Geometry & Topology XX (20XX) 1001–999 **1001**

Pseudo-Anosov mapping classes not arising from Penner's construction

HYUNSHIK SHIN
BALÁZS STRENNER

We show that Galois conjugates of stretch factors of pseudo-Anosov mapping classes arising from Penner's construction lie off the unit circle. As a consequence, we show that for all but a few exceptional surfaces, there are examples of pseudo-Anosov mapping classes so that no power of them arises from Penner's construction. This resolves a conjecture of Penner.

37E30; 57M99, 15A18, 11R32

1 Introduction

Let $S_{g,n}$ be the orientable surface of genus g with n punctures. The mapping class group $\mathrm{Mod}(S_{g,n})$ is the group of isotopy classes of orientation-preserving homeomorphisms of $S_{g,n}$. Thurston's classification theorem [Thu88] states that each element of $\mathrm{Mod}(S_{g,n})$ is either periodic, reducible, or pseudo-Anosov. An element $f \in \mathrm{Mod}(S_{g,n})$ is pseudo-Anosov if there is a representative homeomorphism ψ, a number $\lambda > 1$, and a pair of transverse invariant singular measured foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ such that

$$\psi(\mathcal{F}^u) = \lambda \mathcal{F}^u \quad \text{and} \quad \psi(\mathcal{F}^s) = \lambda^{-1} \mathcal{F}^s.$$

The number λ is called the stretch factor (or dilatation) of f.

Isotopy classes of orientation-preserving Anosov maps of the torus can easily be classified as actions of matrices $M \in \mathrm{SL}(2,\mathbb{Z})$ with $|\mathrm{tr}(M)| > 2$ on $\mathbb{R}^2/\mathbb{Z}^2$. However, it is much harder to give explicit examples of pseudo-Anosov maps on more complicated surfaces.

Thurston gave the first general construction of pseudo-Anosov mapping classes in terms of Dehn twists [Thu88]. After Thurston's work, various other constructions have been developed [AY81, Kra81, Lon85, Pen88, CB88, BH92]. In this paper, we study Penner's construction [Pen88].

Published: XX Xxxember 20XX DOI: 10.2140/gt.20XX.XX.1001

Penner's Construction *Let $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_m\}$ be a pair of multicurves on a surface S. Suppose that A and B are filling, that is, A and B are in minimal position and the complement of $A \cup B$ is a union of disks and once punctured disks. Then any product of positive Dehn twists about a_j and negative Dehn twists about b_k is pseudo-Anosov provided that all $n+m$ Dehn twists appear in the product at least once.*

Penner [Pen91] used this construction to give examples of pseudo-Anosov mapping classes with small stretch factors. (See also [Bau92] and [Lei04] for more work on small stretch factors arising from Penner's construction.)

Pseudo-Anosov maps arising from Penner's construction fix the singularities and separatrices of their invariant foliations, and therefore not all pseudo-Anosov mapping classes arise from Penner's construction. However, since the construction is fairly general, Penner conjectured the following.

Conjecture (Penner, 1988) *Every pseudo-Anosov mapping class has a power that arises from Penner's construction.*

The conjecture is listed as Problem 4 in Chapter 7 of [Far06] and also discussed briefly in Section 14.1.2 of [FM12].

It is a folklore theorem that Penner's construction is true for $S_{1,0}$ and $S_{1,1}$ and that it is false for $S_{0,4}$, but a modified version of the conjecture, allowing half-twists in addition to Dehn twists in Penner's construction, is true. To the best of our knowledge, no proof of this has appeared in the literature. In the appendix, we give a proof by considering the action of the mapping class group on the curve complex. The main result of this paper is the answer to Penner's conjecture in the remaining nontrivial cases.

We call a pseudo-Anosov mapping class and its stretch factor λ *coronal* if λ has a Galois conjugate on the unit circle.

Main Theorem *A coronal pseudo-Anosov mapping class has no power coming from Penner's construction. Moreover, there exists a coronal pseudo-Anosov mapping class on $S_{g,n}$ when $3g+n \geq 5$. In particular, Penner's conjecture is false for $S_{g,n}$ when $3g+n \geq 5$.*

We remark that even the modified version of the conjecture, allowing half-twists in addition to Dehn twists in Penner's construction, is false for $S_{0,n}$ when $n \geq 5$.

The proof of the first part of the Main Theorem is based on the fact that stretch factors of pseudo-Anosov mapping classes arising from Penner's construction appear as Perron–Frobenius eigenvalues of products of certain integral matrices, which depend only on the intersection numbers of curves. We show that such matrix products may not have eigenvalues on the unit circle other than 1, which implies that pseudo-Anosov stretch factors arising from Penner's construction are not coronal.

The key idea is that an eigenvalue on the unit circle corresponds to a rotation on an invariant plane, which we consider a dynamical system. Our topological setting provides a natural quadratic form h which, considered as a height function, plays a role similar to that of Lyapunov functions in stability theory. We show that the products of matrices arising from Penner's construction act by increasing the height, which prohibits rotations on subspaces.

To prove the second part of the Main Theorem, we use known coronal pseudo-Anosov mapping classes on $S_{2,0}$ and $S_{0,5}$ to construct coronal pseudo-Anosov mapping classes on the rest of the surfaces via introducing punctures and taking branched covers.

The Main Theorem provides a number-theoretical obstruction for pseudo-Anosov maps to arise from Penner's construction: if the stretch factor of f has a Galois conjugate on the unit circle, then no power of f can arise from Penner's construction. We do not know whether there are other obstructions.

Question 1.1 *Let f be a pseudo-Anosov mapping class whose stretch factor does not have Galois conjugates on the unit circle. Does f^n arise from Penner's construction for some $n \in \mathbb{N}$?*

Acknowledgements The authors are grateful to Richard Kent and Dan Margalit for numerous helpful conversations and invaluable comments. We also thank Richard Kent for suggesting the term *coronal* and the referee for many helpful comments.

2 Proof of the Main Theorem

2.1 Stretch factors arising from Penner's construction

Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$ be a pair of multicurves on a surface. Introduce the notation

$$(e_1,\ \ldots\ ,\ e_{n+m}) = (a_1,\ \ldots,\ a_n,\ b_1,\ \ldots,\ b_m).$$

The *intersection matrix* of A and B is the symmetric $(n+m)\times(n+m)$ nonnegative integral matrix $\Omega = \Omega(A,B)$ whose (j,k)-entry is the geometric intersection number $i(e_j, e_k)$.

The monoid $\Gamma(\Omega)$ Penner showed that actions of the Dehn twists T_{a_j} and $T_{b_k}^{-1}$ on $A \cup B$ can be described by the matrices

$$Q_i = I + D_i\Omega \quad (1 \le i \le n+m),$$

where I is the $(n+m)\times(n+m)$ identity matrix, and D_i denotes the $(n+m)\times(n+m)$ matrix whose ith entry on the diagonal is 1 and whose other entries are zero. Any product of T_{a_j} and $T_{b_k}^{-1}$, where each a_j and each b_k appear at least once, is pseudo-Anosov, and its stretch factor is given by the Perron–Frobenius eigenvalue of the corresponding product of the matrices Q_i. Therefore one can study pseudo-Anosov stretch factors arising from Penner's construction by studying the monoid

$$\Gamma(\Omega) = \langle Q_i : 1 \le i \le n+m\rangle,$$

generated by the matrices Q_i depending on Ω. For more details, see [Pen88].

The height function h Define the quadratic form $h\colon\ \mathbb{R}^{n+m} \to \mathbb{R}$ by the equation

$$h(\mathbf{v}) = \frac{1}{2}\mathbf{v}^T\Omega\mathbf{v}.$$

Geometrically, the vector $\mathbf{v}$ corresponds to assigning a real number to each curve in A and B. The function h is the sum of the products of the values of intersecting curves over all intersection points.

The multicurves A and B define two transverse cylinder decompositions of the surface. When $\mathbf{v} > 0$, the values assigned to the curves can be thought of as the widths of the cylinders. This way we get a singular flat metric on the surface with a rectangle corresponding to each intersection, and the area of this flat surface is $h(\mathbf{v})$. When $\mathbf{v}$ is not positive, one can still think of $h(\mathbf{v})$ as a signed area. However, it is not clear how this geometric interpretation explains the following interaction between the function $h(\mathbf{v})$ and the matrices Q_i.

Proposition 2.1 $h(Q_i\mathbf{v}) - h(\mathbf{v}) = ||Q_i\mathbf{v} - \mathbf{v}||^2$.

Proof Since all entries on the diagonal of Ω are zero, we have $D_i\Omega D_i = 0$ for all i, and hence we have

$$\frac{1}{2}Q_i^T\Omega Q_i - \frac{1}{2}\Omega = \frac{1}{2}(I+\Omega D_i)\Omega(I+D_i\Omega) - \frac{1}{2}\Omega = \Omega D_i\Omega.$$

It follows that $h(Q_i\mathbf{v}) - h(\mathbf{v}) = ||D_i\Omega\mathbf{v}||^2 = ||Q_i\mathbf{v} - \mathbf{v}||^2$. □

Corollary 2.2 *If $M \in \Gamma(\Omega)$, then $h(M\mathbf{v}) \geq h(\mathbf{v})$ with equality if and only if $M\mathbf{v} = \mathbf{v}$.*

Proposition 2.3 *If $M \in \Gamma(\Omega)$, then M cannot have eigenvalues on the unit circle except 1.*

Proof Assume for contradiction that M has an eigenvalue $\mu \neq 1$ on the unit circle. Then there exists $\mathbf{v} \in \mathbb{R}^{n+m}$ and a sequence $p_i \to \infty$ of positive integer powers such that $M\mathbf{v} \neq \mathbf{v}$ and $M^{p_i}\mathbf{v} \to \mathbf{v}$. (If $\mu \neq -1$, choose $\mathbf{v}$ to be any nonzero vector in the two-dimensional invariant subspace on which M acts by a rotation. If $\mu = -1$, choose $\mathbf{v}$ to be a corresponding eigenvector.) Therefore we have $h(M\mathbf{v}) > h(\mathbf{v})$, and hence $h(M^{p_i}\mathbf{v}) \geq h(M(\mathbf{v}))$ for all p_i by Corollary 2.2. However, we have $h(M^{p_i}\mathbf{v}) \to h(\mathbf{v})$ by continuity, which is a contradiction. □

2.2 Coronal pseudo-Anosov mapping classes

Recall that a pseudo-Anosov mapping class and its stretch factor λ are *coronal* if λ has a Galois conjugate on the unit circle.

Lemma 2.4 *If a pseudo-Anosov mapping class f is coronal, then each power of f is also coronal.*

Proof Let λ be the stretch factor of f. Let σ be an automorphism of the Galois extension $L/\mathbb{Q}$ with $|\sigma(\lambda)| = 1$, where L is the splitting field of the minimal polynomial of λ. For all $k \geq 1$, we have $|\sigma(\lambda^k)| = |\sigma(\lambda)^k| = 1$. Therefore λ^k, the stretch factor of f^k, has a Galois conjugate $\sigma(\lambda^k)$ on the unit circle. □

As a consequence of Proposition 2.3 and Lemma 2.4, we have the following.

Corollary 2.5 (First part of the Main Theorem) *A coronal pseudo-Anosov mapping class has no power coming from Penner's construction.*

To complete the proof of the Main Theorem, we need to show that coronal pseudo-Anosov mapping classes exist on all but a few exceptional surfaces.

Lemma 2.6 *If there exists a coronal pseudo-Anosov mapping class on a surface S, and there is a branched covering $\widetilde{S} \to S$, then there exists a coronal pseudo-Anosov mapping class on $\widetilde{S}$ as well.*

Proof If $f \in \mathrm{Mod}(S)$ is a coronal pseudo-Anosov mapping class, then some power of f can be lifted to a pseudo-Anosov mapping class $\widetilde{f}$ on $\widetilde{S}$ with the same stretch factor as the power of f (see [FLP79, Exposé 13, II.1.]). By Lemma 2.4, $\widetilde{f}$ is also coronal. $\square$

Lemma 2.7 *If there exists a coronal pseudo-Anosov mapping class on a surface S, then there exists a coronal pseudo-Anosov mapping class on the surface $S' = S \setminus \{p\}$ with one more puncture as well.*

Proof Let $f \in \mathrm{Mod}(S)$ be a coronal pseudo-Anosov mapping class with stretch factor λ and let ψ be its representative homeomorphism. Some power ψ^k has a fixed point p (see [FLP79, Proposition 9.20] or [FM12, Theorem 14.19]), and hence ψ^k induces a pseudo-Anosov homeomorphism of $S \setminus \{p\}$ with coronal stretch factor λ^k. $\square$

Proposition 2.8 (Second part of the Main Theorem) *There exists a coronal pseudo-Anosov mapping class on $S_{g,n}$ when $3g+n \geq 5$.*

Proof On $S_{2,0}$ there is a coronal pseudo-Anosov mapping class with stretch factor the Perron root of the polynomial $x^4 - x^3 - x^2 - x + 1$ [Zhi95]. For each $g \geq 3$ there is an unbranched covering of $S_{2,0}$ by $S_{g,0}$. It follows from Lemma 2.6 and Lemma 2.7 that there exists a coronal pseudo-Anosov mapping class on all $S_{g,n}$ with $g \geq 2$ and $n \geq 0$.

For the genus 0 cases, start from a coronal pseudo-Anosov mapping class on $S_{0,5}$ with stretch factor the Perron root of $x^4 - 2x^3 - 2x + 1$ [LT11a]. By Lemma 2.7, there exists a coronal pseudo-Anosov mapping class on $S_{0,n}$ for each $n \geq 5$.

Finally, there is a branched covering $S_{1,2} \to S_{0,5}$, induced by the hyperelliptic involution of $S_{1,2}$ exchanging the two punctures, which yields a coronal pseudo-Anosov mapping class on $S_{1,2}$ by Lemma 2.6. (Technically, here $S_{1,2}$ and $S_{0,5}$ should be considered surfaces with marked points, not punctures. Because the theory of pseudo-Anosov maps and stretch factors is the same on surfaces with punctures and on surfaces with marked points, we can go back and forth between marked points and punctures as is convenient.) By Lemma 2.7, there exists a coronal pseudo-Anosov mapping class on $S_{1,n}$ for $n \geq 2$. $\square$

The Main Theorem immediately follows from Corollary 2.5 and Proposition 2.8.

3 Remarks on the Galois conjugates of stretch factors

Examples of coronal pseudo-Anosov mapping classes The set of coronal pseudo-Anosov mapping classes is presumably much larger than the set of examples constructed above. For example, the minimal pseudo-Anosov stretch factors tend to be coronal. In fact, when $g = 2, 3, 4, 5, 7, 8$, the minimal stretch factor on S_g among pseudo-Anosov mapping classes with orientable foliations are known, and they are all coronal [LT11b]. The minimal pseudo-Anosov stretch factors on the surfaces $S_{0,n}$ for $5 \leq n \leq 9$ are also all coronal with the exception of $n = 8$ [LT11a].

Not only is the set of coronal pseudo-Anosov mapping classes infinite, but so is the set of coronal stretch factors (even modulo taking powers). This follows from the first author's examples of pseudo-Anosov mapping classes on S_g with stretch factor a degree $2g$ Salem number [Shi]. Hironaka's infinite family of pseudo-Anosov mapping classes coming from the fibration of a single 3–manifold [Hir10] also seem to consist mostly of coronal pseudo-Anosov mapping classes whose stretch factors can have arbitrarily high algebraic degree.

The abundance of coronal pseudo-Anosov mapping classes are also suggested by computer experiments of Nathan Dunfield and Giulio Tiozzo on random walks in the group of braids with 10 and 14 strands. Using the standard Artin generators, mean length 25, variance 9, and a sample of 100,000 pseudo-Anosov mapping classes, 94% of the stretch factors had Galois conjugates on the unit circle. Computer experiments also show that a random reciprocal polynomial is very likely to have a root on the unit circle. This may suggest that pseudo-Anosov mapping classes arising from Penner's construction are actually rare.

Location of Galois conjugates It would be interesting to know precise constraints on the location of Galois conjugates of pseudo-Anosov stretch factors arising from Penner's construction. In particular, we wonder if they can at least approach the unit circle or if they are even dense in $\mathbb{C}$. A positive answer would imply that Galois conjugates of all pseudo-Anosov stretch factors are dense in $\mathbb{C}$, which is also suggested by the experiments of Dunfield and Tiozzo.

A Penner's conjecture for the exceptional surfaces

In this appendix, we show that Penner's construction is true for $S_{1,0}$ and $S_{1,1}$ and that it is false for $S_{0,4}$, but a modified version of the conjecture, allowing half-twists in addition to Dehn twists in Penner's construction, is true.

The curve complex Let S be one of these three surfaces. The modified curve complex $\mathcal{C}(S)$ is a graph with vertices the isotopy classes of simple closed curves on S, where two vertices are connected by an edge if they have minimal intersection number (one for $S_{1,0}$ and $S_{1,1}$, and two for $S_{0,4}$). In all three cases, $\mathcal{C}(S)$ is isomorphic to the 1-skeleton of the Farey tessellation $\mathcal{F}$ of the hyperbolic plane (Figure 1). For more details, see [FM12, Section 4.1.1].

The action of $\mathrm{Mod}(S)$ Let us consider the action of $\mathrm{Mod}(S)$ on $\mathcal{C}(S)$, which gives rise to a homomorphism

$$A\colon \mathrm{Mod}(S) \to \mathrm{Isom}^+(\mathcal{F}) \cong \mathrm{PSL}(2, \mathbb{Z})$$

of $\mathrm{Mod}(S)$ to the orientation-preserving isometries of $\mathcal{F}$, once an identification of $\mathcal{C}(S)$ with $\mathcal{F}$ is chosen. We denote the image of an element $f \in \mathrm{Mod}(S)$ by A_f.

Actions of Dehn twists We call an ideal triangle in the complement of $\mathcal{F}$ a *tile*. A rotation of $\mathcal{F}$ about a vertex v of $\mathcal{F}$ to the left by k tiles is defined as the parabolic element of $\mathrm{Isom}^+(\mathcal{F})$ that fixes v and shifts the tiles adjacent to v in counterclockwise direction by k. Rotations to the right are defined analogously.

For a Dehn twist T_c about a curve c in S, the isometry A_{T_c} is parabolic, and it fixes the vertex of $\mathcal{F}$ corresponding to c. Depending on the choice of identification of $\mathcal{C}(S)$ with $\mathcal{F}$, positive Dehn twists can act by rotating $\mathcal{F}$ to the left or to the right. We choose the identification so that positive Dehn twists correspond to rotations to the right and negative Dehn twists correspond to rotations to the left. Note that Dehn twists on $S_{1,0}$ and $S_{1,1}$ act by rotations by one tile, but Dehn twists on $S_{0,4}$ act by rotations by two tiles. It is the half-twists on $S_{0,4}$ that correspond to rotations by one tile.

Actions of pseudo-Anosov elements For a pseudo-Anosov element $f \in \mathrm{Mod}(S)$, the isometry A_f is hyperbolic in $\mathrm{PSL}(2, \mathbb{Z})$ and hence A_f has an invariant geodesic on the hyperbolic plane, called the *axis* γ of A_f. Since f does not fix any curve on S, A_f does not fix any vertex of $\mathcal{F}$. In particular, the endpoints of the axis γ of A_f are not vertices of $\mathcal{F}$. Therefore γ traverses a bi-infinite sequence of triangles in the Farey tessellation, and it cuts two sides of each triangle.

Associated to f, there is a bi-infinite sequence of letters L and R obtained as follows: travel along γ in the direction of the translation, and for each triangle record if the

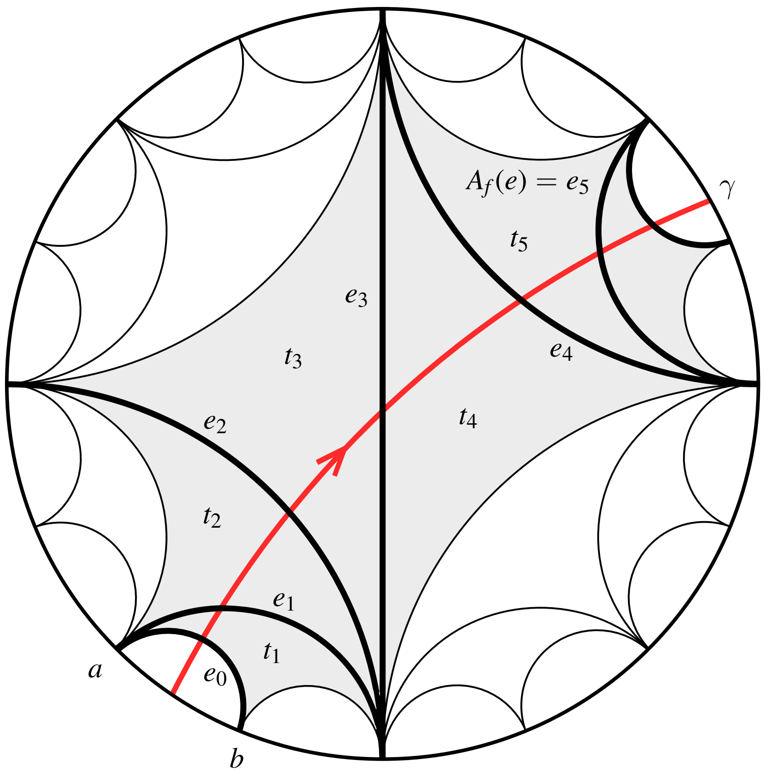

Figure 1: The action A_f of a pseudo-Anosov mapping class f on the Farey tessellation.

common vertex of the cut sides are on the left or the right side of γ. This sequence is periodic, because A_f is a translation along γ. As the following lemma shows, this bi-infinite sequence encodes how the hyperbolic isometry A_f can be written as a composition of parabolic isometries.

Lemma A.1 *Let e_0 be an edge of $\mathcal{F}$ intersecting γ. Let a and b be the endpoints of e_0 on the left and right hand side of γ, respectively. Let $e_1, e_2, \ldots, e_n = A_f(e_0)$ be edges of $\mathcal{F}$ intersected by γ such that e_{k-1} and e_k are different sides of an ideal triangle t_k of $\mathcal{F}$ for all $1 \le k \le n$. (See Figure 1 for an illustration when $n = 5$.) For all $1 \le k \le n$, define s_k to be the letter L or the letter R depending on whether the common vertex of e_{k-1} and e_k is on the left or right side of γ.*

Let τ_a and τ_b be the rotations of $\mathcal{F}$ by one tile to the right about the points a and b, respectively, and introduce the notation

$$\tau(s) = \begin{cases} \tau_a^{-1} & \text{if } s = L \\ \tau_b & \text{if } s = R. \end{cases}$$

Then

$$A_f = \tau(s_1) \circ \cdots \circ \tau(s_n).$$

(By the usual convention for composition of functions, the rotations are applied in right-to-left order.)

Proof For all $1 \le k \le n$, there is a unique $\phi_k \in \mathrm{Isom}^+(\mathcal{F})$ that maps e_0 to e_k and a to the endpoint of e_k lying on the left hand side of γ. We have $\phi_n = A_f$, so we need to prove that $\phi_n = \tau(s_1) \circ \cdots \circ \tau(s_n)$. We will prove by induction that $\phi_k = \tau(s_1) \circ \cdots \circ \tau(s_k)$ for all $1 \le k \le n$.

For $k = 1$, we can easily see that the isometry mapping e_0 to e_1 is τ_a^{-1} or τ_b, depending on whether the common vertex of e_0 and e_1 is a or b.

Now assume that the claim is true for k where $1 \le k < n$, that is, $\phi_k = \tau(s_1) \circ \cdots \circ \tau(s_k)$. We want to show that $\phi_{k+1} = \phi_k \circ \tau(s_{k+1})$. Note that the edges e_0 and $\tau(s_{k+1})(e_0)$ of t_1 meet on the same side of γ as the edges e_k and e_{k+1} of t_{k+1}. Since we have $\tau(s_k)(e_0) = e_k$ by the induction hypothesis, this implies

$$\phi_k(\tau(s_{k+1})(e_0)) = e_{k+1}.$$

The right hand side can also be written as $\phi_{k+1}(e_0)$, therefore $\phi_k \circ \tau(s_{k+1})$ and ϕ_{k+1} map e_0 to the same edge, and their actions on the endpoints also agree. Hence we have $\phi_{k+1} = \phi_k \circ \tau(s_{k+1})$ as claimed. □

Proof of Penner's conjecture for $S_{1,0}$, $S_{1,1}$ and $S_{0,4}$ Let S be one of these three surfaces and let f be any pseudo-Anosov element of $\mathrm{Mod}(S)$. We want to show that f has a power arising from Penner's construction.

Let A_f, γ, e_0, a and b be as above. Choose a and b for the role of the filling curves in the construction. By Lemma A.1, some product of T_a^{-1} and T_b defines an element $h \in \mathrm{Mod}(S)$ such that $A_f = A_h$. Since A_h is hyperbolic, both Dehn twists must appear in this product. Therefore h is a pseudo-Anosov mapping class arising from Penner's construction.

When S is the torus or the once-punctured torus, we have $\mathrm{Mod}(S) \cong \mathrm{SL}(2,\mathbb{Z})$. So $f = \pm h$ and hence $f^2 = h^2$. When $S = S_{0,4}$, then A is surjective with kernel $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ [FM12, Prop. 2.7]. The kernel is generated by two hyperelliptic involutions and only its identity element fixes all four punctures. Thus two elements of $\mathrm{Mod}(S_{0,4})$ that project to the same element of $\mathrm{PSL}(2,\mathbb{Z})$ are equal if they permute the four punctures in the same way. Therefore $f^{12} = h^{12}$, because both maps act trivially on the punctures. (The number 12 is the least common multiple of the orders of elements of the symmetric group on 4 points.) Hence f has a power arising from Penner's construction.

Remark Note that without allowing half-twists, the conjecture is false for $S_{0,4}$. Indeed, an *LR*-sequence corresponding to a product of Dehn twists is a sequence of *LL* and *RR* blocks. So any pseudo-Anosov mapping class that contains the block *LRL* in its sequence does not have a power arising from the construction.

References

[AY81] Pierre Arnoux and Jean-Christophe Yoccoz. Construction de difféomorphismes pseudo-Anosov. *C. R. Acad. Sci. Paris Sér. I Math.*, 292(1):75–78, 1981.

[Bau92] Max Bauer. An upper bound for the least dilatation. *Trans. Amer. Math. Soc.*, 330(1):361–370, 1992.

[BH92] Mladen Bestvina and Michael Handel. Train tracks and automorphisms of free groups. *Ann. of Math. (2)*, 135(1):1–51, 1992.

[CB88] Andrew J. Casson and Steven A. Bleiler. *Automorphisms of surfaces after Nielsen and Thurston*, volume 9 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1988.

[Far06] Benson Farb, editor. *Problems on mapping class groups and related topics*, volume 74 of *Proceedings of Symposia in Pure Mathematics*. American Mathematical Society, Providence, RI, 2006.

[FLP79] *Travaux de Thurston sur les surfaces*, volume 66 of *Astérisque*. Société Mathématique de France, Paris, 1979. Séminaire Orsay, With an English summary.

[FM12] Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.

[Hir10] Eriko Hironaka. Small dilatation mapping classes coming from the simplest hyperbolic braid. *Algebr. Geom. Topol.*, 10(4):2041–2060, 2010.

[Kra81] Irwin Kra. On the Nielsen-Thurston-Bers type of some self-maps of Riemann surfaces. *Acta Math.*, 146(3-4):231–270, 1981.

[Lei04] Christopher J. Leininger. On groups generated by two positive multi-twists: Teichmüller curves and Lehmer's number. *Geom. Topol.*, 8:1301–1359 (electronic), 2004.

[Lon85] Darren D. Long. Constructing pseudo-Anosov maps. In *Knot theory and manifolds (Vancouver, B.C., 1983)*, volume 1144 of *Lecture Notes in Math.*, pages 108–114. Springer, Berlin, 1985.

[LT11a] Erwan Lanneau and Jean-Luc Thiffeault. On the minimum dilatation of braids on punctured discs. *Geom. Dedicata*, 152:165–182, 2011. Supplementary material available online.

[LT11b] Erwan Lanneau and Jean-Luc Thiffeault. On the minimum dilatation of pseudo-Anosov homeromorphisms on surfaces of small genus. *Ann. Inst. Fourier (Grenoble)*, 61(1):105–144, 2011.

[Pen88] Robert C. Penner. A construction of pseudo-Anosov homeomorphisms. *Trans. Amer. Math. Soc.*, 310(1):179–197, 1988.

[Pen91] Robert C. Penner. Bounds on least dilatations. *Proc. Amer. Math. Soc.*, 113(2):443–450, 1991.

[Shi] Hyunshik Shin. Algebraic degrees of stretch factors in mapping class groups. Submitted.

[Thu88] William P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc. (N.S.)*, 19(2):417–431, 1988.

[Zhi95] A. Yu. Zhirov. On the minimum dilation of pseudo-Anosov diffeomorphisms of a double torus. *Uspekhi Mat. Nauk*, 50(1(301)):197–198, 1995.

Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago
Chicago, IL 60607, USA

Department of Mathematics, University of Wisconsin–Madison
Madison, WI 53705, USA

shin@math.uic.edu, strenner@math.wisc.edu

http://www.math.uic.edu/~shin, http://math.wisc.edu/~strenner