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A stabilized covolume method for the Stokes problem ☆

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Abstract

We present a covolume method for the modified Stokes problem using the simplest approximation spaces, Q_1–P_0. This scheme turns out the stabilized covolume method for the Stokes problem. We prove that the covolume method in this paper has a unique solution and O(h) convergence order in H^1 semi-norm for the velocity and in L^2 norm the pressure approximants, respectively. We also present numerical results corresponding to our analysis. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

Various stabilized finite element methods for solving the Stokes problem have been introduced and analyzed successfully [3,7,10,11,15]. For example, Hughes et al. [11] developed stabilized finite element methods using pairs of arbitrary order as approximations for the velocity and the pressure. The stability was achieved by the addition of least-square forms of residual. In the case of continuous pressure approximations, Brezzi and Douglas [3] proved the stability and the convergence for the Stokes problem by adding a penalty term.

On the other hand, finite volume methods or covolume methods have been developed as effective discretization scheme for fluid flow problems [1,9,13,14]. One advantage of these methods is that the discrete equations are derived based on local conservation of mass, momentum or energy over control volumes. In particular, Chou [4] and Chou and Kwak [5,6] proposed new covolume methods which are successfully applied to the generalized Stokes problem. In their works, the formulations are derived through the design of primal and dual partitions of the domain. Various finite element spaces satisfying the inf–sup condition are used as trial function spaces for this incompressible fluid problem. The inf–sup condition plays an important role in their analysis. The test functions are piecewise constant on the dual grid. Thus these methods can be viewed as locally and globally conservative Petrov–Galerkin methods.

The simplest pair of approximation spaces is Q_1–P_0, the conforming piecewise bilinears for the velocity and piecewise constants for the pressure on rectangular elements. It is well known [2] that this pair of approximation spaces does not satisfy the inf–sup condition. Such drawback can be overcome by stabilized techniques [3,7,11,12]. The emphasis of stabilized mixed finite element methods for Q_1–P_0 is the control of the pressure approximation by introducing a pressure jump operator.

In this paper, we introduce a covolume scheme for the stabilized mixed method using Q_1–P_0. Integrating a modified incompressible condition gives us a stabilized covolume formulation. We shall compute the

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approximation of the velocity which locally permits a small compressibility on each primal element. This scheme is different from [5] in that the macro-elements are not necessary. We represent our covolume formulation as Petrov–Galerkin method and obtain linear convergence in H^1 semi-norm for the velocity and in L^2 norm for the pressure approximants.

2. Stabilized mixed finite element methods

We consider the two-dimensional Stokes Problem:

$$-\mu\Delta\mathbf{u}+\nabla p=\mathbf{f}\quad\text{in }\Omega\subset\mathbb{R}^2, \tag{2.1}$$

$$\nabla\cdot\mathbf{u}=0\quad\text{in }\Omega, \tag{2.2}$$

$$\mathbf{u}=\mathbf{0}\quad\text{on }\partial\Omega, \tag{2.3}$$

where μ is the viscosity of the fluid. For the sake of simplicity, we shall assume $\mu=1$ in this paper. Let $H_0^1(\Omega)$ be the space of weakly differentiable functions with zero trace, $H^i(\Omega)$, $i=1,2$ be the usual Sobolev spaces, and $L_0^2(\Omega)$ be the set of all L^2 functions over Ω with zero integral mean. Let us denote $\mathbf{H}_0^1=(H_0^1(\Omega))^2$, $|\cdot|_1$ and $\|\cdot\|_0$ be the usual $(H^1(\Omega))^2$ semi-norm and the L^2 norm, respectively.

The weak formulation associated with (2.1)–(2.3) is: Find $(\mathbf{u},p)\in\mathbf{H}_0^1\times L_0^2(\Omega)$

$$(\nabla\mathbf{u},\nabla\mathbf{v})-(\text{div }\mathbf{v},p)=(\mathbf{f},\mathbf{v})\quad\forall\mathbf{v}\in\mathbf{H}_0^1, \tag{2.4}$$

$$(\text{div }\mathbf{u},q)=0\quad\forall q\in L_0^2(\Omega). \tag{2.5}$$

The unique solvability of this problem is well known [2].

Assume that Ω is a polygonal domain whose sides are parallel to the coordinate axis. Let $\mathscr{R}_h=\cup K$ be a partition of the domain Ω into a union of rectangular elements K. We denote $h=\max h_K$ where h_K is the diameter of K. We shall assume throughout this paper that the primal partition is regular in the usual sense, i.e.

$$Ch^2\leqslant|K|\leqslant h^2\quad\forall K\in\mathscr{R}_h,$$

where $|K|$ is the area of K. We also assume that $\mathscr{R}_h$ is quasi-uniform, i.e., there exists a positive constant C such that

$$h/h_K\leqslant C$$

for all $K\in\mathscr{R}_h$. For simplicity, we assume that the partition $\mathscr{R}_h$ is equally divided along each axis. Let Γ_h be the set of all interior edges of $\mathscr{R}_h$ and h_e the length of $e\in\Gamma_h$.

Define the finite element subspace of the velocity by

$$\mathbf{H}_h=\{\mathbf{v}_h\in\mathbf{H}_0^1:\mathbf{v}_h|_K\in(Q_1(K))^2\ \forall K\in\mathscr{R}_h\},$$

where $Q_1(K)$ denotes the piecewise bilinear functions on the rectangle K. For the finite element subspace for the pressure, define

$$L_h=\{q_h\in L_0^2(\Omega):q_h|_K\text{ is constant }\forall K\in\mathscr{R}_h\}.$$

With these subspaces, $\mathbf{H}_h$ and L_h, a stabilized finite element formulation of (2.4) and (2.5) is: Find $(\tilde{\mathbf{u}}_h,\tilde{p}_h)\in\mathbf{H}_h\times L_h$ such that

$$(\nabla\tilde{\mathbf{u}}_h,\nabla\mathbf{v}_h)-(\text{div }\mathbf{v}_h,\tilde{p}_h)=(\mathbf{f},\mathbf{v}_h)\quad\forall\mathbf{v}_h\in\mathbf{H}_h, \tag{2.6}$$

$$(\text{div }\tilde{\mathbf{u}}_h,q_h)+\beta\sum_{e\in\Gamma_h}h_e\int_e[\tilde{p}_h]_e[q_h]_e\,\mathrm{d}s=0\quad\forall q_h\in L_h, \tag{2.7}$$

where $[\cdot]_e$ stands for the jump operator across $e\in\Gamma_h$ and β is an arbitrary positive constant. Standard mixed finite element methods require a pair of approximation spaces satisfying the inf–sup condition in order to be stable [8]. In this point of view, the pair $(\mathbf{H}_h, L_h)$ is unstable. This phenomenon occurs due to the relatively large degrees of freedom of the pressure approximation in comparison with that of the velocity. This can be overcome by the jump operator in (2.7) which controls pressure jumps across all interior edges. These types of stabilization techniques are developed in [7,11,12].

3. Covolume formulation

In this section, we introduce a covolume method for Q_1–P_0 pair. Two partitions of the problem domain are necessary to describe the covolume method for the Stokes problem. We call the partition $\mathscr{R}_h$, which is defined in Section 2, the primal partition. Next, we construct the dual partition $\mathscr{R}_h^*$. Given the primal partition, we can further subdivide the domain Ω by adding horizontal and vertical grid lines through the midpoints of the elements in $\mathscr{R}_h$. These are the dashed lines in Fig. 1.

Let P_0 be an arbitrary node, a vertex of some rectangles. The dual element based at the interior node P_0 is made up of the gray colored rectangle as in Fig. 1. We make the obvious modification at boundary nodes. Carrying out the construction for every node in the primal partition generates a dual partition for the domain. We denote the dual element based at P as K_P^* and the dual partition as $\mathscr{R}_h^* = \cup K_P^*$.

Associated with the partitions $\mathscr{R}_h$, the trial function spaces for the velocity and the pressure approximations are defined by $\mathbf{H}_h$ and L_h, respectively. In our covolume scheme, the velocity nodes are assigned at the vertices and the pressure nodes at the center of each rectangular element $K \in \mathscr{R}_h$. On the other hand, the test function space $\mathbf{Y}_h$ is defined by the space of certain piecewise constant vector functions

$$\mathbf{Y}_h = \{\mathbf{w} \in (L^2(\Omega))^2 : \mathbf{w}|_{K_P^*} \text{ is a constant vector, } \mathbf{w}|_{K_P^*} = \mathbf{0} \text{ on any boundary dual element } K_P^*\}.$$

Denote by χ_P^* the scalar characteristic function associated with the dual element $K_{P_j}^*$, $j = 1, \dots, N_I$, where N_I is the number of interior nodes of $\mathscr{R}_h$. We see that for any $\mathbf{w}_h \in Y_h$

$$\mathbf{w}_h(x) = \sum_{j=1}^{N_I} \mathbf{w}_h(P_j)\chi_{P_j}^*(x) \quad \forall x \in \Omega.$$

We are now ready to describe a covolume formulation for stabilized mixed methods. This formulation can be achieved by integrating the momentum equation (2.1) over the dual element and the modified continuity equation with the artificial compressible term

$$\operatorname{div} \mathbf{u} = \alpha(x)\Delta p \tag{3.1}$$

over the primal elements. In (3.1), the homogeneous Neumann boundary condition is imposed on the pressure and $\alpha(x) = \beta|K|$, $x \in K$ for an arbitrary positive constant β. Note that $\operatorname{div} \mathbf{u}$ approaches zero as h tends to zero.

Define the bilinear forms $a^* : \mathbf{H}_h \times \mathbf{Y}_h \to \mathbb{R}$, $b^* : \mathbf{Y}_h \times L_h \to \mathbb{R}$ and $c : \mathbf{H}_h \times L_h \to \mathbb{R}$ as follows. For $\mathbf{v}_h \in \mathbf{H}_h$, $\mathbf{w}_h \in \mathbf{Y}_h$, $q_h \in L_h$ and $\mathbf{f} \in (L^2(\Omega))^2$

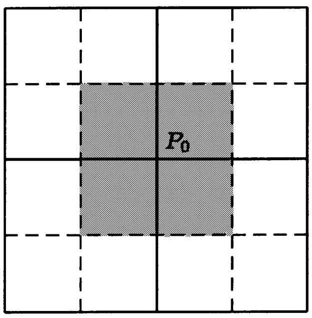

Fig. 1. Primal and dual elements.

$$a^*(\mathbf{v}_h, \mathbf{w}_h) := -\sum_{i=1}^{N_I} \mathbf{w}_h(P_i) \cdot \int_{\partial K^*_{P_i}} \frac{\partial \mathbf{v}_h}{\partial \mathbf{n}}\, \mathrm{d}s, \tag{3.2}$$

$$b^*(\mathbf{v}_h, q_h) := \sum_{i=1}^{N_I} \mathbf{v}_h(P_i) \cdot \int_{\partial K^*_{P_i}} q_h \mathbf{n}\, \mathrm{d}s, \tag{3.3}$$

$$c(\mathbf{v}_h, q_h) := \sum_{j=1}^{N_R} q_h(Q_j) \cdot \int_{K_j} \mathrm{div}\ \ \mathbf{v}_h\, \mathrm{d}x, \tag{3.4}$$

$$(\mathbf{f}, \mathbf{w}_h) := \sum_{i=1}^{N_I} \mathbf{w}_h(P_i) \cdot \int_{K^*_{P_i}} \mathbf{f}\, \mathrm{d}x, \tag{3.5}$$

where N_R denotes the number of elements in $\mathscr{R}_h$ and Q_j is the center of K_j. Eq. (3.2) is the result of integrating the first term of (2.1) against the test functions and using the second Green's identity.

Next, we shall approximate the integral of $\alpha(x)\Delta p$ over K by cell centered finite differences: Integrate $\alpha(x)\Delta p$ against test function $\chi_K \in L_h$, where χ_K stands for characteristic function associated with the primal element K. Use the divergence theorem to get

$$\int_K \alpha(x)\Delta p\, \mathrm{d}x = \int_{\partial K} \alpha(x) \frac{\partial p}{\partial n}\, \mathrm{d}s, \tag{3.6}$$

and then use cell centered finite difference scheme to approximate (3.6). Referring to Fig. 2, we approximate (3.6) as

$$\beta h_x h_y \left(\frac{p_1 - 2p_0 + p_3}{h_x} h_y + \frac{p_2 - 2p_0 + p_4}{h_y} h_x \right), \tag{3.7}$$

where h_x is the width of K and h_y is the height of K and p_i, $i = 0, 1, \ldots, 4$ are the values of p_h in the element K_i. Define the bilinear form $d : L_h \times L_h \to \mathbb{R}$ associated with (3.7) by

$$d(p_h, q_h) = \beta \sum_{j}^{N_R} q_h(P_j) \Big[h_y^2 (2p_h(P_j) - p_h(P_{jW}) - p_h(P_{jE})) + h_x^2 (2p_h(P_j) - p_h(P_{jN}) - p_h(P_{jS})) \Big]$$
$$\forall p_h, q_h \in L_h, \tag{3.8}$$

where $P_{jW,E,N,S}$ stand for the centers of four adjacent rectangles of K_j.

We now can present a covolume formulation for the modified Stokes problem: Find $(\mathbf{u}_h, p_h) \in \mathbf{H}_h \times L_h$ such that

$$a^*(\mathbf{u}_h, \mathbf{v}_h) + b^*(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{Y}_h, \tag{3.9}$$

$$c(\mathbf{u}_h, q_h) + d(p_h, q_h) = 0 \quad \forall q_h \in L_h. \tag{3.10}$$

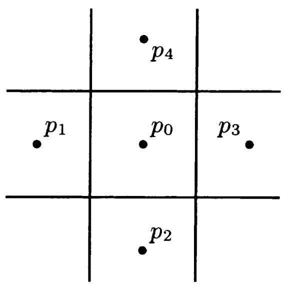

Fig. 2. The value of p_h on each element.

It turns out that we can reformulate this system into a stabilized method as (2.6) and (2.7). Let us start the reformulation by introducing the one-to-one transfer operator γ_h from $\mathbf{H}_h$ onto $\mathbf{Y}_h$ defined by

$$\gamma_h \mathbf{v}_h(x) := \sum_{j=1}^{N} \mathbf{v}_h(P_j)\chi_j^*(x) \quad \forall x \in \Omega$$

for all $\mathbf{v}_h \in \mathbf{H}_h$. With this transfer operator, define the following bilinear forms:

$$a(\mathbf{v}_h, \mathbf{w}_h) := a^*(\mathbf{v}_h, \gamma_h \mathbf{w}_h) \quad \forall \mathbf{v}_h, \mathbf{w}_h \in \mathbf{H}_h,$$
$$b(\mathbf{v}_h, q_h) := b^*(\gamma_h \mathbf{v}_h, q_h) \quad \forall \mathbf{w}_h \in \mathbf{H}_h \;\; \forall q_h \in L_h.$$

For the stability and the convergence analysis of the covolume method, we shall need some lemmas. Lemmas 3.1–3.3 are derived in [5].

Lemma 3.1. *There exists a positive constant C_0 independent of h such that*

$$\|\gamma_h \mathbf{v}_h - \mathbf{v}_h\|_0 \leqslant C_0 h |\mathbf{v}_h|_1 \quad \forall \mathbf{v}_h \in \mathbf{H}_h. \tag{3.11}$$

Lemma 3.2. *For* $\mathbf{v}$, $\mathbf{w} \in \mathbf{H}_h$, *the bilinear form $a(\cdot,\cdot)$ has the following properties*:

(i) *a is symmetric*;

(ii) *a is bounded and coercive*, *i.e.*,

$$|a(\mathbf{v}, \mathbf{w})| \leqslant C|\mathbf{v}|_1 |\mathbf{w}|_1 \quad \forall \mathbf{v}, \mathbf{w} \in \mathbf{H}_h$$

and

$$a(\mathbf{v}, \mathbf{v}) \geqslant C|\mathbf{v}|_1^2 \quad \forall \mathbf{v} \in \mathbf{H}_h$$

for some positive constant C independent of h.

(iii) *$a(\mathbf{v}_h, \mathbf{w}_h)$ differs from $(\nabla \mathbf{v}_h, \nabla \mathbf{w}_h)$ only by a quadrature term*, *i.e.*,

$$a(\mathbf{v}, \mathbf{w}) = (\nabla \mathbf{v}, \nabla \mathbf{w}) + Q(\mathbf{v}, \mathbf{w}), \tag{3.12}$$

where

$$Q(\mathbf{v}, \mathbf{w}) = \frac{1}{24} \sum_K (h_x h_y^3 + h_x^3 h_y)(\mathbf{v}_{xy} \cdot \mathbf{w}_{yx}).$$

Here, $\mathbf{v}_x$ *stands for the partial derivative with respect to x*.

Lemma 3.3. *For all $\mathbf{v}_h \in \mathbf{H}_h$ and $q_h \in L_h$*

$$b(\mathbf{v}_h, q_h) = b^*(\gamma_h \mathbf{v}_h, q_h) = -c(\mathbf{v}_h, q_h). \tag{3.13}$$

The bilinear form $d(\cdot,\cdot)$, corresponding to the artificial compressible term, is not only symmetric but also positive semi-definite. It plays an important role for the stability of our covolume method.

Lemma 3.4. *For p_h, $q_h \in L_h$,*

$$d(p_h, q_h) = \beta \sum_{e \in \Gamma_h} h_e \int_e [p_h]_e [q_h]_e \, \mathrm{d}s \tag{3.14}$$

where $[p_h]_e$ is the jump of p_h across the edge e.

Proof. Let K_i, $i = 1, 2, 3, 4$ be four elements having the common edge e_i with K_0 as Fig. 2. Setting $q_h = \chi_K$, we have

$$\beta \sum_{i=1}^{4} h_e \int_{e_i} [p_h]_e [q_h]_e \, \mathrm{d}s = \beta h_x^2((p_0 - p_1) + (p_0 - p_3)) + \beta h_y^2((p_0 - p_2) + (p_0 - p_4)) = d(p_h, q_h)|_K.$$

Hence the summation over all K gives (3.14). □

By Lemmas 3.2–3.4, the covolume scheme (3.9) and (3.10) becomes: Find $(\mathbf{u}_h, p_h) \in \mathbf{H}_h \times L_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) - c(\mathbf{v}_h, p_h) = (\mathbf{f}, \gamma_h \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{H}_h, \tag{3.15}$$

$$c(\mathbf{u}_h, q_h) + d(p_h, q_h) = 0 \quad \forall q_h \in L_h. \tag{3.16}$$

Now the stability and convergence analysis of (3.15) and (3.16) can be carried out in the framework of the stabilized finite element methods.

Define a mesh-dependent norm on $\mathbf{H}_h \times L_h$ by

$$|||(\mathbf{v}_h, q_h)|||^2 = |\mathbf{v}_h|_2 + |||[q_h]|||_{0,\Gamma_h}^2,$$

where

$$|||[q_h]|||_{0,\Gamma_h}^2 = \beta \sum_{e \in \Gamma_h} h_e \int_e [q_h]_e^2 \, \mathrm{d}s.$$

Note that if $|||[q_h]|||_{0,\Gamma_h}^2 = 0$, then $q_h =$ constant on Ω. From the fact that $q_h \in L_0^2$, we have $q_h \equiv 0$.

Let us introduce a bilinear form Φ defined on $\mathbf{H}_h \times L_h$ by

$$\Phi((\mathbf{v}_h, q_h); (\mathbf{w}_h, r_h)) = a(\mathbf{v}_h, \mathbf{w}_h) - c(\mathbf{w}_h, q_h) + c(\mathbf{v}_h, r_h) + d(q_h, r_h)$$

for $(\mathbf{v}_h, q_h)$, $(\mathbf{w}_h, r_h) \in \mathbf{H}_h \times L_h$. The covolume formulation (3.15) and (3.16) can be rewritten in the following form: Find $(\mathbf{u}_h, p_h) \in \mathbf{H}_h \times L_h$ such that

$$\Phi((\mathbf{u}_h, p_h); (\mathbf{v}_h, q_h)) = (\mathbf{f}, \gamma_h \mathbf{v}_h) \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{H}_h \times L_h.$$

It is easy to see that

$$\Phi((\mathbf{v}_h, q_h); (\mathbf{v}_h, q_h)) \geqslant C|||(\mathbf{v}_h, q_h)|||^2 \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{H}_h \times L_h$$

for some positive constant C. It follows from the coercivity of the bilinear form Φ that the problem (3.15) and (3.16) is uniquely solvable.

4. Convergence analysis

We now prove the main theorem of this paper.

Theorem 4.1. *Let the primal partition family of the domain Ω be quasi-uniform, and $(\mathbf{u}_h, p_h)$ be the solution of the problem* (3.15) *and* (3.16), *and $(\mathbf{u}, p)$ solve the problem* (2.4) *and* (2.5). *Then there exists a positive constant C independent of h such that*

$$|\mathbf{u} - \mathbf{u}_h|_1 + \|p - p_h\|_0 \leqslant Ch(\|\mathbf{u}\|_2 + \|p\|_1 + 1), \tag{4.1}$$

provided that $\mathbf{u} \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)$, $p \in H^1(\Omega)$.

Proof. We first introduce an auxiliary Stokes approximation problem: Find $(\tilde{\mathbf{u}}_h, \tilde{p}_h) \in \mathbf{H}_h \times L_h$ such that

$$(\nabla \tilde{\mathbf{u}}_h, \nabla \mathbf{v}_h) - c(\mathbf{v}_h, \tilde{p}_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{H}_h, \tag{4.2}$$

$$c(\tilde{\mathbf{u}}_h, q_h) + d(\tilde{p}_h, q_h) = 0 \quad \forall q_h \in L_h. \tag{4.3}$$

This problem is the same stabilized finite element method as (2.6) and (2.7). The following convergent result for this problem is well known [7]:

$$|\mathbf{u} - \tilde{\mathbf{u}}_h|_1 + \|p - \tilde{p}_h\|_0 \leqslant Ch(\|\mathbf{u}\|_2 + \|p\|_1), \tag{4.4}$$

for some positive constant C, provided that $\mathbf{u} \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)$, $p \in H^1(\Omega)$.

Subtracting (4.3) from (3.16), we have

$$c(\mathbf{u}_h - \tilde{\mathbf{u}}_h, q_h) = -d(p_h - \tilde{p}_h, q_h) = -\beta \sum_{e \in \Gamma_h} h_e \int_e [p_h - \tilde{p}_h]_e [q_h]_e \, \mathrm{d}s \quad \forall q_h \in L_h. \tag{4.5}$$

Subtracting (4.2) from (3.15), we have

$$(\nabla \mathbf{u}_h - \nabla \tilde{\mathbf{u}}_h, \nabla \mathbf{v}_h) - c(\mathbf{v}_h, p_h - \tilde{p}_h) = (\mathbf{f}, \gamma_h \mathbf{v}_h - \mathbf{v}_h) - Q(\mathbf{u}_h, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{H}_h. \tag{4.6}$$

Define

$$\tilde{\mathbf{e}}_{u,h} := \mathbf{u}_h - \tilde{\mathbf{u}}_h, \quad \tilde{e}_{p,h} := p_h - \tilde{p}_h.$$

Replace $\mathbf{v}_h$ in (4.6) with $\tilde{\mathbf{e}}_h$ and use (4.5) to obtain

$$|\tilde{\mathbf{e}}_{u,h}|_1^2 + \beta \sum_{e \in \Gamma_h} h_e \int_e [e_{p,h}]_e^2 \, \mathrm{d}s = (\mathbf{f}, \gamma_h \tilde{\mathbf{e}}_{u,h} - \tilde{\mathbf{e}}_{u,h}) - Q(\mathbf{u}_h, \tilde{\mathbf{e}}_{u,h}). \tag{4.7}$$

Observe that

$$-Q(\mathbf{u}_h, \tilde{\mathbf{e}}_{u,h}) = -Q(\tilde{\mathbf{e}}_{u,h}, \tilde{\mathbf{e}}_{u,h}) - Q(\tilde{\mathbf{u}}_h, \tilde{\mathbf{e}}_{u,h}) \leqslant |Q(\tilde{\mathbf{u}}_h, \tilde{\mathbf{e}}_{u,h})|.$$

From the positiveness of the second part on the left-hand side of (4.7) and Lemma 3.1, we obtain

$$|\tilde{\mathbf{e}}_{u,h}|_1^2 \leqslant C_0 h \|\mathbf{f}\|_0 |\tilde{\mathbf{e}}_{u,h}|_1 + |Q(\tilde{\mathbf{u}}_h, \tilde{\mathbf{e}}_{u,h})|. \tag{4.8}$$

Let us estimate the bound for $|Q(\tilde{\mathbf{u}}_h, \tilde{\mathbf{e}}_{u,h})|$. We shall need a further partition of the domain. Divide each rectangular element K into two triangles by connecting the diagonal with positive slope. Denote the resulting triangular partition as $\mathscr{T}_h$. Let $\mathbf{u}_I$ be a piecewise linear interpolation of $\mathbf{u}$ such that $\mathbf{u}_I$ is a piecewise linear functions on each $T \in \mathscr{T}_h$. For this $\mathbf{u}_I$, it holds that $Q(\mathbf{u}_I, \mathbf{w}_h) = 0$ for all $\mathbf{w}_h \in \mathbf{H}_h$ and $|\mathbf{u} - \mathbf{u}_I|_1 \leqslant Ch\|\mathbf{u}\|_2$. By inverse estimates, we obtain

$$|Q(\mathbf{v}_h - \mathbf{u}_I, \mathbf{w}_h)| \leqslant C|\mathbf{v}_h - \mathbf{u}_I|_1 |\mathbf{w}_h|_1 \quad \forall \mathbf{v}_h, \mathbf{w}_h \in \mathbf{H}_h. \tag{4.9}$$

Using (4.9), we have

$$|Q(\tilde{\mathbf{u}}_h, \tilde{\mathbf{e}}_{u,h})| = |Q(\tilde{\mathbf{u}}_h - \mathbf{u}_I, \tilde{\mathbf{e}}_{u,h})| \leqslant C|\tilde{\mathbf{u}}_h - \mathbf{u}_I|_1 |\tilde{\mathbf{e}}_{u,h}|_1 \leqslant C(|\tilde{\mathbf{u}}_h - \mathbf{u}|_1 + |\mathbf{u} - \mathbf{u}_I|_1)|\tilde{\mathbf{e}}_{u,h}|_1 \leqslant Ch|\tilde{\mathbf{e}}_{u,h}|_1. \tag{4.10}$$

Hence we have the following estimate:

$$|\tilde{\mathbf{e}}_{u,h}|_1 \leqslant Ch, \tag{4.11}$$

for some constant C which depends on $\mathbf{f}$, $\mathbf{u}$, but not on h. The estimates (4.11), (4.4) and the triangle inequality give

$$|\mathbf{u} - \mathbf{u}_h|_1 \leqslant Ch(\|\mathbf{u}\|_2 + \|p\|_1 + 1). \tag{4.12}$$

For the estimation of the pressure, let us consider the following problem: Find $(\Psi, \chi) \in \mathbf{H}_0^1 \times L_0^2(\Omega)$ such that

$$-\Delta\Psi + \nabla\chi = 0 \quad \text{in } \Omega, \tag{4.13}$$
$$\operatorname{div}\ \Psi = e_{p,h} \quad \text{in } \Omega, \tag{4.14}$$
$$\Psi = 0 \quad \text{on } \partial\Omega. \tag{4.15}$$

Since $e_{p,h} \in L_0^2(\Omega)$, it is well known [7] that the problem (4.13)–(4.15) has a unique solution which satisfies the a priori estimate

$$|\Psi|_1 + \|\chi\|_0 \leqslant C\|e_{p,h}\|_0. \tag{4.16}$$

Let $\Psi^I \in \mathbf{H}_h$ be a piecewise bilinear interpolation of Ψ. Since $\mathscr{R}_h$ is quasi-uniform, it is easy to derive

$$\left(\sum_{e\in\Gamma_h} h_e^{-1} \int_e |(\Psi - \Psi^I)\cdot\mathbf{n}|^2\,\mathrm{d}s\right)^{1/2} \leqslant C|\Psi|_1. \tag{4.17}$$

by the inverse inequality. Then it follows from (4.14) that

$$\|e_{p,h}\|_0^2 = (\operatorname{div}\ \Psi, e_{p,h}) = \sum_e \int_e \Psi\cdot\mathbf{n}[e_{p,h}]_e\,\mathrm{d}s = \sum_e \int_e (\Psi - \Psi^I)\cdot\mathbf{n}[e_{p,h}]_e\,\mathrm{d}s + \sum_e \int_e \Psi^I\cdot\mathbf{n}[e_{p,h}]_e\,\mathrm{d}s$$
$$:= I_1 + I_2.$$

From (4.7) and (4.11), we have

$$\sum_{e\in\Gamma_h} h_e \int_e [e_{p,h}]_e^2\,\mathrm{d}s \leqslant Ch^2. \tag{4.18}$$

Using Cauchy–Schwarz inequality, (4.16)–(4.18), we obtain

$$|I_1| \leqslant \left(\sum_e h_e^{-1}\int_e |(\Psi - \Psi^I)\cdot\mathbf{n}|^2\,\mathrm{d}s\right)^{1/2}\left(\sum_e h_e \int_e [e_{p,h}]_e^2\,\mathrm{d}s\right)^{1/2} \leqslant Ch|\Psi|_1 \leqslant Ch\|e_{p,h}\|_0. \tag{4.19}$$

In order to estimate I_2, we make use of the divergence theorem

$$I_2 = \sum_{i=1}^{N_R} e_{p,h}(Q_i)\int_{K_i} \operatorname{div}\ \Psi^I\,\mathrm{d}x = c(\Psi^I, e_{p,h}).$$

From (4.6), we have

$$|I_2| = |(\nabla\tilde{\mathbf{e}}_{u,h}, \nabla\Psi^I) + Q(\tilde{\mathbf{e}}_{u,h}, \Psi^I) + Q(\tilde{\mathbf{u}}_h, \Psi^I) - (\mathbf{f}, \gamma_h\Psi^I - \Psi^I)| \leqslant C|\tilde{\mathbf{e}}_{u,h}|_1|\Psi^I|_1 + Ch|\Psi^I|_1$$
$$+ C_0 h\|\mathbf{f}\|_0|\Psi^I|_1 \leqslant Ch|\Psi^I|_1 \leqslant Ch|\Psi|_1 \leqslant Ch\|e_{p,h}\|_0. \tag{4.20}$$

Here, we used the convergence result for the velocity approximants $|\tilde{\mathbf{e}}_{u,h}|_1 \leqslant Ch$ and the estimate for $|Q(\tilde{\mathbf{u}}_h, \Psi^I)|$ similar to (4.10). Combining (4.19) with (4.20), we have

$$\|p_h - \tilde{p}_h\|_0 \leqslant Ch.$$

The triangle inequality gives

$$\|p - p_h\|_0 \leqslant Ch(\|\mathbf{u}\|_2 + \|p\|_1 + 1),$$

for some constant C which depends on $\mathbf{f}$, $\mathbf{u}$, but not on h. □

Remark 1. We can also apply the stabilized covolume method to the P_1–P_0 pair, the conforming piecewise linears and piecewise constants for triangular partitions.

Table 1
Covolume method for $\beta = 0.1$

h	$\lvert \mathbf{u}_h - \mathbf{u}^I \rvert_1$	$\lVert p - p_h \rVert_0$
1/4	2.0790×10^{-1}	6.2154×10^{-1}
1/8	8.0375×10^{-2} (2.5751)	2.7405×10^{-1} (2.2680)
1/16	2.6304×10^{-2} (3.0693)	1.2237×10^{-1} (2.2395)
1/32	8.8081×10^{-3} (3.2554)	5.7487×10^{-2} (2.1287)
1/64	2.4544×10^{-3} (3.2921)	4.6143×10^{-2} (2.0526)

Table 2
Stabilized FEM for $\beta = 0.1$

h	$\lvert \mathbf{u}_h - \mathbf{u}^I \rvert_1$	$\lVert p - p_h \rVert_0$
1/4	2.1881×10^{-1}	6.1111×10^{-1}
1/8	8.2364×10^{-2} (2.6479)	2.7158×10^{-1} (2.2502)
1/16	2.6637×10^{-2} (3.1022)	1.2195×10^{-1} (2.4044)
1/32	8.1441×10^{-3} (3.2707)	5.7425×10^{-2} (2.1236)
1/64	2.4672×10^{-3} (3.3009)	2.7999×10^{-2} (2.0501)

Table 3
Covolume method for $\beta = 0.01$

h	$\lvert \mathbf{u}_h - \mathbf{u}^I \rvert_1$	$\lVert p - p_h \rVert_0$
1/4	1.0837×10^{-1}	4.6168×10^{-1}
1/8	3.5854×10^{-2} (3.0225)	2.2433×10^{-1} (2.0580)
1/16	9.6045×10^{-3} (3.7730)	1.1099×10^{-1} (2.0212)
1/32	2.4796×10^{-3} (3.8734)	5.5315×10^{-2} (2.0065)
1/64	6.3868×10^{-4} (3.8824)	2.7631×10^{-2} (2.0019)

Table 4
Stabilized FEM for $\beta = 0.01$

h	$\lvert \mathbf{u}_h - \mathbf{u}^I \rvert_1$	$\lVert p - p_h \rVert_0$
1/4	1.2385×10^{-1}	4.5445×10^{-1}
1/8	3.9638×10^{-2} (3.1245)	2.2311×10^{-1} (2.0369)
1/16	1.0580×10^{-2} (3.7465)	1.1083×10^{-1} (2.0131)
1/32	2.7224×10^{-3} (3.8863)	5.5295×10^{-2} (2.0043)
1/64	6.9813×10^{-4} (3.8996)	2.7629×10^{-2} (2.0013)

5. Numerical results

We solve the Stokes problem on the unit square $\Omega = [0,1] \times [0,1]$ with the following exact solution

$$
\begin{aligned}
u(x,y) &= 60x^2(x-1)^2 y(y-1)(2y-1),\\
v(x,y) &= 60x(x-1)(2x-1)y^2(y-1)^2,\\
p(x,y) &= 15(x-1/2)(y-1/2).
\end{aligned}
$$

We test both the covolume method and the stabilized finite element method for Q_1–P_0.

The matrix system corresponding to these two method is of the form

$$\begin{bmatrix} A & -B^{t} \\ B & D \end{bmatrix} \tag{5.1}$$

where A is the symmetric positive matrix associated with the Laplacian, B the divergence and D is the stabilized term or artificial compressible term. Multiplying the second row of (5.1) by -1 gives the symmetric version of the system. Thus we may use the preconditioned conjugate gradient method as an iteration method.

Tables 1–4 represent the numerical results for h and various constant parameter $\beta = 0.1$, 0.01. The discrete H^1 semi-norm of the velocity are computed by $a(\mathbf{u}_h - \mathbf{u}^I, \mathbf{u}_h - \mathbf{u}^I)^{1/2}$ for the bilinear interpolation $\mathbf{u}^I$ of the exact solution $\mathbf{u}$ in case of the covolume method. In case of the stabilized finite element method, we compute $\|\nabla(\tilde{\mathbf{u}}_h - \mathbf{u}^I)\|_0$. The numbers in the parentheses stand for the convergence order at the mesh size h against $2h$. Tables 1 and 3 show the results of the covolume method while Tables 2 and 4 show that of the stabilized finite element method. The tables show that convergence orders are nearly same for the two methods.

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