4.3 and Theorem 5.1 in [GT11], we have the following result (see also [GHKL13, Proposition 3.6]).

Proposition 2.2. *Let $f \in \mathrm{Mod}(S_{g,n})$ be a pseudo-Anosov element and let τ be its invariant train track obtained by Bestvina–Handel algorithm. Let r*

be the number of real branches and let q be the integer such that $M_{\mathcal{R}}^q$ has a positive diagonal entry, where $M_{\mathcal{R}}$ is the transition matrix of real branches. If we set $k = 2qr + 24|\chi(S_{g,n})| - 8n$, then for any real branch β of τ, the path $f^k(\beta)$ traverses every branch of τ. Furthermore, combining with the nesting lemma, if we set $w := k + 6|\chi(S_{g,n})| - 2n$, then we have

$$\ell_{\mathcal{C}}(f) \geq \frac{1}{w}.$$

3. Lower bound for Torelli groups

Let X be a compact oriented manifold and let $\phi : X \to X$ be a continuous map. We define the graph of ϕ by $\text{graph}(\phi) = \{(x, \phi(x)) \in X \times X| \ x \in X\}$. Let Δ be the diagonal of $X \times X$, that is, $\Delta = \{(x, x)| \ x \in X\} \subset X \times X$. The *Lefschetz number* of ϕ, denoted by $L(\phi)$, is defined by the algebraic intersection number $\hat{i}(\Delta, \text{graph}(\phi))$. Since the Lefschetz number is a homotopy invariant, it is well-defined for the homotopy class f of ϕ. It can be computed by

$$L(f) = \sum_{i \geq 0} (-1)^i Tr(f_*^{(i)}),$$

where $f_*^{(i)}$ is the action on $H_i(X; \mathbb{R})$ induced by f. For complete discussion, see [GP74] or [BT82].

In [Tsa09], Tsai gives the following crucial lemma.

Lemma 3.1. *For any pseudo-Anosov element $f \in \text{Mod}(S_{g,n})$ equipped with a Markov partition, if $L(f) < 0$, then there is a rectangle R of the Markov partition such that the interior of $f(R)$ and R intersect, i.e., the transition matrix of the Markov partition has a positive diagonal entry.*

Now we are ready to obtain the lower bound for the asymptotic translation length of a pseudo-Anosov element $f \in \mathcal{I}_g$.

Theorem 3.2. *Let $g \geq 2$ and let f be a pseudo-Anosov mapping class in the Torelli group $\mathcal{I}_g$. Then we have*

$$\ell_{\mathcal{C}}(f) \geq \frac{1}{96g - 96}.$$

Proof. For a closed surface S_g of genes $g \geq 2$, $H_i(S_g; \mathbb{R}) = 0$ for all $i \geq 3$. Note that $H_0(S_g; \mathbb{R}) \simeq H_2(S_g; \mathbb{R}) \simeq \mathbb{R}$ and $f_*^{(i)}$ is the identity for $i = 0$ or 2. Since f is in the Torelli group, $f_*^{(1)}$ is also the identity on $H_1(S_g; \mathbb{R}) \simeq \mathbb{R}^{2g}$. Therefore we have the Lefschetz number $L(f) = 1 - 2g + 1 = 2 - 2g < 0$ for all $g \geq 2$. By Lemma 3.1, there is a positive diagonal entry in the Markov partition matrix $M_{\mathcal{R}}$ of f, that is, the transition matrix of real branches of the train track. Hence in Proposition 2.2, we have $q = 1$. Since the number of real branches satisfies $r \leq 9|\chi(S_g)|$, we have $w \leq 48|\chi(S_g)|$ and

$$\ell_{\mathcal{C}}(f) \geq \frac{1}{96g - 96}.$$

(a) g even

(b) g odd
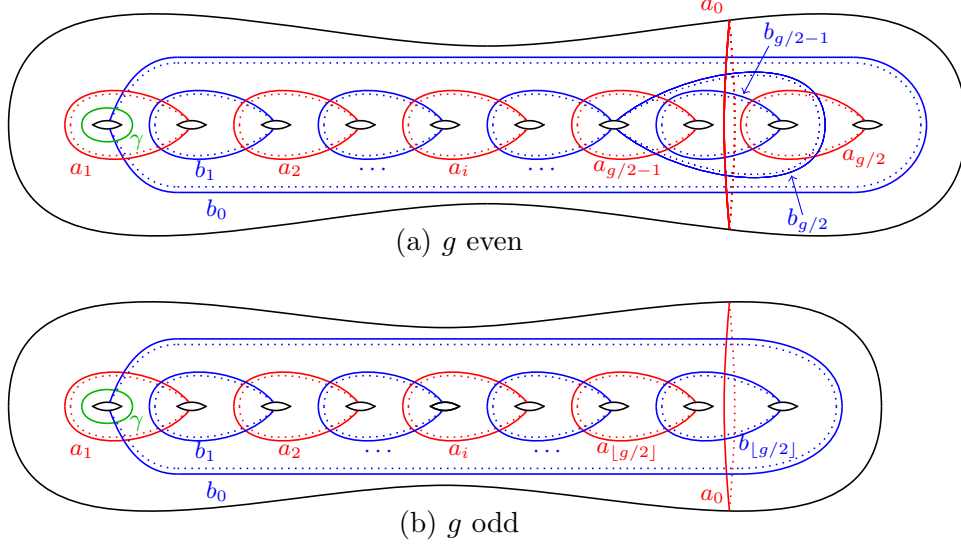
FIGURE 1. Multicurves in the closed surface S_g of genus g

$\square$

4. UPPER BOUND FOR TORELLI GROUPS

In this section, we give an upper bound for $L_C(\mathcal{I}_g)$ using an explicit family of pseudo-Anosov elements f_g in $\mathcal{I}_g$. For a simple closed curve a, let T_a be the left-handed Dehn twist about a. We apply elements of the mapping class group from right to left.

Theorem 4.1. *For all $g \geq 13$,*

$$L_C(\mathcal{I}_g) \leq \frac{8}{g - 12}.$$

Proof. It is enough to find a pseudo-Anosov element f_g in $\mathcal{I}_g$ such that $\ell_{\mathcal{C}}(f_g) \leq 8/(g - 12)$ for all large enough g.

Let us assume that $g \geq 13$ and let $n = \lfloor g/2 \rfloor$. Let $A = \{a_0, \cdots, a_n\}$ and $B = \{b_0, \cdots, b_n\}$ be multicurves as in Figure 1 so that $A \cup B$ fills the surface S_g. Let us define a mapping class $f_g \in \mathrm{Mod}(S_g)$ by

$$f_g = T_B^{-1} T_A,$$

where $T_A = \prod_{i=0}^{n} T_{a_i}$ and $T_B^{-1} = \prod_{i=0}^{n} T_{b_i}^{-1}$ are multi-twists, i.e., products of Dehn twists. Then f_g is a pseudo-Anosov mapping class since it is arising from Penner's construction (for Penner's construction, see [FM12, Theorem

14.4] or [Pen88]). Moreover, since each curve in A and B is a separating curve, f_g lies in the Torelli group $\mathcal{I}_g$ for each g.

We will show that there exists a simple closed curve δ such that

$$d_{\mathcal{C}}(\delta, f_g^{\lceil \frac{g}{4} \rceil - 3}(\delta)) \leq 2,$$

and this implies

$$\ell_{\mathcal{C}}(f_g) \leq \frac{2}{\lceil \frac{g}{4} \rceil - 3} \leq \frac{8}{g - 12}.$$

To show this, we follow the same notation as in the proof of Theorem 6.1 in [GT11]. For a finite collection of curves $\{c_j\}_{j=1}^k \subset A \cup B$ such that $c_1 \cup \cdots \cup c_k$ is connected, let $\mathcal{N}(c_1 \cdots c_k)$ be the regular neighborhood of $c_1 \cup \cdots \cup c_k$. Consider the curve $\delta = a_i$ and its image under the iteration of f_g. One can see that

$$f_g(a_i) \subset \mathcal{N}(b_{i-1}a_ib_i)$$
$$f_g^2(a_i) \subset \mathcal{N}(b_{i-2}a_{i-1}b_{i-1}a_ib_ia_{i+1}b_{i+1})$$
$$f_g^3(a_i) \subset \mathcal{N}(b_{i-3}a_{i-2}\cdots a_{i+2}b_{i+2})$$
$$\vdots$$
$$f_g^j(a_i) \subset \mathcal{N}(b_{i-j}a_{i-j+1}\cdots a_{i+j-1}b_{i+j-1})$$

as long as the inequalities $i - j \geq 1$ and $i + j - 1 \leq \lceil g/2 \rceil - 2$ are satisfied. If we choose $i = \lceil \frac{g}{4} \rceil$ and $j = \lceil \frac{g}{4} \rceil - 3$, then both inequalities are satisfied. In this case, both a_i and $f_g^j(a_i)$ are disjoint from the simple closed curve γ in Figure 1. Hence we have $d_{\mathcal{C}}(a_i, f_g^{\lceil \frac{g}{4} \rceil - 3}(a_i)) \leq 2$, and this completes the proof.

$\square$

Finally, We conclude that Theorem 1.1 follows from Theorem 3.2 and Theorem 4.1.

5. Lower bound for pure braid group

Now we consider the pure braid group PB_n as a subgroup of $\mathrm{Mod}(D_n) \simeq B_n$. As we mentioned in the introduction, we cannot use the Lemma 3.1 for PB_n since it doesn't satisfy the criterion of the lemma (in particular, please see the definition of Lefschetz number for a punctured surface in [Tsa09]). So to obtain the lower bound for Theorem 1.2, we need to use another argument to show that there is a uniform constant q, independent of n, such that $M_{\mathcal{R}}^q$ is a positive matrix, where $M_{\mathcal{R}}$ is the transition matrix for real branches of the invariant train track.

Theorem 5.1. *Let f be a pseudo-Anosov element in* PB_n. *Then we have*

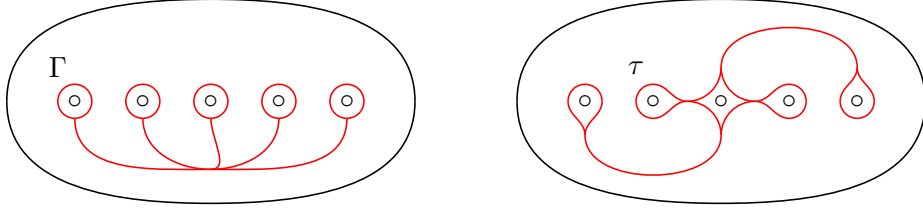$$\ell_{\mathcal{C}}(f) > \frac{1}{158n - 168}.$$

FIGURE 2. An embedded graph in D_n and a train track by Bestvina–Handel algorithm
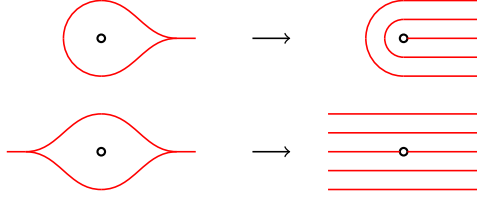

FIGURE 3. Monogon, bigon, and their foliations

Proof. When we obtain τ using the algorithm in [BH95], we can start from a graph Γ in D_n as in Figure 2. The train track τ fills the surface D_n, that is, the complement $D_n \setminus \tau$ is the union of (topologically) disks, once-punctured disks, or boundary-parallel annulus. Therefore each puncture is contained in a distinct ideal polygon, including a monogon and a bigon. Note that a puncture in a monogon and a bigon is a 1-pronged singularity and a regular point, respectively, of the invariant foliation of the pseudo-Anosov f (see Figure 3). We will show that the integer q in Proposition 2.2 can be chosen to be a uniform constant, independent of n.

Let k_1 be the number of punctures contained in monogons of τ, and let k_2 be the number of punctures contained in bigons of τ. Also, let $\mathcal{S}$ be the set of singularities of the foliation whose index, i.e., the number of separatrices, is greater than or equal to 3. By the Euler–Poincaré formula (see [FLP79]),

$$k_1 + \sum_{s \in \mathcal{S}} (2 - P_s) = 2\chi(S^2) = 4,$$

where P_s is the index of the singularity $s \in \mathcal{S}$ of the foliation. Since $P_s \geq 3$, we have

$$k_1 = 4 + \sum_{s \in \mathcal{S}} (P_s - 2) \geq 4 + |\mathcal{S}|.$$

On the other hand, since each puncture of D_n either is a 1-pronged singularity, is contained in a bigon, or lies in $\mathcal{S}$, we have that $k_1 + k_2 + |\mathcal{S}| \geq n$. Combining the two inequalities, we obtain

$$k_1 \geq \frac{4 + n - k_2}{2}.$$

Case 1. Suppose $k_2 < \frac{1}{2}n$. Then $k_1 > \frac{4+n-n/2}{2} = \frac{1}{4}n + 2$. Now we show that there is a monogon in τ where at most 23 real branches are attached at the cusp of the monogon. This implies that the integer q in Proposition 2.2 can be chosen to be 23. It is because $f \in \mathrm{PB}_n$ fixes each puncture and hence for each real branch β attached to this monogon, $f(\beta)$ must passes through one of the other attached real branches. If there are at most 23 attached real branches, then $f^{23}(\beta)$ must passes through β.

Suppose on the contrary that each monogon in τ containing a puncture has at least 24 real branches attached. Then there are at least $12k_1 \geq 3n+24$ real branches. However, as we explained in Section 2.1, there are at most $3|\chi(D_n)| = 3n - 3$ real branches, which is a contradiction.

Case 2. Suppose $k_2 \geq \frac{1}{2}n$. The idea is very similar to case 1. Suppose there are at least 12 real branched attached at the cusps of each bigon of τ containing a puncture. Then there are at least $6k_2 \geq 3n$ real branches in τ which is again a contradiction. This implies that there is at least one bigon with at most 11 real branched attached. Hence by the same logic, one can take q to be 11.

In conclusion, for either case 1 or case 2, one can choose $q = 23$. By Proposition 2.2, we have $w \leq 158n - 168$. This finishes the proof.

$\square$

Remark. We don't expect that the lower bounds of Theorem 3.2 and Theorem 5.1 are optimal. We emphasize that the main purpose of this paper is to describe the asymptotic behavior of the asymptotic translation length of pseudo-Anosov elements in $\mathcal{I}_g$ and PB_n.

Another quick remark is about the the Euler–Poincaré formula used in the proof above. In [FLP79], the formula is only stated for compact surfaces. By replacing the puncture with a boundary component, a p-pronged singularity at a puncture is equivalent to p-many 3-pronged singularities on the boundary of type (B) (see section 5.1 of [FLP79] for the definition). Hence the Euler–Poincaré formula for a compact surface with boundary immediately yields the version that we used in this paper.

6. Upper bound of pure braid group

Obtaining the upper bound of Theorem 1.2 is completely analogous to the proof of Theorem 4.1.

Theorem 6.1. *For all $n \geq 4$,*

$$L_{\mathcal{C}}(\mathrm{PB}_n) \leq \frac{2}{n-3}.$$

Proof. It suffices to find a pseudo-Anosov element f_n in PB_n such that $\ell_{\mathcal{C}}(f_n) \leq \dfrac{2}{n-3}$. Let us assume that $n \geq 4$. Label the punctures of D_n with $p_1, \cdots, p_n$, and for each i, consider the simple closed curves a_i bounding two puncture p_i and p_{i+1} shown in Figure 4.
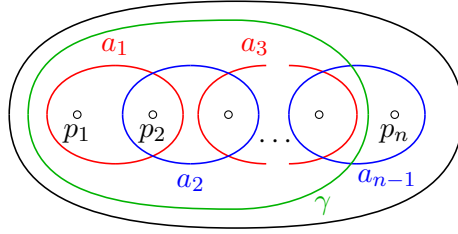
FIGURE 4. Simple closed curves in D_n

We consider a mapping class $f_n \in \mathrm{Mod}(D_n)$ given by

$$f_n = \prod_{i=1}^{n-1} T_{a_i}^{\epsilon_i},$$

where $\epsilon_i = (-1)^{i+1}$. Then f_n is a pseudo-Anosov element since it is arising from Penner's construction. Moreover, $f_n \in \mathrm{PB}_n$ because each curve a_i is a separating curve.

We claim that $d_{\mathcal{C}}(a_1, f_n^{n-1}(a_1)) \leq 2$. We use the same notation as in the proof of Theorem 4.1. Under the iteration of f_n, one can see that

$$f_n(a_1) \subset \mathcal{N}(a_1 a_2)$$
$$f_n^2(a_1) \subset \mathcal{N}(a_1 a_2 a_3)$$
$$\vdots$$
$$f_n^{n-3}(a_1) \subset \mathcal{N}(a_1 a_2 \cdots a_{n-2}).$$

Therefore, we can see that both a_1 and $f_n^{n-3}(a_1)$ are disjoint from the simple closed curve γ in Figure 4. Therefore, we have $d_{\mathcal{C}}(a_1, f_n^{n-3}(a_1)) \leq 2$ and

$$\ell_{\mathcal{C}}(f) \leq \frac{2}{n-3}.$$

$\square$

Finally, Theorem 1.2 follows from Theorem 5.1 and Theorem 6.1.

7. PURE MAPPING CLASS GROUPS

In this section, we briefly discuss the case of the pure mapping class group $\mathrm{PMod}(S_{g,n})$. Theorem 1.3 of [Val14] showed that the minimal translation length of the mapping class group $\mathrm{Mod}(S_{g,n})$ behaves like $1/n$ when g is fixed and $g \geq 2$. The lower bound can be computed using the same method as in Section 3 and to get the negativity of Lefschetz number, the assumption that $g \geq 2$ is required. For the upper bound, Valdivia constructed an appropriate sequence of pseudo-Anosov maps in Section 4 of [Val14]. One can observe that these are in fact contained in $\mathrm{PMod}(S_{g,n})$. Hence, Theorem 1.3 of [Val14] and its proof directly imply Theorem 1.3 for $g \geq 2$.

The case $g = 0$ is our Theorem 1.2. For the rest of this section, we have shown that the computation in Section 5 can be easily adapted to the case of $\mathrm{PMod}(S_{g,n})$ for any fixed $g \geq 0$.

Theorem 7.1. *Let $g \geq 0$ be fixed. For a pseudo-Anosov element $f \in \mathrm{PMod}(S_{g,n})$,*

$$\ell_{\mathcal{C}}(f) \geq \frac{1}{1296g + 638n - 1296},$$

provided that $n > 38g - 38$.

Proof. Let $f \in \mathrm{PMod}(S_{g,n})$ and let τ be an invariant train track of f obtained by Bestvina–Handel algorithm. Recall that we only need to compute q so that $M_{\mathcal{R}}^q$ has a positive diagonal entry. Here we follow the same notations of the proof of Theorem 5.1. The Euler-Poincaré formula says

$$k_1 + \sum_{s \in \mathcal{S}} (2 - P_s) = 2\chi(S_g) = 4 - 4g.$$

Combining with $k_1 + k_2 + |\mathcal{S}| \geq n$, we obtain

$$k_1 \geq \frac{n - k_2 + 4 - 4g}{2}.$$

We divide this into two cases as in Section 5.

Case 1. Suppose $k_2 < \frac{1}{2}n$. Then $k_1 > \frac{1}{4}n + 2 - 2g$. Now we show that there is a monogon in τ where at most 31 real branches are attached at the cusp of the monogon. As before, this implies that the integer q in Proposition 2.2 can be chosen to be 31.

Suppose on the contrary that each monogon in τ containing a puncture has at least 32 real branches attached. Then there are at least $16k_1 \geq 4n + 32 - 32g$ real branches. For all $n > 38g - 38$, we have $4n + 32 - 32g > 3n + (38g - 38) - (32g - 32) = 3n + 6g - 6$. This is a contradiction because the number of real branches is at most $3|\chi(S_{g,n})| = 3n + 6g - 6$.

Case 2. Suppose $k_2 \geq \frac{1}{2}n$.

Suppose further that each of k_2 bigons has at least 16 real branches attached. Then there are at least $8k_2 \geq 4n$ real branches in τ, and $4n > 3n + 6g - 6$ for all $n > 6g - 6$. This implies that there is at least one bigon with at most 15 real branched attached. Hence by the same logic, one can take q to be 15.

In conclusion, for either case 1 or case 2, one can choose $q = 31$. By Proposition 2.2, we have $w \leq 1296g + 638n - 1296$.   □

When $g > 1$, of course one gets a better lower bound just using the Lefschetz number argument as in [Val14], but the above proof can be adopted for all $g \geq 0$.

References

[AT15]   T. Aougab and S. J. Taylor. Pseudo-Anosovs optimizing the ratio of Te-
           ichmüller to curve graph translation length. *ArXiv e-prints*, October 2015.
[BB05]   Joan S. Birman and Tara E. Brendle. Braids: a survey. In *Handbook of knot
           theory*, pages 19–103. Elsevier B. V., Amsterdam, 2005.
[BH95]   M. Bestvina and M. Handel. Train-tracks for surface homeomorphisms. *Topol-
           ogy*, 34(1):109–140, 1995.
[Bow08]   Brian H. Bowditch. Tight geodesics in the curve complex. *Invent. Math.*,
           171(2):281–300, 2008.
[BSW18]   Hyungryul Baik, Hyunshik Shin, and Chenxi Wu. Upper bound on translation
           length on curve graph and fibered face. *ArXiv e-prints*, 2018.
[BT82]   Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*, vol-
           ume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin,
           1982.
[FLM08]   Benson Farb, Christopher J. Leininger, and Dan Margalit. The lower central
           series and pseudo-Anosov dilatations. *Amer. J. Math.*, 130(3):799–827, 2008.
[FLP79]   A. Fathi, F. Laudenbach, and V. Poenaru. *Travaux de Thurston sur les sur-
           faces*, volume 66 of *Astérisque*. Société Mathématique de France, Paris, 1979.
           Séminaire Orsay, With an English summary.
[FM12]   Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49
           of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ,
           2012.
[GHKL13]   V. Gadre, E. Hironaka, R. P. Kent, IV, and C. J. Leininger. Lipschitz constants
           to curve complexes. *Math. Res. Lett.*, 20(4):647–656, 2013.
[GP74]   Victor Guillemin and Alan Pollack. *Differential topology*. Prentice-Hall, Inc.,
           Englewood Cliffs, N.J., 1974.
[GT11]   Vaibhav Gadre and Chia-Yen Tsai. Minimal pseudo-Anosov translation lengths
           on the complex of curves. *Geom. Topol.*, 15(3):1297–1312, 2011.
[KS17]   E. Kin and H. Shin. Small asymptotic translation lengths of pseudo-Anosov
           maps on the curve complex. *ArXiv e-prints*, July 2017.
[MM99]   Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves. I.
           Hyperbolicity. *Invent. Math.*, 138(1):103–149, 1999.
[Pen88]   Robert C. Penner. A construction of pseudo-Anosov homeomorphisms. *Trans.
           Amer. Math. Soc.*, 310(1):179–197, 1988.
[PH92]   R. C. Penner and J. L. Harer. *Combinatorics of train tracks*, volume 125 of
           Annals of Mathematics Studies. Princeton University Press, Princeton, NJ,
           1992.
[Tsa09]   Chia-Yen Tsai. The asymptotic behavior of least pseudo-Anosov dilatations.
           Geom. Topol., 13(4):2253–2278, 2009.
[Val14]   Aaron D. Valdivia. Asymptotic translation length in the curve complex. *New
           York J. Math.*, 20:989–999, 2014.
[Val17]   Aaron D. Valdivia. Lipschitz constants to curve complexes for punctured sur-
           faces. *Topology Appl.*, 216:137–145, 2017.

Department of Mathematical Sciences, KAIST, 291 Daehak-ro Yuseong-gu,
Daejeon, 34141, South Korea
  E-mail address: hrbaik@kaist.ac.kr

Department of Mathematical Sciences, KAIST, 291 Daehak-ro Yuseong-gu,
Daejeon, 34141, South Korea
  E-mail address: hshin@kaist.ac.kr