TAKE HOME FINAL

MAS584 INTRODUCTION TO GEOMETRIC TOPOLOGY

1. About the exam

(1) The due date of the exam is 2017.06.16 (by mid-night).
(2) Please either type or scan your answer sheet, and then hand it in via email. Alternatively, I will put a box in front of my office in the morning on 2017.06.16.
(3) Each question has 10 points, but note that some questions are extremely challenging, so do not get discouraged if you cannot solve them all. There are 16 questions in total.
(4) You can get bonus credit by doing exercises given in class. One exercise can give you 10 points and maximum points you can get from exercises is 40 points. Hence in principle, the maximum point of this exam is 200 points.

2. Quasi-isometrical embeddings

Problem 2.1. *Let $f:(X,d_X)\to(Y,d_Y)$ to be a quasi-isometry. Define a quasi-inverse to f and show that it is a quasi-isometry from (Y,d_Y) to (X,d_X).*

Let H be a finintely generated subgroup of a finitely generated group G. Let d_H, d_G denote word metrics associated to some choices of finite generating sets of H, G. Then, we define the distortion $\mathrm{Dist}_H^G:\mathbb{N}\to\mathbb{N}$ of H in G as follows

$$\mathrm{Dist}_H^G(n)=\max\{d_H(1,h):h\in H, d_G(1,h)\le n\}.$$

We say H is *undistorted* in G if there exists $C>0$ such that $\mathrm{Dist}_H^G(n)\le Cn$ for all $n\in\mathbb{N}$.

Problem 2.2. *Let G,H be as above. Show that H is undistorted in G if and only if $H\hookrightarrow G$ is a quasi-isometric embedding.*

A subspace Y of a geodesic metric space X is *k-quasi-convex* when there exists some $k>0$ such that every geodesic in X that connects a pair of points in Y lies in the k-neighborhood of Y. We simply say, Y is quasi-convex if is k-quasi-convex for some k.

Similarly, if H is a subgroup of a group G with a fininte generatring set A, then H is *quasi-convex* in G if it is quasi-convex as a subspace of the Cayley graph $C_A(G)$ of G with respect to A.

Problem 2.3. *Show that the above definition of quasi-convexity of a subgroup does not depend on the finite generatring set of the ambient group G if G is δ-hyperbolic.*

Problem 2.4. *Suppose G is a group with finite generating set A, and H is quasi-convex subgroup of G. Then H is finitely generated and the inclusion $H\hookrightarrow G$ is a quasi-isometric embedding.*

Hint for Problem 2.4: Say H is k-quasi-convex. Then define B to be the set of all elements of H a distance at most $2k+1$ from 1 in $C_A(G)$, and show that H is generated by B.

In a δ-hyperbolic geodesic space, a quasi-geodesic (a quasi-isometric embedding of a line segment) between two points is always uniformly close to any geodesic that connects those two points. More precisely, if X,d is a δ-hyperbolic geodesic space, then there exists a number $N=N(\delta)$ such that for $\rho:[0,r]\to X$ a quasi-geodesic connecting $x=\rho(0)$ and $y=\rho(r)$ and for any geodesic γ from x to y, (the image of) ρ is contained in N-neighborhood of γ. This is difficult to show (not a theoretically hard, but technically very messy), so we will accept this as a fact.

Problem 2.5. *Let H be a finitely generated subgroup of a δ-hyperbolic group G. Using the above fact and some of above problems, show that the followings are equivalent.*

- *H is quasi-convex,*
- *the inclusion $H \hookrightarrow G$ is a quasi-isometric embedding,*
- *H is undistorted in G.*

Problem 2.6. *Show that quasi-convex subgroups of hyperbolic groups are hyperbolic.*

3. Groups acting on hyperbolic spaces

Problem 3.1. *Show that $\mathbb{H}^3$ is δ-hyperbolic for some δ.*

Problem 3.2. *Show that for a closed oriented connected hyperbolic 3-manifold M, there is no π_1-injective 2-torus T in M.*

Recall that the Gromov boundary of a hyperbolic geodesic metric space is defined to be the set of equivalence classes of geodesic rays where two rays are equivalent if they stay in a bounded distance.

Problem 3.3. *Let G be a countable group acting on a hyperbolic geodesic metric space X by isometries. Show that if the orbits are not bounded, then the orbit contains at least one accumulation point on the Gromov-boundary of X. The easiet way to do this is using the Gromov product, and one can define Gromov boundary in this term. See first few pages of Section 2 of https://arxiv.org/abs/math/0202286.*

When a group G acts on a hyperbolic geodesic metric space X, the limit set $\Lambda(G)$ is defined to be the set $\overline{G\cdot x}\cap \partial X$ where ∂X is the Gromov boundary of X and the closure is taken in the compactification $X\cup\partial X$.

Problem 3.4. *For a discrete subgroup G of $Isom(\mathbb{H}^2)$, show that the cardinality of $\Lambda(G)$ is either 0, 1, 2, or infinite.*

When $|\Lambda(G)| = 1$, the action is called parabolic or horocyclic. Such an action does not say much about G.

Let G be a finitely generated infinite group with a finite generating set X. Let $\Gamma = \Gamma(G,X)$ be the corresponding Cayley graph. For each $n \in \mathbb{N} = \{0,1,2,\dots\}$, let Γ_n to be the graph obtained from Γ by adding an edge between any two vertices of distance $\leq 2^n$.

Now form a bigger graph Γ_∞ from the set $\{\Gamma_n : n \in \mathbb{N}\}$ by adding an edge between vertices in Γ_n and Γ_{n+1} which originally represent the same element of G.

Problem 3.5. *Show that G naturally acts on Γ_∞ by isometries, and Γ_∞ is δ-hyperbolic. Discuss the cardinality of $\partial\Gamma_\infty$ (and what conclusion can you derive?).*

Recall that for a close orientable connected hyperbolic surface S, $\pi_1(S)$ acts on $\mathbb{H}^2$ as a deck transformation group. This action extends continuously to the boundary of $\mathbb{H}^2$ which is homeomorphic to the circle. Let's consider this action of $\pi_1(S)$ on $S^1 = \partial\mathbb{H}^2$. Since this is an action by (orientation-preserving) homeomorphisms, we have an induced action on $S^1\times S^1\times S^1$. Define Δ to be the subset of $S^1\times S^1\times S^1$ where at least two points are the same (called big-diagonal). Again, since the action is by homeomorphisms, Δ is preserved, so we have $\pi_1(S)$-action on $S^1\times S^1\times S^1-\Delta$.

Problem 3.6. *Show that $\pi_1(S)$ acts properly discontinuously on $S^1\times S^1\times S^1-\Delta$.*

Hint for Problem 3.6: If G acts on some topologcal space X not properly continuously, this means there exists a sequence (x_i) of points of X which converges to some $a\in X$, and a sequence (g_i) of elements of G such that g_ix_i converges to some other point $b\in X$. Now let $X = S^1\times S^1\times S^1-\Delta$, see what this means in terms of the dynamics of G on S^1.

4. Geometric objects associated to surfaces

Let Σ_g be the closed surface of genus g.

Let $\mathscr{S}_g$ to be the set of isotopy classes of essential simple closed curves in Σ_g. We define the *curve complex* $\mathscr{C}_g$ of Σ_g as follows. Let $\mathscr{S}_g$ be the vertex set and put an edge between two vertices whenever they can be represented disjointly, and add a simplex whenever one can. The distance between two vertices are defined as the shortest edge path between the vertices in the 1-skeleton of the curve complex (as usual, each edge is assumed to have length 1). The 1-skeleton is commonly called the *curve graph* of Σ_g.

Problem 4.1. *Show that there are two vertices in the curve graph of Σ_g of distance at least 3 apart.*

Note that the mapping class group acts on the curve graph isometrically. In fact, the curve graph is always δ-hyperbolic (actually δ can be taken to be 17) and the action of the mapping class group is acylindrical, i.e, $\forall \varepsilon > 0, \exists R, N > 0$ such that $\forall x, y$ with $d(x,y) > R$, there are at most N elements $g \in G$ satisfying $d(x, gx) < \varepsilon$ and $d(y, gy) < \varepsilon$.

A measured foliation $\mathscr{F}$ is *uniquely ergodic* if $\mathscr{F}$, as a topologicla foliation, carries a unique transverse measure up to a multiplicative factor. Suppose $\mathscr{F}$ is minimal and uniquely ergodic. Then we have the following fact: if (α_i) is an infinite sequence of simpe closed curves which converges to $\mathscr{F}$ in Hausdorff topology, and (β_i) is another sequence of simple closed curves such that α_i are disjoint from β_i for each i and converges to Λ for some closed subset of Σ, then in fact Λ must coincide with $\mathscr{F}$.

Problem 4.2. *Using the fact mentioned in the above pargraph, show that the curve graph has infinite diameter.*

Problem 4.3. *Show that the set of measured geodesic laminations are in 1-1 correspondence with the set of measured foliations in Σ_g up to Whitehead equivalence.*

Problem 4.4. *Can you find a surface of finite type (not necessarily compact or closed) whose mapping class group acts linearly on some finite-dimensional vector space? It not, how about piecewise-linear action?*