MAS430: Combinatorial Topology HW 1 (2017 Fall)

Due date: 2017.9.20 in class

In this homework set, we will see several application of our knowledge of the fundamental group of S^1.

In class, we already saw what a retraction is. Let X be a topological space and A be a subspace of X. Then a continuous map $r : X \to A$ is a retraction if $r(a) = a$ for each $a \in A$.

We showed that $r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$ is surjective for $a_0 \in A$. A very similar argument can show that if $\iota : A \to X$ is the inclusion map, ι_* is injective (as given as Lemma 55.1 in the book).

An immediate corollary of either of above facts is Theorem 55.2 in the book, which says that there is no retraction from B^2 to S^1 where B^2 is the closed disk. In general, B^n denotes the closed ball of dimension n (so the boundary of B^n is S^{n-1}, the $(n-1)$-dimensional sphere.

The fundamental group of a space consists of path-homotopy classes of loops. But what is a loop? A loop in the space X is simply a continuous map $h : S^1 \to X$. When does such a map h represents a non-trivial element of $\pi_1(X)$?

Lemma 55.3 gives us a criterion. It basically says that h represents a trivial element in $\pi_1(X)$ if and only if it is a restriction of some continuous map $k : B^2 \to X$ to the boundary of B^2 (in this case, we say that h extends to a continuous map k).

Question 1. *Read the proof of Lemma 55.3 in the book.*

Question 2. *Let $h : S^2 \to X$ be a continuous map. Prove or disprove that h is nullhomotopic if and only if h extends to a continuous map $k : B^3 \to X$.*

In fact, the set of homotopy classes of the maps $(S^2, a) \to (X, b)$ is called the second homotopy group $\pi_2(X, b)$ (the fudamental group is sometimes callsed the first homotopy group). This can be generalized to any dimension.

Question 3. *For two continuous maps $f, g : (S^2, a) \to (X, b)$, define their product, and show that this gives us a group operation on $\pi_2(X, b)$.*

Question 4. *Read the rest of Section 55 and solve Problem 4 (on p.353).*

Another application of our knowledge on $\pi_1(S^1)$ is the fundamental theorem of algebra.

Question 5. *Read Section 56 and solve Problem 1 (on p.356).*

We need some group theory too. From our course website, you can dowload a wonderful lecture note on basic group theory written by Juan Alonso.

Question 6. *Read the following sections of the note by Juan Alonso: Sections 1, 2, 6, 7 (except 7.6), 8.1, 8.3, 9.1, and 9.3.*