Algebraic degrees of stretch factors in mapping class groups

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We explicitly construct pseudo-Anosov maps on the closed surface of genus g with orientable foliations whose stretch factor λ is a Salem number with algebraic degree $2g$. Using this result, we show that there is a pseudo-Anosov map whose stretch factor has algebraic degree d, for each positive even integer d such that $d \leq g$.

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1 Introduction

Let S_g be a closed surface of genus $g \geq 2$. The *mapping class group* of S_g, denoted $\mathrm{Mod}(S_g)$, is the group of isotopy classes of orientation preserving homeomorphisms of S_g. An element $f \in \mathrm{Mod}(S_g)$ is called a *pseudo-Anosov* mapping class if there are transverse measured foliations $(\mathcal{F}^u, \mu_u)$ and $(\mathcal{F}^s, \mu_s)$, a number $\lambda(f) > 1$, and a representative homeomorphism ϕ such that

$$\phi(\mathcal{F}^u, \mu_u) = (\mathcal{F}^u, \lambda(f)\mu_u) \text{ and } \phi(\mathcal{F}^s, \mu_s) = (\mathcal{F}^s, \lambda(f)^{-1}\mu_s).$$

In other words, ϕ stretches along one foliation by $\lambda(f)$ and the other by $\lambda(f)^{-1}$. The number $\lambda(f)$ is called the *stretch factor* (or *dilatation*) of f.

A pseudo-Anosov mapping class is said to be orientable if its invariant foliations are orientable. Let $\lambda_H(f)$ be the spectral radius of the action of f on $H_1(S_g; \mathbb{R})$. Then

$$\lambda_H(f) \leq \lambda(f),$$

and the equality holds if and only if the invariant foliations for f are orientable (see [5]). The number $\lambda_H(f)$ is called the *homological stretch factor* of f.

Question *Which real numbers can be stretch factors?*

It is a long-standing open question. Fried [4] conjectured that $\lambda > 1$ is a stretch factor if and only if all conjugate roots of λ and $1/\lambda$ are strictly greater than $1/\lambda$ and strictly less than λ in magnitude.

Thurston [12] showed that a stretch factor λ is an algebraic integer whose algebraic degree has an upper bound $6g-6$. More specifically, λ is the largest root in absolute value of a monic palindromic polynomial. Thurston gave a construction of mapping classes of $\mathrm{Mod}(S_g)$ generated by two multitwists and he mentioned that his construction can make a pseudo-Anosov mapping class whose stretch factor has algebraic degree $6g-6$. However, he did not give specific examples.

What happens if we fix the genus g? To simplify the question, we may ask which algebraic degrees are possible on S_g.

Question *What degrees of stretch factors can occur on S_g?*

Very little is known about this question. Using Thurston's construction, it is easy to find quadratic integers as stretch factors. Neuwirth and Patterson [10] found non-quadratic examples, which are algebraic integers of degree 4 and 6 on surfaces of genus 4 and 6, respectively. Using interval exchange maps, Arnoux and Yoccoz [1] gave the first generic construction of pseudo-Anosov maps whose stretch factor has algebraic degree g on S_g for each $g \geq 2$.

Main Theorems

In this paper, we give a generic construction of pseudo-Anosov mapping classes with stretch factor of algebraic degree $2g$.

Let c_i and d_j be simple closed curves on S_g as in Figure 1. For $k \geq 3$, let us define

$$f_{g,k} = T_{A_{g,k}} T_{B_g},$$

where $T_{A_{g,k}} = \left(T_{c_1} T_{c_2} \cdots T_{c_{g-1}}\right)\left(T_{c_g}\right)^k$ and $T_{B_g} = T_{d_1} \cdots T_{d_g}$. Here, T_α is the Dehn twist about α. We will show that $f_{g,k}$ is a pseudo-Anosov mapping class and its stretch factor $\lambda(f_{g,k})$ is a special algebraic integer, called Salem number. A *Salem number* is an algebraic integer $\alpha > 1$ whose Galois conjugates other than α have absolute value less than or equal to 1 and at least one conjugate lies on the unit circle.

Theorem A *For each $g \geq 2$ and $k \geq 3$, $f_{g,k}$ is a pseudo-Anosov mapping class and satisfies the following properties:*

(1) $\lambda(f_{g,k}) = \lambda_H(f_{g,k})$,

(2) $\lambda(f_{g,k})$ *is a Salem number, and*

(3) $\lim_{g\to\infty} \lambda(f_{g,k}) = k-1$.

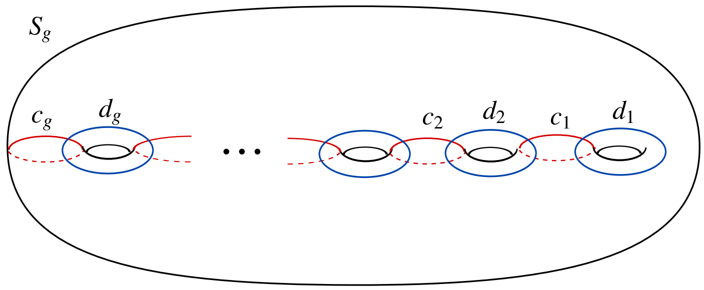

Figure 1: Simple closed curves on S_g

In particular, we will prove that for $k = 4$, the algebraic degree of stretch factor is $2g$. It is known that the degree of the stretch factor of a pseudo-Anosov mapping class $f \in \mathrm{Mod}(S_g)$ with orientable foliations is bounded above by $2g$ (see [12]). Therefore our examples give the maximum degrees of stretch factors for orientable foliations in $\mathrm{Mod}(S_g)$ for each $g \geq 2$.

Theorem B *Let $f_g \in \mathrm{Mod}(S_g)$ be the mapping class given by*

$$f_g = f_{g,4} = T_{A_{g,4}} T_{B_g}.$$

Then the minimal polynomial of the stretch factor $\lambda(f_g)$ is

$$p_g(x) = x^{2g} - 2\left(\sum_{j=1}^{2g-1} x^j\right) + 1.$$

This implies

$$\deg \lambda(f_g) = 2g.$$

The hard part is to show the irreducibility of $p_g(x)$, which is proved in section 7.

In general, for each $k \geq 3$, the Salem stretch factor of $f_{g,k}$ is the root of the polynomial

$$p_{g,k}(x) = x^{2g} - (k-2)\left(\sum_{j=1}^{2g-1} x^j\right) + 1.$$

It can be shown that $p_{g,k}(x)$ is irreducible for each $k \geq 4$, but since the main purpose of this paper is degree realization, we will prove only for $k = 4$ case that the algebraic degree of the stretch factor is $2g$.

Using a branched cover construction, we use Theorem B to deduce the following partial answer to our question about algebraic degrees.

Corollary 5 *For each positive integer $h \leq g/2$, there is a pseudo-Anosov mapping class $\widetilde{f_h} \in \mathrm{Mod}(S_g)$ such that $\deg(\lambda(\widetilde{f_h})) = 2h$ and $\lambda(\widetilde{f_h})$ is a Salem number.*

Obstructions.

There are three known obstructions for the existence of algebraic degrees. For any pseudo-Anosov $f \in \mathrm{Mod}(S_g)$, we have:

(1) $\deg \lambda(f) \geq 2$,

(2) $\deg \lambda(f) \leq 6g-6$, and

(3) if $\deg \lambda(f) > 3g-3$, then $\deg \lambda(f)$ is even.

The third obstruction is due to Long [8] and we have another proof in section 5. It turns out these are the only obstructions for $g = 2$. However it is not known whether there are other obstructions of algebraic degrees for $g \geq 3$. By computer search, odd degree stretch factors are rare compared to even degrees. We conjecture that every even degree $d \leq 6g-6$ can be realized as the algebraic degree of stretch factors.

Conjecture *On S_g, there exists a pseudo-Anosov mapping class with a stretch factor of algebraic degree d for each positive even integer $d \leq 6g-6$.*

In section 6, we show that the conjecture is true for $g = 2, 3, 4,$ and 5.

Outline

In section 2 we will give the basic definitions and results about Thurston's consturction. We will prove Theorem A in section 3 by the theory of Coxeter graphs. In section 4, we construct pseudo-Anosov mapping classes via branched covers. In section 5, we explain some properties of odd degree stretch factors. Section 6 contains examples of even degree stretch factors for $g = 2, 3, 4$ and 5. Section 7 is where we prove Theorem B, that is, we prove that the minimal polynomial of $\lambda(f_g)$ has degree $2g$.

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2 Background

Thurston's construction

We recall Thurston's construction of mapping classes [12]. For more details on this material, see [3] or [6].

Suppose $A = \{a_1, \ldots, a_n\}$ is a set of pairwise disjoint simple closed curves, called a *multicurve*. We denote the product of Dehn twists $\prod_{i=1}^{n} T_{a_i}$ by T_A. This product is called a *multitwist*.

Suppose $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$ are multicurves in a surface S so that $A \cup B$ *fills* S, that is, the complement of $A \cup B$ is a disjoint union of disks and once-punctured disks. Let N be the $n \times m$ matrix whose (j, k)-entry is the geometric intersection number $i(a_j, b_k)$ of a_j and b_k. Let $\nu = \nu(A \cup B)$ be the largest eigenvalue in magnitude of the matrix NN^t. If $A \cup B$ is connected, then NN^t is primitive and by the Perron–Frobenius theorem ν is a positive real number greater than 1 (see [3, p. 392 - 395] for more detail).

Thurston constructed a singular Euclidean structure on S with respect to which $\langle T_A, T_B \rangle$ acts by affine transformations given by the representation $\rho : \langle T_A, T_B \rangle \to \mathrm{PSL}(2, \mathbb{R})$

$$\rho(T_A) = \begin{pmatrix} 1 & -\nu^{1/2} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho(T_B) = \begin{pmatrix} 1 & 0 \\ \nu^{1/2} & 1 \end{pmatrix}.$$

In particular, an element $f \in \langle T_A, T_B \rangle$ is pseudo-Anosov if and only if $\rho(f)$ is a hyperbolic element in $\mathrm{PSL}(2, \mathbb{R})$ and then the stretch factor $\lambda(f)$ is equal to the bigger eigenvalue of $\rho(f)$. For instance, for a mapping class $f = T_A T_B$,

$$\rho(T_A T_B) = \begin{pmatrix} 1 & -\nu^{1/2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \nu^{1/2} & 1 \end{pmatrix} = \begin{pmatrix} 1-\nu & -\nu^{1/2} \\ \nu^{1/2} & 1 \end{pmatrix},$$

and the stretch factor $\lambda(T_A T_B)$ is the bigger root of the characteristic polynomial

$$\lambda^2 - \lambda(\nu - 2) + 1,$$

provided that $\nu - 2 > 2$.

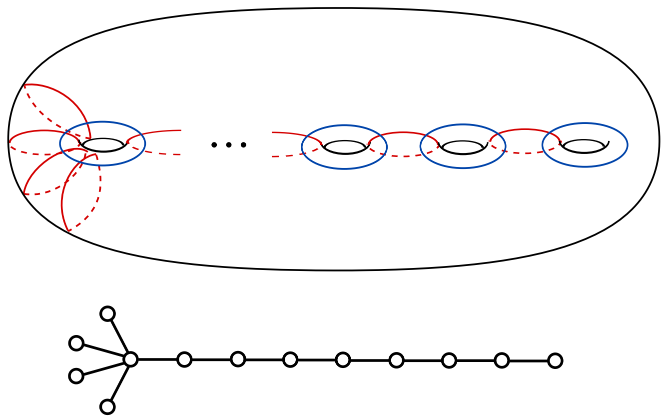

Figure 2: Multicurves and configuration graph $\mathcal{G}(A_{g,k} \cup B_g)$

3 Proof by the theory of Coxeter graphs

We will prove Theorem A in this section.

For the set C of simple closed curves on the surface S_g, the *configuration graph* for C, denoted $\mathcal{G}(C)$, is the graph with a vertex for each simple closed curve and an edge for every point of intersection between simple closed curves.

Let $f_{g,k}$ be a mapping class on S_g defined by

$$f_{g,k} = T_{A_{g,k}} T_{B_g}, \ k \geq 3,$$

as in Theorem A. By regarding the multiple power of T_{c_g} as the product of Dehn twists about parallel (isotopic) simple closed curves $c_{g_1}, \ldots, c_{g_k}$, let us define the multicurves

$$A_{g,k} = \{c_1, \ldots, c_{g-1}, c_{g_1}, \ldots, c_{g_k}\} \ \text{and} \ B_g = \{d_1, \ldots, d_g\}.$$

Then the configuration graph $\mathcal{G}(A_{g,k} \cup B_g)$ is a tree as in Figure 2,

3.1 Coxeter graphs and mapping class groups

We say that a finite graph $\mathcal{G}$ is a *Coxeter graph* if there are no self-loops or multiple edges. For given multicurves A and B such that $A \cup B$ fills the surface S, suppose that the configuration graph $\mathcal{G} = \mathcal{G}(A \cup B)$ is a Coxeter graph. Leininger proved the following theorem.

Theorem 1 ([6] Theorem 8.1 and Theorem 8.4) *Let $\mathcal{G}(A \cup B)$ be a non-critical dominant Coxeter graph. Then $T_A T_B$ is a pseudo-Anosov mapping class with stretch factor λ such that*

$$\lambda^2 + \lambda(2 - \mu^2) + 1 = 0,$$

where μ is the spectral radius of the graph $\mathcal{G}$.

For the definitions and pictures of critical and dominant graphs, see [6, Section 1]

For the multicurves $A_{g,k}$ and B_g in Theorem A, $\mathcal{G}(A_{g,k} \cup B_g)$ is a non-critical dominant Coxeter graph for each $k \geq 3$. Therefore by Theorem 1 the mapping class $f_{g,k} = T_A T_B$ is pseudo-Anosov for each $k \geq 3$.

3.2 Orientability

Suppose that $\mathcal{G}$ is a connected Coxeter graph with the set Σ of vertices. There is an associated quadratic form $\Pi_{\mathcal{G}}$ on R^{Σ} and a faithful representation

$$\Theta : \mathscr{C}(\mathcal{G}) \to \mathrm{O}(\Pi_{\mathcal{G}}),$$

where $\mathscr{C}(\mathcal{G})$ is a Coxeter group with generating set Σ, $\mathrm{O}(\Pi_{\mathcal{G}})$ is the orthogonal group of the quadratic form $\Pi_{\mathcal{G}}$, and each generator $s_i \in \Sigma$ is represented by a reflection. Leininger also proved the following theorem.

Theorem 2 ([6] Theorem 8.2) *Let $\mathcal{G}(A \cup B)$ be a Coxeter graph and suppose that A and B can be oriented so that all intersections of A with B are positive. Then there exists a homomorphism*

$$\eta : \mathbb{R}^{\Sigma} \to H_1(S; \mathbb{R})$$

such that

$$(T_A T_B)_* \circ \eta = -\eta \circ \Theta(\sigma_A \sigma_B),$$

where $\sigma_A \sigma_B$ is an element in $\mathscr{C}(\mathcal{G})$ corresponding to $T_A T_B$.
Moreover, $\Theta(\sigma_A \sigma_B)|_{\ker(\eta)} = -I$ and η preserves spectral radii.

Theorem 2 implies that if A and B can be oriented as in the theorem, then the stretch factor of a pseudo-Anosov mapping class is equal to the spectral radius of the action on homology. For multicurves $A_{g,k}$ and B_g in Theorem A, they can be oriented so that all intersections are positive as in Figure 3. Therefore we have

$$\lambda(f_{g,k}) = \lambda_H(f_{g,k})$$

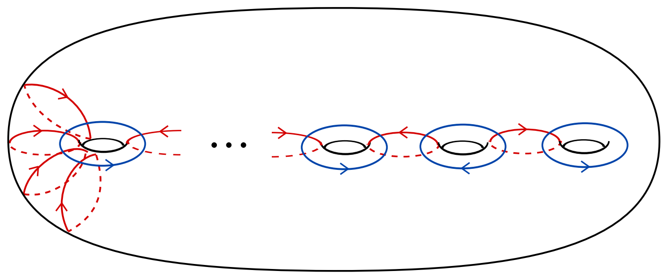

Figure 3: Orientation of positive intersections

and the invariant foliations for $f_{g,k}$ are orientable.

It is also possible to directly compute the action on the first homology. Consider the mapping class $f_g = T_{A_{g,4}} T_{B_g}$ as in Theorem B. Let us choose a basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for $H_1(S_g)$ as in Figure 4.

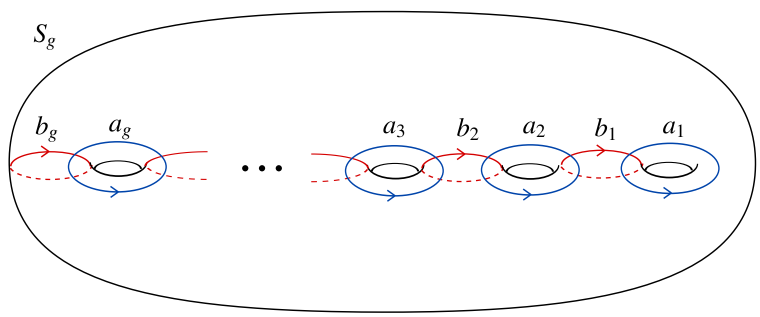

Figure 4: A basis for $H_1(S_g)$.

By computing images of each basis element under f_g, we can get the action on $H_1(S_g)$

$$\begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & -1 \\ 4 & 0 & 0 & 0 & \cdots & -3 \end{pmatrix}.$$

By induction, the characteristic polynomial $h_g(x)$ of the homological action is

$$h_g(x) = x^{2g} + 2\left(\sum_{j=1}^{2g-1} (-1)^j x^j\right) + 1.$$

Since the largest root of $h_g(x)$ in magnitude is a negative real number, we can deduce that the stretch factor $\lambda(f_g)$ is the root of $h_g(-x)$. Specifically, $\lambda(f_g)$ is the root of

$$p_g(x) = x^{2g} - 2\left(\sum_{j=1}^{2g-1} x^j\right) + 1.$$

In a similar way, one can get the polynomial for $\lambda(f_{g,k})$, which is

$$p_{g,k}(x) = x^{2g} - (k-2)\left(\sum_{j=1}^{2g-1} x^j\right) + 1.$$

3.3 Salem numbers and spectral properties of starlike trees

The configuration graph $\mathcal{G}(A_{g,k} \cup B_g)$ for $f_{g,k}$ is a special type of graphs, called a starlike tree, and its relation to Salem numbers is studied in [9]. A *starlike tree* is a tree with at most one vertex of degree > 2. Let $T = T(n_1, n_2, \dots, n_k)$ be the starlike tree with k arms of $n_1, n_2, \dots, n_k$ edges.

Theorem 3 ([9] Corollary 9) *Let $T = T(n_1, n_2, \dots, n_k)$ be a starlike tree and let μ be the spectral radius of T. Suppose that μ is not an integer and T is a non-critical dominant graph. Then $\lambda > 1$, defined by $\sqrt{\lambda} + 1/\sqrt{\lambda} = \mu$, is a Salem number.*

The configuration graph $\mathcal{G}(A_{g,k} \cup B_g)$ in Theorem A is a non-critical dominant starlike tree

$$T(2g-2, \underbrace{1, 1, \dots, 1}_{k\text{-times}}),\ k \geq 3$$

and we will denote it by $T(2g-2, k\cdot 1)$. The fact that the spectral radius of $T(2g-2, k\cdot 1)$ is not an integer follows from the following theorem.

Theorem 4 ([11]) *If μ is the spectral radius of the starlike tree $T(n, k\cdot 1)$, then*

$$\sqrt{k+1} < \mu < \frac{k}{\sqrt{k-1}}$$

for $n \geq 1$ and $k \geq 3$.

Thus for the starlike tree $T(n, k \cdot 1)$, the spectral radius satisfies

$$k + 1 < \mu^2 < \frac{k^2}{k-1} = k + 1 + \frac{1}{k-1}.$$

Therefore μ is not an integer and by Theorem 3 $\lambda(f_{g,k})$ is a Salem number.

Moreover, the proof of Corollary 2.1 of Lepović–Gutman [7] implies that

$$\lim_{g\to\infty} \lambda(f_{g,k}) = k - 1.$$

For completeness, we reprove this here.

Recall that $\lambda(f_{g,k})$ is the largest root of

$$p_{g,k}(x) = x^{2g} - (k-2)\left(\sum_{j=1}^{2g-1} x^j\right) + 1.$$

By multiplying $p_{g,k}(x)$ by $x-1$, the stretch factor $\lambda(f_{g,k})$ is the largest root in magnitude of

$$q_{g,k}(x) = x^{2g+1} - (k-1)x^{2g} + (k-1)x - 1.$$

We have $q_{g,k}(k-1) = (k-1)^2 - 1 > 0$, and for any fixed positive integer m,

$$q_{g,k}\left(k - 1 - \frac{1}{10^m}\right) = \left(k - 1 - \frac{1}{10^m}\right)^{2g}\left(-\frac{1}{10^m}\right) + (k-1)\left(k - 1 - \frac{1}{10^m}\right) - 1$$

Hence $q_{g,k}\left(k - 1 - 10^{-m}\right) < 0$ for sufficiently large values of g and therefore $p_{g,k}(x)$ has a root on the interval $(k - 1 - 10^{-m}, k - 1)$. This implies

$$\lim_{g\to\infty} \lambda(f_{g,k}) = k - 1.$$

This completes the proof of Theorem A.

Remark A positive integer cannot be a stretch factor (which is an algebraic integer of degree 1). However, Theorem A implies that for sufficiently large genus g there is a stretch factor which is a Salem number arbitrarily close to a given integer $k - 1$ for each $k \geq 3$.

4 Branched Covers

Lifting a pseudo-Anosov mapping class via a covering map is one way to construct another pseudo-Anosov mapping class. If there is a branched cover $\widetilde{S} \to S$ and a pseudo-Anosov mapping class $f \in \mathrm{Mod}(S)$, then there is some $k \in \mathbb{N}$ such that $\mathrm{Mod}(\widetilde{S})$ has a pseudo-Anosov element $\widetilde{f}$ which is a lift of f^k and hence $\lambda(\widetilde{f}) = \lambda(f)^k$.

Corollary 5 *Let $g \geq 2$. For each positive integer $h \leq g/2$, there is a pseudo-Anosov mapping class $\widetilde{f_h} \in \mathrm{Mod}(S_g)$ such that $\deg(\lambda(\widetilde{f_h})) = 2h$ and $\lambda(\widetilde{f_h})$ is a Salem number.*

Proof Let

$$h = \begin{cases} \frac{g-2m}{2}, & \text{if } g \text{ is even, } m = 0, 1, \ldots, (g-2)/2, \\ \\ \frac{g-1-2m}{2}, & \text{if } g \text{ is odd, } m = 0, 1, \ldots, (g-3)/2. \end{cases}$$

Then h is an integer such that $1 \leq h \leq g/2$.

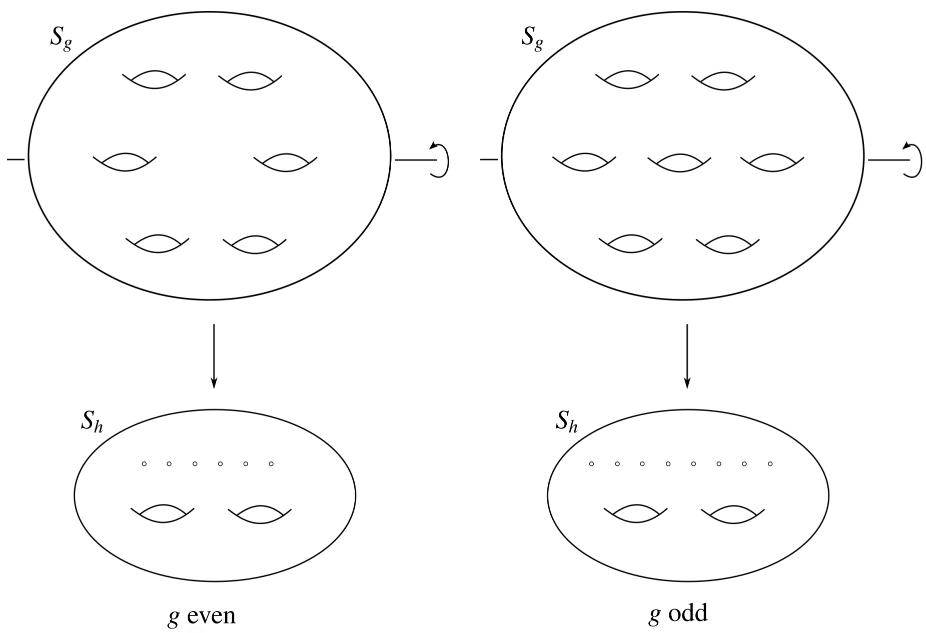

Figure 5: A branched cover

Construct a branched cover $S_g \to S_h$ as in Figure 5. For $h \geq 2$, S_h has a pseudo-Anosov mapping class $f_h \in \mathrm{Mod}(S_h)$ as in the Theorem B whose stretch factor has

$\deg(\lambda(f_h)) = 2h$. For some k, ${f_h}^k$ lifts to S_g and the lift has stretch factor $\lambda(f_h)^k$. We claim that $\deg\left(\lambda(f_h)^k\right) = 2h$. To see this, let λ_i, $1 \leq i \leq 2h$, be the roots of the minimal polynomial of $\lambda(f_h)$ and let us define a polynomial

$$p(x) = \prod_{i=1}^{2h} \left(x - \lambda_i^k\right).$$

Then $p(x)$ is an integral polynomial because the following elementary symmetric polynomials in $\lambda_1, \ldots, \lambda_{2h}$

$$\sum \lambda_i, \ \sum_{i<j} \lambda_i\lambda_j, \ \sum_{i<j<l} \lambda_i\lambda_j\lambda_l, \ \cdots, \ \lambda_1\lambda_2\cdots\lambda_{2h}$$

are all integers and hence each coefficient of $p(x)$

$$\sum \lambda_i^k, \ \sum_{i<j} \lambda_i^k\lambda_j^k, \ \sum_{i<j<l} \lambda_i^k\lambda_j^k\lambda_l^k, \ \cdots, \ \lambda_1^k\lambda_2^k\cdots\lambda_{2h}^k$$

are integers as well. Therefore $p(x)$ is divided by the minimal polynomial of $\lambda(f_h)^k$. Due to the proof of Theorem B in section 7, $\lambda(f_h)^k$ is also a Salem number and $p(x)$ does not have a cyclotomic factor. This implies that $p(x)$ is irreducible and $\deg\left(\lambda(f_h)^k\right) = 2h$.

If $h = 1$, S_h is a torus and it admits a Anosov mapping class f whose stretch factor $\lambda(f)$ has algebraic degree 2. Then similar arguments as above tells us that there is a lift of some power of f to S_g whose stretch factor has $\deg(\lambda(f^k)) = 2$.

Therefore there is a pseudo-Anosov map $\widetilde{f_h} \in \mathrm{Mod}(S_g)$ with $\deg(\lambda(\widetilde{f_h})) = 2h$ for each $h \leq g/2$. In other words, every positive even degree $d \leq g$ is realized as the algebraic degree of a stretch factor on S_g. $\square$

5 Stretch factors of odd degrees

Long proved the following degree obstruction and McMullen communicated to us the following proof. First we will give a definition of the reciprocal polynomial. Given a polynomial $p(x)$ of degree d, we define the reciprocal polynomial $p^*(x)$ of $p(x)$ by $p^*(x) = x^d p(1/x)$. It is a well-known property that $p^*(x)$ is irreducible if and only if $p(x)$ is irreducible.

Theorem 6 ([8]) *Let $f \in \mathrm{Mod}(S_g)$ be a pseudo-Anosov mapping class having stretch factor $\lambda(f)$. If $\deg(\lambda(f)) > 3g-3$, then $\deg(\lambda(f))$ is even.*

Proof Since f acts by a piecewise integral projective transformation on the $6g-6$ dimensional space $\mathcal{PMF}$ of projective measured foliations on S_g, and since $\lambda(f)$ is an eigenvalue of this action, $\lambda(f)$ is an algebraic integer with $\deg(\lambda(f)) \leq 6g-6$. Also, since f preserves the symplectic structure on $\mathcal{PMF}$, it follows that $\lambda(f)$ is the root of palindromic polynomial $p(x)$ whose degree is bounded above by $6g-6$.

Let $q(x)$ be the minimal polynomial of $\lambda(f)$ and let $q^*(x)$ be the reciprocal polynomial of $q(x)$. Then either $q(x) = q^*(x)$ or they have no common roots, because if there is at least one common root ζ of $q(x)$ and $q^*(x)$, then both $q(x)$ and $q^*(x)$ are the minimal polynomial of ζ and hence $q(x) = q^*(x)$. Suppose $\deg(q(x)) > 3g-3$. If $q(x)$ and $q^*(x)$ have no common roots, then their product $q(x)\,q^*(x)$ is a factor of $p(x)$ since $q^*(x)$ is the minimal polynomial of $1/\lambda(f)$. This is a contradiction because $\deg(p(x)) \leq 6g-6$ but $\deg\big(q(x)\,q^*(x)\big) > 6g-6$. Therefore we must have $q(x) = q^*(x)$ and this implies that $q(x)$ is an irreducible palindromic polynomial. Hence $\deg(q(x))$ is even since roots of $q(x)$ come in pairs, λ_i and $1/\lambda_i$. □

It follows from the previous proof that if the minimal polynomial $p(x)$ of λ has odd degree, then $p(x)$ is not palindromic and in fact the minimal palindromic polynomial containing λ as a root is $p(x)p^*(x)$.

We will now show that the stretch factors of degree 3 have an additional special property. A *Pisot number*, also called a *Pisot–Vijayaraghavan number* or a *PV number*, is an algebraic integer greater than 1 such that all its Galois conjugates are strictly less than 1 in absolute value.

Proposition 7 *Let $f \in \mathrm{Mod}(S_g)$. If $\deg(\lambda(f)) = 3$, then $\lambda(f)$ is a Pisot number.*

Proof Let $\lambda_1 > 1$ be the stretch factor of a pseudo-Anosov mapping class with algebraic degree 3, and let $p(x)$ be the minimal polynomial of λ_1. Let λ_1, λ_2, and λ_3 be the roots of $p(x)$. Then the degree of $p(x)p^*(x)$ is 6 and it has pairs of roots $(\lambda_1, 1/\lambda_1), (\lambda_2, 1/\lambda_2), (\lambda_3, 1/\lambda_3)$, where λ_1 is the largest root in absolute value. We claim that the absolute values of λ_2 and λ_3 are strictly less than 1.

Suppose one of them has absolute value greater than or equal to 1, say $|\lambda_2| \geq 1$. The constant term $\lambda_1\lambda_2\lambda_3$ of $p(x)$ is ± 1 since it is the factor of a palindromic polynomial with constant term 1. Hence $|\lambda_1\lambda_2\lambda_3| = 1$ and we have

$$\frac{1}{|\lambda_3|} = |\lambda_1\lambda_2| \geq |\lambda_1|,$$

which is a contradiction to the fact that the stretch factor λ_1 is strictly greater than all other roots of the palindromic polynomial $p(x)p^*(x)$. This proves the claim and hence the stretch factor of degree 3 is a Pisot number. □

We now explain two constructions of mapping classes $f \in \mathrm{Mod}(S_g)$ whose degree of $\lambda(f)$ is odd.

1. As we mentioned, Arnoux–Yoccoz [1] gave examples of a pseudo-Anosov mapping class on S_g whose stretch factor has algebraic degree g. In particular for odd g, this gives examples of mapping classes with odd degree stretch factors. They proved that these stretch factors are all Pisot numbers.

2. For genus 2, there is a pseudo-Anosov mapping class f whose stretch factor has algebraic degree 3 (see section 6). This is the only possible odd degree on S_2 by Long's obstruction. It is also true that $\deg(\lambda(f)^k) = 3$ for each k because the stretch factor is a Pisot number (Proposition 7). There is a cover $S_g \to S_2$ for each g, so the lift of some power of f has a stretch factor with algebraic degree 3 on S_g.

Proposition 8 *For each genus g, the stretch factor with algebraic degree 3 can occur on S_g.*

Question *Are there stretch factors with odd algebraic degree that are not Pisot numbers?*

6 Examples of even degrees

Tables 1 through 4 give explicit examples of pseudo-Anosov mapping classes whose stretch factors realize various degrees. We will follow the notation of the software `Xtrain` by Brinkmann. More specifically, a_i, b_i, c_i, and d_i are Dehn twists along standard curves and A_i, B_i, C_i, and D_i are the inverse twists as in [2]. The only missing degree on S_3 is degree 5. We do not know if there is a degree 5 example or there is another degree obstruction.

deg	$f \in \mathrm{Mod}(S_2)$	Minimal polynomial	$\lambda(f)$
2	$a_0a_0d_0C_0D_1C_0$	x^2-3x+1	$\lambda = 2.618$
3	$a_0d_0d_0C_0C_0D_1$	x^3-3x^2-x-1	$\lambda = 3.383$
4	$a_0d_0d_0d_1c_0d_0$	$x^4-x^3-x^2-x+1$	$\lambda = 1.722$
6	$a_0a_0d_0A_0C_0D_1$	$x^6-x^5-4x^3-x+1$	$\lambda = 2.015$

Table 1: Examples of genus 2

deg	$f \in \mathrm{Mod}(S_3)$	Minimal polynomial	$\lambda(f)$
2	$a_1c_0d_0c_0d_2C_1D_1$	x^2-4x+1	3.732
3	$a_0c_0d_0C_1D_1D_2$	x^3-2x^2+x-1	1.755
4	$a_1c_0d_0a_1c_1d_1d_2$	$x^4-x^3-2x^2-x+1$	1.722
6	$a_0c_0d_0d_2C_1D_1$	$x^6-3x^5+3x^4-7x^3+3x^2-3x+1$	2.739
8	$a_0c_0d_0d_1C_1D_2$	$x^8-x^7-2x^5-2x^3-x+1$	1.809
10	$a_1c_0d_0d_1C_1A_2D_2$	$x^{10}-x^9-2x^8+2x^7-2x^5+2x^3-2x^2-x+1$	1.697
12	$a_1c_1c_0d_1d_2A_0D_0$	$x^{12}-x^{11}-x^9-x^8+x^7+x^5-x^4-x^3-x+1$	1.533

Table 2: Examples of genus 3

deg	$f \in \mathrm{Mod}(S_4)$	deg	$f \in \mathrm{Mod}(S_4)$
4	$a_0a_0a_1c_0d_0c_1d_1c_2d_2c_3d_3$	12	$a_0B_1d_0c_0d_1c_1d_2c_2d_3c_3$
6	$a_0B_2A_3d_0c_0d_1c_1d_2c_2d_3c_3$	14	$a_0d_0B_0d_0c_0d_1c_1d_2c_2d_3c_3$
8	$a_0A_1d_0c_0d_1c_1d_2c_2d_3c_3$	16	$A_0d_0c_0d_1c_1d_2c_2d_3c_3$
10	$a_0b_1A_2d_0c_0d_1c_1d_2c_2d_3c_3$	18	$a_0B_1A_2d_0c_0d_1c_1d_2c_2d_3c_3$

Table 3: Examples of genus 4

deg	$f \in \mathrm{Mod}(S_5)$	deg	$f \in \mathrm{Mod}(S_5)$
6	$b_3d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$	16	$a_1B_2d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$
8	$a_0a_1d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$	18	$a_1B_0d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$
10	$a_1A_4d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$	20	$a_1A_0d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$
12	$b_2C_2d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$	22	$a_2A_1d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$
14	$a_1B_1d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$	24	$c_2A_2d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$

Table 4: Examples of genus 5

7 Irreducibility of Polynomials

In this section, we will prove Theorem B. It is enough to show that the polynomial

$$p_n(x) = x^{2n} - 2\left(\sum_{j=1}^{2n-1} x^j\right) + 1.$$

is irreidcible for $n \geq 2$. We will show that $p_n(x)$ does not have a cyclotomic polynomial factor. It then follows from Kronecker's theorem that $p_n(x)$ is irreducible.

Suppose $p_n(x)$ has the mth cyclotomic polynomial factor for some $m \in \mathbb{N}$. Then $e^{2\pi i/m}$ is a root of $p_n(x)$. Multiplying $p_n(x)$ by $x-1$ yields

$$x^{2n+1} - 3x^{2n} + 3x - 1$$

and hence we have

$$e^{2(2n+1)\pi i/m} - 3e^{4n\pi i/m} + 3e^{2\pi i/m} - 1 = 0. \tag{1}$$

Consider the real part and the complex part of (1). Then we have the system of equations

$$\begin{cases} \cos\left(\frac{2(2n+1)\pi}{m}\right) - 3\cos\left(\frac{4n\pi}{m}\right) + 3\cos\left(\frac{2\pi}{m}\right) - 1 = 0 \\ \sin\left(\frac{2(2n+1)\pi}{m}\right) - 3\sin\left(\frac{4n\pi}{m}\right) + 3\sin\left(\frac{2\pi}{m}\right) = 0 \end{cases}$$

Using double-angle formula for the first cosine and sum-to-product formula for the last two cosines, the first equation gives

$$2\sin\left(\frac{(2n+1)\pi}{m}\right)\left[3\sin\left(\frac{(2n-1)\pi}{m}\right) - \sin\left(\frac{(2n+1)\pi}{m}\right)\right] = 0.$$

Similarly the second equation gives

$$2\cos\left(\frac{(2n+1)\pi}{m}\right)\left[\sin\left(\frac{(2n+1)\pi}{m}\right) - 3\sin\left(\frac{(2n-1)\pi}{m}\right)\right] = 0.$$

Since sine and cosine have no common zeros, we must have

$$\sin\left(\frac{(2n+1)\pi}{m}\right) - 3\sin\left(\frac{(2n-1)\pi}{m}\right) = 0.$$

For $m \le 5$, by direct calculation we can see that $p_n(e^{2\pi i/m}) \ne 0$. So we may assume that $m \ge 6$. Let $\varphi = (2n-1)\pi/m$ and then we can write the above equation as

$$\sin\left(\varphi + \frac{2\pi}{m}\right) - 3\sin(\varphi) = 0. \tag{2}$$

Since $\sin\big(\varphi + 2\pi/m\big)$ is a real number between -1 and 1, we have

$$-\frac{1}{3} \le \sin(\varphi) \le \frac{1}{3}. \tag{3}$$

Let $\psi = \sin^{-1}(1/3)$. Then note that $\psi < \pi/6$. Equation (3) gives the restriction on φ, which is

$$-\psi \le \varphi \le \psi \ \text{ or } \ \pi - \psi \le \varphi \le \pi + \psi.$$

Another observation from (2) is that both $\sin\big(\varphi + 2\pi/m\big)$ and $\sin(\varphi)$ must have the same sign.

We claim that φ has to be on the either first or third quadrant. Suppose φ is on the second quadrant, that is, $\pi - \psi < \varphi < \pi$. Note that $m \ge 6$ implies $2\pi/m \le \pi/3$.

Since φ is above the x-axis, $\varphi + 2\pi/m$ also has to be above the x-axis due to (2) and hence the only possibility is that $\varphi + 2\pi/m$ is between φ and π. Then

$$0 < \sin\left(\varphi + \frac{2\pi}{m}\right) < \sin(\varphi) \implies \sin\left(\varphi + \frac{2\pi}{m}\right) < 3\sin(\varphi),$$

which is a contradiction to (2). Similar arguments hold if φ is on the fourth quadrant. Therefore the possible range for φ is

$$0 < \varphi \leq \psi \text{ or } \pi < \varphi \leq \pi + \psi.$$

Suppose φ is on the first quadrant. Then so is $\varphi + 2\pi/m$ because

$$0 < \varphi + \frac{2\pi}{m} \leq \psi + \frac{\pi}{3} < \frac{\pi}{2}.$$

We can write

$$\varphi = \frac{(2n-1)\pi}{m} \equiv \frac{j\pi}{m} \pmod{2\pi}$$

for some positive integer j, i.e., $0 < j\pi/m < \pi/2$.

If $j \geq 2$, Using the subadditivity of $\sin(x)$ on the first quadrant

$$\sin(x+y) \leq \sin(x) + \sin(y),$$

we have

$$\begin{aligned}
\sin\left(\varphi + \frac{2\pi}{m}\right) - 3\sin(\varphi) &\leq \left(\sin(\varphi) + \sin\left(\frac{2\pi}{m}\right)\right) - 3\sin(\varphi) \\
&= \sin\left(\frac{2\pi}{m}\right) - 2\sin(\varphi) \\
&= \sin\left(\frac{2\pi}{m}\right) - 2\sin\left(\frac{j\pi}{m}\right) < 0,
\end{aligned}$$

which contradicts (2).

If $j = 1$, using triple-angle formula

$$\begin{aligned}
\sin\left(\varphi + \frac{2\pi}{m}\right) - 3\sin(\varphi) &= \sin\left(\frac{3\pi}{m}\right) - 3\sin\left(\frac{\pi}{m}\right) \\
&= \left(3\sin\left(\frac{\pi}{m}\right) - 4\sin^3\left(\frac{\pi}{m}\right)\right) - 3\sin\left(\frac{\pi}{m}\right) \\
&= -4\sin^3\left(\frac{\pi}{m}\right) < 0,
\end{aligned}$$

which contradicts (2) again. Therefore there is no possible φ on the first quadrant. By using the same arguments, the fact that φ is on the third quadrant gives a contradiction. Therefore we can conclude that $p(x)$ does not have a cyclotomic factor.

We now show that $p_n(x)$ is irreducible over $\mathbb{Z}$. Suppose $p_n(x)$ is reducible and write $p_n(x) = g(x)h(x)$ with non-constant functions $g(x)$ and $h(x)$. There is only one root of $p_n(x)$ whose absolute value is strictly greater than 1. Therefore one of $g(x)$ or $h(x)$ has all roots inside the unit disk. By Kronecker's theorem, this polynomial has to be a product of cyclotomic polynomials, which is a contradiction because $p_n(x)$ does not have a cyclotomic polynomial factor. Therefore $p_n(x)$ is irreducible.

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