MAS430 : Combinatorial Topology HW 5

Due : 2017. 11. 8. in class.

Exercises

In the following exercises, assume the hypotheses of the Seifert-van Kampen theorem.

1. Suppose that the homomorphism i_* induced by inclusion $i : U \cap V \to X$ is trivial.

(a) Show that j_1 and j_2 induce an epimorphism

$$h : (\pi_1(U, x_0)/N_1) * (\pi_1(V, x_0)/N_2) \longrightarrow \pi_1(X, x_0),$$

where N_1 is the least normal subgroup of $\pi_1(U, x_0)$ containing image i_1, and N_2 is the least normal subgroup of $\pi_1(V, x_0)$ containing image i_2.

(b) Show that h is an isomorphism. [*Hint:* Use Theorem 70.1 to define a left inverse for h.]

2. Suppose that i_2 is surjective.

(a) Show that j_1 induces an epimorphism

$$h : \pi_1(U, x_0)/M \longrightarrow \pi_1(X, x_0),$$

where M is the least normal subgroup of $\pi_1(U, x_0)$ containing $i_1(\ker i_2)$. [*Hint:* Show j_1 is surjective.]

(b) Show that h is an isomorphism. [*Hint:* Let $H = \pi_1(U, x_0)/M$. Let $\phi_1 : \pi_1(U, x_0) \to H$ be the projection. Use the fact that $\pi_1(U \cap V, x_0)/\ker i_2$ is isomorphic to $\pi_1(V, x_0)$ to define a homomorphism $\phi_2 : \pi_1(V, x_0) \to H$. Use Theorem 70.1 to define a left inverse for h.]

3. (a) Show that if G_1 and G_2 have finite presentations, so does $G_1 * G_2$.

(b) Show that if $\pi_1(U \cap V, x_0)$ is finitely generated and $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ have finite presentations, then $\pi_1(X, x_0)$ has a finite presentation. [*Hint:* If N' is a normal subgroup of $\pi_1(U, x_0) * \pi_1(V, x_0)$ that contains the elements $i_1(g_i)^{-1} i_2(g_i)$ where g_i runs over a set of generators for $\pi_1(U \cap V, x_0)$, then N' contains $i_1(g)^{-1} i_2(g)$ for arbitrary g.]