MAS430: Combinatorial Topology HW2 & HW3 (2017 Fall)

Due date: 2017.10.18 in class

This note consists of two homework sets both are due on the same date. If you cannot make it to the class on Oct. 18th or cannot finish it by the time of the class, please visit my office on the 4th floor of E6-1 (room 4407) to drop off your HW in the box next to my office door by the midnight of Oct. 18.

1. HW2

Question 1. *Show that if A is a deformation retract of X, and B is a deformation retract of A, then B is a deformation retract of X.*

Question 2. *Recall that a space X is said to be contractible if the identity map from X to itself is nullhomotopic. Show that X is contractible if and only if X has the homotopy type of a one-point space.*

Question 3. *Show that a retract of a contractible space is contractible.*

Question 4. *Let A be a subspace of X; let $j : A \to X$ be the inclusion map, and let $f : X \to A$ be a continuous map. Suppose there is a homotopy $H : X \times I \to X$ between the map $j \circ f$ and the identity map of X.*

(a) *Show that if f is a retraction, then j_* is an isomorphism.*
(b) *Show that H maps $A \times I$ into A, then j_* is an isomorphism.*
(c) *Give an example in which j_* is not an isomorphism.*

Question 5. *Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$.*

(a) *Show that the homomorphism $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is injective.*
(b) *Let $H = p_*(\pi_1(E, e_0))$. Show that the lifting correspondence ϕ induces an injective map $\Phi : \pi_1(B, b_0)/H \to p^{-1}(b_0)$ of the collections of right cosets of H into $p^{-1}(b_0)$. Show that this map is bijective if E is path-connected.*
(c) *Let f be a loop in B based at b_0. Show that $[f] \in H$ if and only if f lifts to a loop in E based at e_0.*

2. HW3

Question 6. *Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. Show that $H_0 = p_*(\pi_1(E, e_0))$ is a normal subgroup of $\pi_1(B, b_0)$ if and only if for every pair of points e_1, e_2 of $p^{-1}(b_0)$, there exists an equivalence $h : E \to E$ with $h(e_1) = e_2$.*

Question 7. *Let $p : X \times Y \to X$ and $q : X \times Y \to Y$ be the projection mappings. Then we get an induced mapping $p_* \times q_* : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$. Show that this map is an isomrphism.*

For the additive group of integers $\mathbb{Z}$ and a positive integer m, the subgroup of $\mathbb{Z}$ generated by m means the group of integer multiples of m with the addition. For instance, if $m = 2$, then one gets the group of even numbers with addition.

Question 8. *We know that $\pi_1(S^1, b_0)$ is isomorphic to $\mathbb{Z}$. Find a covering space of S^1 corresponding to the subgroup of $\mathbb{Z}$ generated by a positive integer m.*

Question 9. *Let $T = S^1 \times S^1$ be the torus. As shown in the preceding question, there is an isomorphism of $\pi_1(T, (b_0, b_0))$ with $\mathbb{Z} \times \mathbb{Z}$ induced by projections of T onto its two factors.*

(a) *Find a covering space of T corresponding to the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by the element $(m, 0)$, where m is a positive integer.*
(b) *Find a covering space of T corresponding to the trivial subgroup of $\mathbb{Z} \times \mathbb{Z}$.*
(c) *Find a covering space of T corresponding to the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by $(m, 0)$ and $(0, n)$, where m, n are integers.*