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A COVOLUME METHOD BASED ON ROTATED BILINEARS FOR THE GENERALIZED STOKES PROBLEM*

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Abstract. We introduce a covolume or marker and cell (MAC) method for approximating the generalized Stokes problem on an axiparallel domain. Two grids are needed, the primal grid made up of rectangles and the dual grid of quadrilaterals. The velocity is approximated by nonconforming rotated bilinear elements with degrees of freedom at midpoints of rectangular elements and the pressure by piecewise constants. The error in the velocity in the H_h^1 norm and the pressure in the L^2 norm are of first order, provided that the exact velocity is in H^2 and the exact pressure in H^1.

Key words. covolume methods, MAC methods, saddle point problems, Stokes problem

AMS subject classifications. Primary 65N15, 65N30, 76D07; Secondary 35B45, 35J50

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1. Introduction.

The MAC (marker and cell) method of Harlow and Welch [16] on rectangular grids and its variants on unstructured grids have been very popular in the computational fluid dynamics circle. However, rigorous and systematic analysis of this class of methods are relatively scarce. Porsching [23] is one of the first papers proving convergence of the MAC method applied to linear Navier–Stokes systems. Later, Porsching [24] and Chou [5], [6], generalized the MAC scheme to two-fluid problems on flow domains with unstructured grids by the theory of network model. The emphasis of these papers was on obtaining a conservation of mass or energy through the design of primal and dual partitions. However, no convergence analysis was done for the *full* discretized systems. Another approach was taken in (Nicolaides [19], [20]) where rigorous analysis was given to the so-called covolume methods. The partitions used were the Delaunay–Voronoi mesh systems, which differ from those used in the above papers. Nicolaides' approach represents a major advance in dealing with the div-curl systems since the usual vector operators (div, curl, Laplacian, etc.) were generalized to irregular networks. See also (Choudhury and Nicolaides [11], Nicolaides and Wu [22]). As for the implementation issues resulting from his methodology, Hall et al. [14], [15] have demonstrated covolume methods can be effectively implemented by their dual variable method [1]. For the status of the covolume methods up to 1995, see the review article by Nicolaides, Porsching, and Hall [21] where the two approaches mentioned above—the network approach and the vector analysis on irregular grids—are classified under the single name, the covolume method. We shall follow this convention from now on. Some analysis of the covolume method on three-dimensional problems can be found in the above review and references therein. See also Chou and Kwak [9] where the lowest-order Raviart–Thomas space is used in the covolume mixed method for elliptic problems. A simple three-dimensional case is discussed there.

Chou [7] introduced a covolume method on unstructured triangular grids based on the first approach above and proved the convergence of the nonconforming piecewise linear velocities. More importantly from the practical viewpoint, the method

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was implemented by the augmented Lagrangian method via a network model on saddle point problems. Its corresponding method using conforming piecewise bilinear velocities on macroelements was given and analyzed in Chou and Kwak [8]. While mathematically sound (the inf-sup condition satisfied), it seems unnatural physically to let the pressure be constant over the four cells that make up a macroelement, although this practice is not uncommon in the finite element analysis (Boland and Nicolaides [3], Gunzburger [13]). Furthermore, there is no natural network model that could be derived from the conforming bilinears where velocities are basically assigned at vertices. (The network model is, among other things, to ease the construction of weakly divergence free basis.) The purpose of this paper is to introduce and analyze a nonconforming covolume method on rectangular grids where the velocity degrees of freedom can be assigned at midpoints of the common sides of rectangular elements and at the same time for which the network model approach is most natural.

Consider the generalized Stokes problem in two dimensions for steady flow of a heavily viscous fluid:

$$\alpha_0 \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega \subset R^2, \tag{1.1}$$

$$\mathrm{div} \mathbf{u} = 0 \qquad \text{in } \Omega, \tag{1.2}$$

$$\mathbf{u} = 0 \qquad \text{on } \partial\Omega \tag{1.3}$$

where $\alpha_0 \geq 0, \nu > 0$. When $\alpha_0 = 0$ we have the Stokes problem, and the case of $\alpha_0 \neq 0$ usually arises as part of the solution process for the Navier–Stokes equation (p. 359, Quarteroni and Valli [25]). We shall assume $\nu = 1$ in this paper, as $\nu \mathbf{u}$ can be used as a transformed variable. Let $H^i(\Omega), i = 1, 2$ be the usual Sobolev spaces, $L_0^2(\Omega)$ be the set of all L^2 functions over Ω with zero integral mean, and let $H_0^1(\Omega) := \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega\}$. Define the bilinear forms

$$\tilde{a}(\mathbf{u}, \mathbf{v}) := \sum_{i,j} \left(\frac{\partial u_i}{\partial x_j}, \frac{\partial v_i}{\partial x_j} \right) + \alpha_0 (\mathbf{u}, \mathbf{v}), \qquad \mathbf{u}, \mathbf{v} \in \mathbf{H}_0^1 := H_0^1(\Omega)^2 \tag{1.4}$$

$$\tilde{b}(\mathbf{v}, q) := -(q, \mathrm{div}\mathbf{v}), \qquad \mathbf{v} \in \mathbf{H}_0^1, q \in L_0^2, \tag{1.5}$$

where $(\cdot, \cdot)$ is the L^2 inner product. The weak formulation associated with (1.1)–(1.3) is to find $(\mathbf{u}, p) \in \mathbf{H}_0^1 \times L_0^2$ such that

$$\tilde{a}(\mathbf{u}, \mathbf{v}) + \tilde{b}(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0^1, \tag{1.6}$$

$$\tilde{b}(\mathbf{u}, q) = 0 \quad \forall q \in L_0^2. \tag{1.7}$$

The approximation of this system using the mixed finite element method is well documented in (Brezzi and Fortin [2]). We now describe a MAC-like method. The method is motivated by the MAC technique for incompressible flow problems and will be viewed as a Petrov–Galerkin method as far as error analysis is concerned.

First we need to partition the problem domain, which for simplicity, is assumed to be polygonal with sides parallel to the axes. Let $\mathcal{R}_h = \cup K_C$ be a partition of the domain Ω into a union of rectangular elements K_C whose barycenters are C. Here h is the maximum over all h_{K_c}, the maximal side of K_c. The nodes of an element are defined to be the midpoints of its sides. These points are represented in Fig. 1 as $P_i, i = 1, \ldots, 4$. We shall assume throughout the paper that the primal partition family $\{\mathcal{R}_h\}$ is regular; there exists a positive constant C independent of h such that

$$Ch^2 \leq |K| \leq h^2 \qquad \forall K \in \mathcal{R}_h,$$

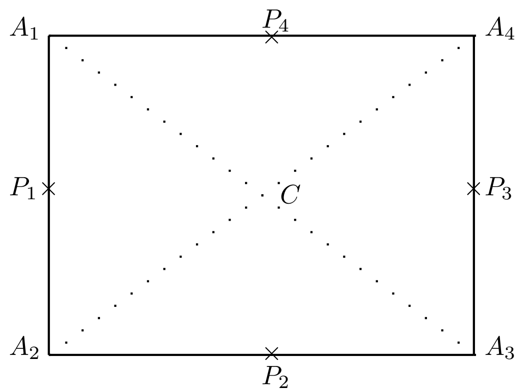

FIG. 1. *An element K, its degree of freedom, and its dual partition.*

where $|K|$ is the area of K. Denote by $\bar{\mathbf{w}}_h(P_j)$ the mean value of $\mathbf{w}_h$ over the edge e_j of K, where P_j is the midpoint of e_j, i.e.,

$$\bar{\mathbf{w}}_h(P_j) := \frac{1}{|e_j|}\int_{e_j} \mathbf{w}_h(x)dx.$$

The notation indicates the mean value is associated with the node and will be used as degrees of freedom. We shall say $\bar{\mathbf{w}}_h$ is continuous at node P_j of an edge e if K_1 and K_2 share the edge e and

$$\int_e \mathbf{w}_h|_{\partial K_1} ds = \int_e \mathbf{w}_h|_{\partial K_2} ds.$$

The *trial* function space $\mathbf{H}_h$ associated with the approximation of the fluid velocity space $\mathbf{H}_0^1$ is defined as

$$\mathbf{H}_h = \left\{ \begin{array}{l} \mathbf{v}_h : \mathbf{v}_h|_K \in Q_1^2(K) \quad \forall \text{ rectangle } K \in \mathcal{R}_h, \\ \bar{\mathbf{v}}_h = \mathbf{0} \text{ at all boundary nodes, and is continuous at all nodes} \end{array} \right\} \tag{1.8}$$

where $Q_1(K)$ denote the space of functions of the form $a + bx_1 + cx_2 + d(x_1^2 - x_2^2)$ on K. The choice of nonconforming elements is motivated by the fact that the usual conforming rectangular elements do not admit a nonzero divergence-free velocity—the so-called locking phenomenon. Furthermore, the space $Q_1(K)$, unlike the usual bilinears, can use mean values over the four edges as degrees of freedom. This space seems to be introduced and analyzed first by (Rannacher and Turek [26]) where it is termed the nonparametric rotated bilinears and later independently by (Chen [4]). Note that a function in $Q_1(K)$ is not bilinear in the variables x_1, x_2 but bilinear in the new variables $\xi := 1/\sqrt{2}(x_1 + x_2), \eta := 1/\sqrt{2}(x_1 - x_2)$. The word "rotated bilinears" is used to reflect this fact.

Next, we construct the dual partition $\mathcal{R}_h^*$ and its associated test function space. Divide each rectangle of the primal partition into four subtriangles as in Fig. 1. The dual grid is defined as a union of quadrilaterals, each of which is made up of two subtriangles. For example, in Fig. 1 the dual element based at the node P_1 is made up of the two triangles that share the edge A_1A_2 as a common base. We do the obvious modification at a boundary node. Carrying out the construction for every node in the primal partition generates a dual partition for the domain. We denote the dual element based at P as K_P^* and the dual partition as $\mathcal{R}_h^* = \cup K_P^*$. Define the associate *test function space* $\mathbf{Y}_h$ as the space of certain piecewise constant vector functions:

$$\mathbf{Y}_h = \{\mathbf{q} \in (L^2(\Omega))^2 : \mathbf{q}|_{K_P^*} \text{ is a constant vector,} \\ \text{and } \mathbf{q}|_{K_P^*} = \mathbf{0} \text{ on any boundary dual element } K_P^*\}.$$

Denote by χ_j^* the scalar characteristic function associated with the dual element $K_{P_j}^*,\quad j = 1, \dots, N_I$, where N_I is the number of interior nodes of $\mathcal{R}_h$. We see that for any $\mathbf{v}_h \in \mathbf{Y}_h$

$$\mathbf{v}_h(x) = \Sigma_{j=1}^{N_I} \mathbf{v}_h(P_j)\chi_j^*(x) \quad \forall x \in \Omega. \tag{1.9}$$

As for the approximate pressure space $L_h \subset L_0^2(\Omega)$, we define it to be the set of all piecewise constants with respect to the primal partition since in the MAC scheme the pressure was assigned at the centers of rectangular elements. Finally, note that the test function spaces are chosen to reflect the fact that in a MAC-like method the momentum (1.1) is integrated over the dual element and the continuity (1.2) over the primal element.

Define

$$a^S(\mathbf{u}_h, \mathbf{v}_h) := -\sum_{i=1}^{N_I} \int_{\partial K_{P_i}^*} \frac{\partial \mathbf{u}_h}{\partial n} \cdot \mathbf{v}_h \, d\sigma, \tag{1.10}$$

$$= -\sum_{i=1}^{N_I} \mathbf{v}_h(P_i) \cdot \int_{\partial K_{P_i}^*} \frac{\partial \mathbf{u}_h}{\partial n} \, d\sigma, \tag{1.11}$$

where $\partial \mathbf{u}_h / \partial n$ is the vector field containing two (outward) normal derivatives of $\mathbf{u}_h$. Equation (1.10) is motivated by integrating the second term of (1.1) against a test function and then formally applying the second Green's identity. Let N_R denote the number of rectangles in the primal partition.

$$a^N(\mathbf{u}_h, \mathbf{v}_h) := \alpha_0(\mathbf{u}_h, \mathbf{v}_h), \tag{1.12}$$

$$a(\mathbf{u}_h, \mathbf{v}_h) := a^S(\mathbf{u}_h, \mathbf{v}_h) + a^N(\mathbf{u}_h, \mathbf{v}_h), \tag{1.13}$$

$$b(\mathbf{v}_h, p_h) := \sum_{i=1}^{N_I} \mathbf{v}_h(P_i) \cdot \int_{\partial K_{P_i}^*} p_h \, \mathbf{n} \, d\sigma, \tag{1.14}$$

$$c(\mathbf{u}_h, q_h) := -\sum_{k=1}^{N_R} q_h(C_k) \int_{K_{C_k}} \text{div} \mathbf{u}_h \, dx, \tag{1.15}$$

$$(\mathbf{f}, \mathbf{v}) = \sum_{i=1}^{N_I} \mathbf{v}_h(P_i) \cdot \int_{K_{P_i}^*} \mathbf{f} \, dx. \tag{1.16}$$

The weak formulation of the approximate problem to (1.6)–(1.7) is to find $(\mathbf{u}_h, p_h) \in \mathbf{H}_h \times L_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \mathbf{Y}_h, \tag{1.17}$$

$$c(\mathbf{u}_h, q_h) = 0 \quad \forall q_h \in L_h. \tag{1.18}$$

Note that there are as many unknowns as equations; the number of unknowns being $2N_I + N_R$. (We did not count the zero-mean pressure condition.)

It turns out that we can reformulate this system into a saddle point problem as (1.6)–(1.7). Convergence analysis can thus be done in the frame of the nonconforming mixed finite element method. We outline how the convergence analysis is done. Introduce the one-to-one *transfer* operator γ_h from $\mathbf{H}_h$ *onto* $\mathbf{Y}_h$ by

$$\gamma_h \mathbf{u}_h(x) := \sum_{j=1}^{N_I} \bar{\mathbf{u}}_h(P_j)\chi_j^*(x) \quad \forall x \in \Omega, \tag{1.19}$$

where $\bar{\mathbf{u}}_h(P_j)$ is the average of $\mathbf{u}_h$ over the edge for which P_j is the midpoint. (The idea of using operators to connect test and trial spaces in the Petrov–Galerkin method seems to be first introduced by (Li [18]) for elliptic problems on triangular-hexagonal grids and by (Wang [27]) for the Stokes problem in the $C^2(\Omega)$ setting [17].) Define the following bilinear forms:

$$A^S(\mathbf{z}_h, \mathbf{w}_h) := a^S(\mathbf{z}_h, \gamma_h \mathbf{w}_h) \quad \forall \mathbf{z}_h, \mathbf{w}_h \in \mathbf{H}_h, \tag{1.20}$$

$$A^N(\mathbf{z}_h, \mathbf{w}_h) := a^N(\mathbf{z}_h, \gamma_h \mathbf{w}_h) \quad \forall \mathbf{z}_h, \mathbf{w}_h \in \mathbf{H}_h, \tag{1.21}$$

$$A(\mathbf{z}_h, \mathbf{w}_h) := A^S(\mathbf{z}_h, \mathbf{w}_h) + A^N(\mathbf{z}_h, \mathbf{w}_h), \tag{1.22}$$

$$B(\mathbf{w}_h, q_h) := b(\gamma_h \mathbf{w}_h, q_h) \quad \forall \mathbf{w}_h \in \mathbf{H}_h; \forall q_h \in L_h. \tag{1.23}$$

It is shown in section 2 that the two bilinear forms B and c are identical.

Thus the approximation problem (1.17)–(1.18) becomes: Find $(\mathbf{u}_h, p_h) \in \mathbf{H}_h \times L_h$ such that

$$A(\mathbf{u}_h, \mathbf{w}_h) + B(\mathbf{w}_h, p_h) = (\mathbf{f}, \gamma_h \mathbf{w}_h) \qquad \forall \mathbf{w}_h \in \mathbf{H}_h, \tag{1.24}$$

$$B(\mathbf{u}_h, q_h) = 0 \quad \forall q_h \in L_h. \tag{1.25}$$

Since $\mathbf{H}_h$ is nonconforming, the gradient and divergence operator on it must be defined piecewise:

$$(\nabla_h \mathbf{w}_h)|_K := \nabla(\mathbf{w}_h|_K),$$

$$(\operatorname{div}_h \mathbf{w}_h)|_K := \operatorname{div}(\mathbf{w}_h|_K).$$

On the space $\mathbf{H}_h$ we define

$$|\mathbf{w}_h|_{1,h}^2 := (\nabla_h \mathbf{w}_h, \nabla_h \mathbf{w}_h) = \sum_K (\nabla \mathbf{w}_h, \nabla \mathbf{w}_h)_K, \tag{1.26}$$

$$(\nabla \mathbf{w}_h, \nabla \mathbf{z}_h)_K := \sum_{i=1}^{2} (D_i \mathbf{w}_h, D_i \mathbf{z}_h)_K,$$

$$||\mathbf{w}_h||_{1,h}^2 := |\mathbf{w}_h|_{1,h}^2 + ||\mathbf{w}_h||_0^2,$$

where $(\cdot,\cdot)_K$ is the $L_2(K)^2$ inner product, and D_i denotes the partial derivatives on K. Below we shall use ∇ for ∇_h and div for div_h when there is no danger of confusion.

We derive the main error estimate result in Theorem 3.1, which states that there exists a constant $C > 0$ independent of h such that

$$||\mathbf{u}_h - \mathbf{u}||_{1,h} + ||p - p_h||_0 \le Ch(||\mathbf{u}||_2 + ||p||_1 + 1),$$

provided that $\mathbf{u} \in \mathbf{H}^2(\Omega)$ and $p \in H^1(\Omega)$ and that the primal partition family is regular. We note that the estimate is optimal both in order and in regularity; the regularity assumption on the exact pressure is minimum, most covolume methods assume $p \in H^2(\Omega)$ [20]. The basic ideas of the proof of the main theorem are as follows. In Lemma 2.3, we show that $A^S(\mathbf{z}_h, \mathbf{w}_h)$ is symmetric and equal to $(\nabla \mathbf{z}_h, \nabla \mathbf{w}_h)$. Thus we can introduce a symmetric approximation problem in [26] to the Stokes problem whose solutions are convergent to the exact solution. Comparing this nearby symmetric system with (1.24)–(1.25) then simplifies the convergence analysis since both systems are finite dimensional.

2. Saddle-point form and inf-sup condition. In this section we prove several important properties of the following bilinear forms: For $\mathbf{z}_h \in \mathbf{H}_h, \mathbf{v}_h \in \mathbf{Y}_h$ and $q_h \in L_h$,

$$a^S(\mathbf{z}_h, \mathbf{v}_h) = -\sum_{i=1}^{N_I} \mathbf{v}_h(P_i) \cdot \int_{\partial K^*_{P_i}} \frac{\partial \mathbf{z}_h}{\partial n}\, d\sigma, \tag{2.1}$$

$$b(\mathbf{v}_h, p_h) = \sum_{i=1}^{N_I} \mathbf{v}_h(P_i) \cdot \int_{\partial K^*_{P_i}} p_h\, \mathbf{n}\, d\sigma, \tag{2.2}$$

$$c(\mathbf{z}_h, q_h) = -\sum_{k=1}^{N_R} q_h(C_k) \int_{K_{C_k}} \mathrm{div} \mathbf{z}_h\, dx, \tag{2.3}$$

$$(\mathbf{f}, \mathbf{v}_h) = \sum_{i=1}^{N_I} \mathbf{v}_h(P_i) \cdot \int_{K^*_{P_i}} \mathbf{f}\, dx. \tag{2.4}$$

First, since $\mathbf{H}_h$ is not a subspace of $\mathbf{H}^1_0$, the Poincaré inequality does not come from the Sobolev space.

LEMMA 2.1. *The Poincaré inequality holds on* $\mathbf{H}_h$*: there exists a positive constant* C *independent of* h *such that*

$$||\mathbf{w}_h||_0 \le C|\mathbf{w}_h|_{1,h} \quad \forall \mathbf{w}_h \in \mathbf{H}_h.$$

Proof. For any $K_1, K_2 \in \mathcal{R}_h$ and $e := K_1 \cap K_2$

$$\int_e \mathbf{w}_h|_{K_1} ds = \int_e \mathbf{w}_h|_{K_2} ds$$

and hence by applying the integral mean value theorem to the difference of the two integrands we see that there exists $x \in e$ such that $w_h|_{\partial K_1}(x) = w_h|_{\partial K_2}(x)$, where w_h is any component of $\mathbf{w}_h$. Note that the point x needs not be the same for the two components. We will work on a typical component below. Thus on each edge e there is a w_h-continuity point and on each boundary edge there is a zero w_h-value point.

Let $x \in K$ be given. Then it is not hard to see that we can reach from x to a zero w_h-value point on the boundary by joining a sequence of w_h-continuity points. Fig. 2 gives a typical path: here the starting point is $x = x^{(0)}$, $x^{(3)}$ is a zero w_h-value point

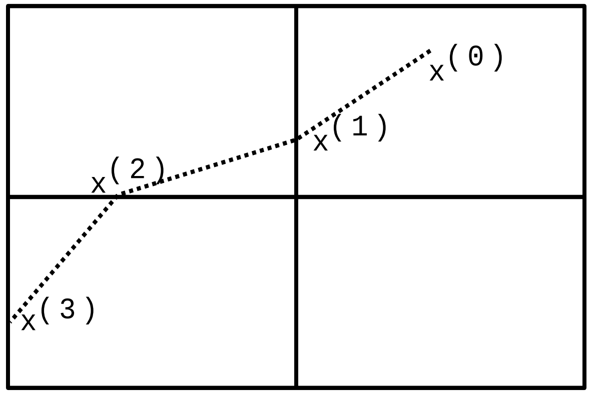

FIG. 2. *A path for Poincaré inequality.*

on the boundary, and $x^{(1)}, x^{(2)}$ are w_h-continuity point. In general, let the sequence be $\{x^{(i)}\}_{i=0}^{l}$ and let C_0 be a constant such that $l \le C_0/h$. Then using the mean value theorem and the Cauchy–Schwarz inequality, we have

$$\begin{aligned} |w_h(x)|^2 = |w_h(x^{(0)})|^2 &= |\sum_{i=0}^{l-1} \Big(w_h(x^{(i)}) - w_h(x^{(i+1)}) \Big) |^2 \\ &= |\sum_{i=0}^{l-1} \Big(\nabla w_h(\bar{x}_i)(x^{(i)} - x^{(i+1)}) \Big) |^2 \\ &\le l \sum_{i=0}^{l-1} |\nabla w_h(\bar{x}_i)|^2 (\sqrt{2}h)^2 \\ &\le C_1 l \left(\sum_{i=0}^{l-1} |\nabla w_h|^2_{0,K_i} \right), \end{aligned}$$

where $\bar{x}_i$ is some point on the line segment $x^{(i)}x^{(i+1)}$ and the last inequality is derived as follows. With $v(x) := Dw_h(x) = D_k w_h(x), k = 1$ or 2, the partial derivatives of w_h, we have by the Taylor's theorem, the fact that ∇v is constant, and the arithmetic-geometrical mean inequality

$$\int_{K_i} v^2(\bar{x}_i) \le 2 \int_{K_i} v^2(x) + |\nabla v(x)|^2 |(\bar{x}_i - x)|^2 dx.$$

Now

$$\begin{aligned} \int_{K_i} v^2(\bar{x}_i) &\le 2 \int_{K_i} v^2(x)dx + 2h^2 \int_{K_i} |\nabla v(x)|^2 dx \\ &\le 2 \int_{K_i} v^2(x)dx + 2C_3 \int_{K_i} |v(x)|^2 dx, \end{aligned}$$

where the last inequality is obtained by an inverse inequality.

Returning to the main proof, we have

$$\int_K |w_h(x)|^2 dx \le C_1 l h^2 \sum_{i=0}^{l-1} |\nabla w_h|^2_{0,K_i}.$$

As K runs horizontally, the same K_i appears at most l times, and hence using the property $lh \leq C_0$, we have

$$\int_\Omega |w_h(x)|^2 dx \leq C|w_h|^2_{1,h},$$

where $C = C_1 C_0^2$. This completes the proof. ◻

The following simple lemma of line integral conversion will be used often throughout the paper.

LEMMA 2.2. *For each element K in the primal partition, divide K into four subtriangles as shown in Fig. 1. Let g be a continuous function in the interior of each of these subtriangles. In addition, assume that the mean values of g on the boundary edges are zero. Then*

$$\sum_{i=1}^{N_I} \int_{\partial K^*_{P_i}} g(x)\, d\sigma = \sum_{K\in R_h} I_K \tag{2.5}$$

where

$$\begin{aligned} I_K &= \int_{A_1CA_4} g(x)d\sigma + \int_{A_2CA_1} g(x)d\sigma + \int_{A_3CA_2} g(x)d\sigma + \int_{A_4CA_3} g(x)d\sigma \\ &= \sum_{j=1}^{4} \int_{A_jC+CA_{j-1}} g(x)d\sigma. \end{aligned}$$

Here and below we adopt the convention $A_{j+4} = A_j, j = 0, 1, 2, 3$ when a subindex is out of bound.

Proof. The proof is straightforward. ◻

We next show that $A^S(\mathbf{z}_h, \mathbf{w}_h)$ is symmetric and is equal to the $\mathbf{H}^1$ (semi)inner product.

LEMMA 2.3. *The following facts hold.*

$$A^S(\mathbf{z}_h, \mathbf{w}_h) = A^S(\mathbf{w}_h, \mathbf{z}_h) = (\nabla \mathbf{z}_h, \nabla \mathbf{w}_h). \tag{2.6}$$

The bilinear form A of (1.22) *is bounded.*

Proof. Let h_1 and h_2 denote the width and height of a typical rectangle K, respectively. Using the divergence theorem on each subtriangle, we see by Lemma 2.2 that

$$\begin{aligned} A^S(\mathbf{z}_h, \mathbf{w}_h) &= -\sum_K \sum_{j=1}^{4} \bar{\mathbf{w}}_h(P_j) \int_{A_{j+1}CA_j} \frac{\partial \mathbf{z}_h}{\partial n} d\sigma \\ &= \sum_K \sum_{j=1}^{4} \bar{\mathbf{w}}_h(P_j) \int_{A_jA_{j+1}} \frac{\partial \mathbf{z}_h}{\partial n} d\sigma \\ &= \sum_K \sum_{j=1,2} -\bar{\mathbf{w}}(P_j) \int_{A_jA_{j+1}} \frac{\partial \mathbf{z}_h}{\partial x_j} d\sigma + \bar{\mathbf{w}}_h(P_{j+2}) \int_{A_{j+2}A_{j+3}} \frac{\partial \mathbf{z}_h}{\partial x_j} d\sigma \\ &\quad \text{(noting that } \tfrac{\partial \mathbf{z}_h}{\partial x_1} \text{ is linear in } x_1 \text{ and constant in } x_2 \text{ etc.,)} \\ &= \sum_K \sum_{j=1,2} h_k \left(\bar{\mathbf{w}}(P_{j+2}) \frac{\partial \mathbf{z}_h}{\partial x_j}(P_{j+2}) - \bar{\mathbf{w}}_h(P_j) \frac{\partial \mathbf{z}_h}{\partial x_j}(P_j) \right) \\ &\quad \text{(where } k = 2 \text{ when } j = 1 \text{ and } k = 1 \text{ when } j = 2,\text{)} \end{aligned}$$

$$
\begin{aligned}
&= \sum_K \int_K \frac{\partial \mathbf{w}_h}{\partial x_1}\frac{\partial \mathbf{z}_h}{\partial x_1}dx + \int_K \mathbf{w}_h \frac{\partial^2 \mathbf{z}_h}{\partial x_1^2}dx \\
&+ \sum_K \int_K \frac{\partial \mathbf{w}_h}{\partial x_2}\frac{\partial \mathbf{z}_h}{\partial x_2}dx + \int_K \mathbf{w}_h \frac{\partial^2 \mathbf{z}_h}{\partial x_2^2}dx \\
&= (\nabla \mathbf{z}_h, \nabla \mathbf{w}_h).
\end{aligned}
$$

The last two equalities were derived as follows. Let $Q_1 = (\cdot, x_2)$ and $Q_3 = (\cdot, x_2)$ be two arbitrary points with the same x_2 coordinate and on the same vertical edge as P_1 and P_3, respectively. Then we have, letting $\mathbf{z} = \mathbf{z}_h$,

$$\mathbf{w}_h \mathbf{z}_{x_1}|_{Q_1}^{Q_3} = \int_{Q_1Q_3} (\mathbf{w}_h \mathbf{z}_{x_1})_{x_1} dx_1.$$

Integrating along x_2 and noting that $\mathbf{z}_{x_1}$ is constant in x_2, we see the left-hand side equals

$$\mathbf{z}_{x_1}(P_3) \int_{e_3} \mathbf{w}_h dx_2 - \mathbf{z}_{x_1}(P_1) \int_{e_1} \mathbf{w}_h dx_2,$$

where e_3 and e_1 are the vertical edges to which P_3 and P_1 belong, respectively. Do the same for $\mathbf{w}_h \mathbf{z}_{x_2}|_{Q_2}^{Q_4}$. The extra terms cancel each other out because $\mathbf{z}_{x_1x_1} = -\mathbf{z}_{x_2x_2}$.

Finally, by (2.6), the Poincaré inequality, and the boundedness of γ_h, we see that the bilinear form A is bounded. ∎

We next show an approximation property of the transfer operator γ_h.

LEMMA 2.4. *There exists a positive constant C_0 independent of h and K such that*

$$||\gamma_h \mathbf{w}_h - \mathbf{w}_h||_0 \le C_0 h \, |\mathbf{w}_h|_{1,h} \quad \forall \mathbf{w}_h \in \mathbf{H}_h. \tag{2.7}$$

Proof. Divide the dual element K^* by two triangles Δ_1 and Δ_2 which share an edge of K. Then since $\gamma_h : \mathbf{H}_h(\Delta_1) \to \mathbf{Y}_h(\Delta_1)$ preserves constant functions, an application of the Bramble–Hilbert lemma [12] shows there exists a constant C independent of h such that

$$||\gamma_h \mathbf{w}_h - \mathbf{w}_h||_{0,\Delta_1}^2 \le Ch^2 \, |\mathbf{w}_h|_{1,\Delta_1}^2 \quad \forall \mathbf{w}_h \in \mathbf{H}_h(\Delta_1).$$

Now summing over all triangles yields the result. ∎

Remark 2.1. Before proving the next lemma, we want to point out one peculiarity of the space $Q_1(K) = \{a + bx_1 + cx_2 + d(x_1^2 - x_2^2)\}$. This space is *nonparametric* in the sense that there is not a single reference element for all $K \in \mathcal{R}_h$. Any two finite element spaces $(K, Q_1(K),$ mean values$)$ are not affine equivalent since $Q_1(K)$ is not preserved by affine transformations except for the translation. Consequently, in proving a statement such as "there exists a positive constant independent of h, then such and such is true," one cannot in general use the standard argument of transforming things back to a reference element. This makes the proof of the following lemma very tedious.

LEMMA 2.5. *The bilinear form A is coercive: there exists a positive constant C independent of h such that*

$$A(\mathbf{w}_h, \mathbf{w}_h) \ge C|\mathbf{w}_h|_{1,h}^2.$$

Also

$$(\mathbf{w}_h, \gamma_h \mathbf{w}_h) \ge \frac{1}{12}(\mathbf{w}_h, \mathbf{w}_h) \quad \forall \mathbf{w}_h \in \mathbf{H}_h. \tag{2.8}$$

Proof. To show the coerciveness, it suffices to prove (2.8). Let K be a rectangle centered at $(0,0)$ whose width and height are h_1 and h_2, respectively. Then referring to Fig. 1, denoting by M_i the midpoints A_iC, and using the midpoint quadrature for quadratic polynomials over triangles, we get

$$\begin{aligned} I_K &= (\mathbf{w}_h, \gamma_h \mathbf{w}_h)_K = \int_K \mathbf{w}_h \cdot \gamma_h \mathbf{w}_h dx \\ &= \textstyle\sum_{j=1}^4 \bar{\mathbf{w}}_h(P_j) \int_{\Delta A_j A_{j+1} C} \mathbf{w}_h(x) dx \\ &= \tfrac{|\Delta|}{3} \textstyle\sum_{j=1}^4 \bar{\mathbf{w}}_h(P_j)[\mathbf{w}_h(P_j) + \mathbf{w}_h(M_j) + \mathbf{w}_h(M_{j+1})]. \end{aligned}$$

With $\mathbf{w}_h = a + bx_1 + cx_2 + d(x_1^2 - x_2^2), a, b, c, d \in R^2$, we get after some lengthy calculations that

$$I_K = \frac{h_1 h_2}{12}\left(12a^2 + 3ad(h_1^2 - h_2^2) + \frac{d^2}{24}(5h_1^4 - 6h_1^2h_2^2 + 5h_2^4) + b^2h_1^2 + c^2h_2^2\right).$$

Now we compute $(\mathbf{w}_h, \mathbf{w}_h)_K$. By the symmetry of some of the integrands with respect the origin, we see that

$$\begin{aligned} \int_K \mathbf{w}_h^2 dx &= \int_K \left(a^2 + b^2x_1^2 + c^2x_2^2 + d^2(x_1^4 - 2x_1^2x_2^2 + x_2^4) + 2ad(x_1^2 - x_2^2)\right) dx_1 dx_2 \\ &= \frac{h_1h_2}{12}\left(12a^2 + b^2h_1^2 + c^2h_2^2 + d^2\left[\frac{3}{20}h_1^4 - \frac{h_1^2h_2^2}{6} + \frac{3}{20}h_2^4\right] + 2ad(h_1^2 - h_2^2)\right). \end{aligned}$$

Thus

$$\begin{aligned} \frac{12}{h_1h_2}\left[12(\mathbf{w}_h, \gamma\mathbf{w}_h)_K - (\mathbf{w}_h, \mathbf{w}_h)_K\right] &\geq 132a^2 + 34ad(h_1^2 - h_2^2) \\ &\quad + d^2\left[\frac{47h_1^4}{20} - \frac{17h_1^2h_2^2}{6} + \frac{47h_2^4}{20}\right], \end{aligned}$$

which as a quadratic form in a, d is nonnegative since the discriminant is negative. Hence $12(\mathbf{w}_h, \gamma\mathbf{w}_h)_K - (\mathbf{w}_h, \mathbf{w}_h)_K$ is nonnegative. ◻

LEMMA 2.6.

$$B(\mathbf{w}_h, q_h) = b(\gamma_h \mathbf{w}_h, q_h) = c(\mathbf{w}_h, q_h) \qquad \forall \mathbf{w}_h \in \mathbf{H}_h, q_h \in L_h.$$

Proof. Using the divergence theorem on each dotted subtriangles of K, we have

$$B(\mathbf{w}_h, q_h) = \sum_K I_K,$$

where

$$\begin{aligned} I_K &= \sum_{j=1}^4 \bar{\mathbf{w}}_h(P_j) \int_{A_{j+1}CA_j} q_h \mathbf{n} d\sigma \\ &= -\sum_{j=1}^4 q_h \int_{A_jA_{j+1}} \bar{\mathbf{w}}_h(P_j) \cdot \mathbf{n} d\sigma \\ &= -\sum_{j=1}^4 q_h \int_{A_jA_{j+1}} \mathbf{w}_h \cdot \mathbf{n} d\sigma \\ &= -\int_{\partial K} q_h \mathbf{w}_h \cdot \mathbf{n} d\sigma \\ &= -\int_K q_h \operatorname{div} \mathbf{w}_h dx = c(\mathbf{w}_h, q_h)|_K. \end{aligned}$$ ◻

The following inf-sup condition is proved in [26].

LEMMA 2.7. *There exists a positive constant β independent of h such that*

$$\sup_{\mathbf{w}_h \neq \mathbf{0}} \frac{B(\mathbf{w}_h, q_h)}{|\mathbf{w}_h|_{1,h}} \geq \beta ||q_h||_0. \tag{2.9}$$

3. Error estimates. We now prove the main theorem of this paper.

THEOREM 3.1. *Let the rectangular partition family $\mathcal{R}_h$ of the domain Ω be regular, let $\{\mathbf{u}_h, p_h\}$ be the solution of the problem* (1.24)–(1.25)*, and $\{\mathbf{u}, p\}$ be the solution of the problem* (1.6)–(1.7)*. Then there exists a positive constant C independent of h such that*

$$|\mathbf{u} - \mathbf{u}_h|_{1,h} + ||p - p_h||_0 \leq Ch(||\mathbf{u}||_2 + ||p||_1 + 1), \tag{3.1}$$

and

$$||\mathbf{u} - \mathbf{u}_h||_0 \leq Ch(||\mathbf{u}||_2 + ||p||_1 + 1), \tag{3.2}$$

provided that $\mathbf{u} \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega), p \in H^1(\Omega)$.

Proof. Lemmas 2.3 and 2.5 guarantee the existence and uniqueness of the solution $\{\mathbf{u}_h, p_h\}$. We first introduce an auxiliary symmetric Stokes approximation problem to (1.6)–(1.7): Find $(\tilde{\mathbf{u}}_h, \tilde{p}_h) \in \mathbf{H}_h \times L_h$ such that

$$(\nabla \tilde{\mathbf{u}}_h, \nabla \mathbf{w}_h) + \alpha_0(\tilde{\mathbf{u}}_h, \mathbf{w}_h) + B(\mathbf{w}_h, \tilde{p}_h) = (\mathbf{f}, \mathbf{w}_h) \quad \forall \mathbf{w}_h \in \mathbf{H}_h, \tag{3.3}$$

$$B(\tilde{\mathbf{u}}_h, q_h) = 0 \quad \forall q_h \in L_h. \tag{3.4}$$

This scheme has been studied in (p. 102, [26]), and we have the following convergence result:

$$|\mathbf{u} - \tilde{\mathbf{u}}_h|_{1,h} + \alpha_0^{1/2} ||\mathbf{u} - \tilde{\mathbf{u}}_h||_0 + ||p - \tilde{p}_h||_0 \leq Ch(||\mathbf{u}||_2 + ||p||_1), \tag{3.5}$$

provided that $\mathbf{u} \in \mathbf{H}^2(\Omega), p \in H^1(\Omega)$. It should be pointed out that (3.5) is shown in [26] only for the case of $\alpha_0 = 0$, but their proof carries over easily if one replaces their $a_h(\mathbf{v}, \mathbf{w})$ (which equals our $A^s(\mathbf{v}, \mathbf{w})$) with $A^s(\mathbf{v}, \mathbf{w}) + \alpha_0(\mathbf{v}, \mathbf{w})$, bearing in mind that the Poincaré inequality is satisfied. On the other hand,

$$A^S(\mathbf{u}_h, \mathbf{w}_h) + \alpha_0(\mathbf{u}_h, \gamma_h \mathbf{w}_h) + B(\mathbf{w}_h, p_h) = (\mathbf{f}, \gamma_h \mathbf{w}_h) \quad \forall \mathbf{w}_h \in \mathbf{H}_h, \tag{3.6}$$

$$B(\mathbf{u}_h, q_h) = 0 \quad \forall q_h \in L_h. \tag{3.7}$$

Subtracting (3.4) from (3.7) gives

$$B(\mathbf{u}_h - \tilde{\mathbf{u}}_h, q_h) = 0 \quad \forall q_h \in L_h. \tag{3.8}$$

Using Lemma 2.3 and subtracting (3.3) from (3.6) gives

$$\begin{aligned} (\nabla(\mathbf{u}_h - \tilde{\mathbf{u}}_h), \nabla \mathbf{w}_h) + \alpha_0(\mathbf{u}_h, \gamma_h \mathbf{w}_h) - \alpha_0(\tilde{\mathbf{u}}_h, \mathbf{w}_h) + B(\mathbf{w}_h, p_h - \tilde{p}_h) \\ = (\mathbf{f}, \gamma_h \mathbf{w}_h) - (\mathbf{f}, \mathbf{w}_h) \quad \forall \mathbf{w}_h \in \mathbf{H}_h. \end{aligned} \tag{3.9}$$

Define

$$\tilde{\mathbf{e}}_h := \mathbf{u}_h - \tilde{\mathbf{u}}_h.$$

Replace the $\mathbf{w}_h$ in (3.9) by $\tilde{\mathbf{e}}_h$ and use (3.8) to obtain

$$|\tilde{\mathbf{e}}_h|_{1,h}^2 + \alpha_0(\tilde{\mathbf{e}}_h, \gamma_h \tilde{\mathbf{e}}_h) = (\mathbf{f}, \gamma_h \tilde{\mathbf{e}}_h - \tilde{\mathbf{e}}_h) + \alpha_0(\tilde{\mathbf{u}}_h, \tilde{\mathbf{e}}_h - \gamma_h \tilde{\mathbf{e}}_h). \tag{3.10}$$

Using (2.8) on the second term of the left-hand side, Lemma 2.4, and $||\tilde{\mathbf{u}}_h||_0 \leq M$, we obtain

$$|\tilde{\mathbf{e}}_h|_{1,h}^2 \leq ||\mathbf{f}||_0 C_0 h |\tilde{\mathbf{e}}_h|_{1,h} + C_0 \alpha_0 M h |\tilde{\mathbf{e}}_h|_{1,h}, \tag{3.11}$$

or

$$|\tilde{\mathbf{e}}_h|_{1,h} \leq Ch. \tag{3.12}$$

We can use the inf-sup condition on (3.9) and the same techniques as above to derive

$$||p_h - \tilde{p}_h||_0 \leq C_1 h.$$

An application of the triangle inequality then proves (3.1). As for the L^2-estimate (3.2) we proceed as follows. If $\alpha_0 \neq 0$ then (2.8), (3.10), and (3.12) imply

$$C_1 \alpha_0 ||\tilde{\mathbf{e}}_h||_0^2 \leq \alpha_0(\tilde{\mathbf{e}}_h, \gamma_h \tilde{\mathbf{e}}_h) \leq C_2 h |\tilde{\mathbf{e}}_h|_{1,h} \leq C_3 h^2,$$

and, hence,

$$||\tilde{\mathbf{e}}_h||_0 \leq Ch, \tag{3.13}$$

which upon combining with (3.5) gives (3.2).

If $\alpha_0 = 0$ then setting $\mathbf{z}_h = \tilde{\mathbf{e}}_h$ in Lemma 2.1 and using (3.12) derive the last inequality again. However, this time we cannot use (3.5) directly. Now, let $\pi_h \mathbf{u} \in \mathbf{H}_0^1$ be the piecewise continuous bilinear interpolant of $\mathbf{u}$. Using the Poincaré inequality, approximation properties of the interpolant, and (3.5), we have

$$\begin{aligned} ||\mathbf{u} - \tilde{\mathbf{u}}_h||_0 &\leq ||\mathbf{u} - \pi_h \mathbf{u}||_0 + ||\pi_h \mathbf{u} - \tilde{\mathbf{u}}_h||_0 \\ &\leq ||\mathbf{u} - \pi_h \mathbf{u}||_0 + C_4 |\pi_h \mathbf{u} - \tilde{\mathbf{u}}_h|_{1,h} \\ &\leq ||\mathbf{u} - \pi_h \mathbf{u}||_0 + C_4 |\pi_h \mathbf{u} - \mathbf{u}|_{1,h} + C_4 |\mathbf{u} - \tilde{\mathbf{u}}_h|_{1,h} \\ &\leq (C_5 h^2 + C_6 h) ||\mathbf{u}||_2 + C_4 |\mathbf{u} - \tilde{\mathbf{u}}_h|_{1,h} \\ &\leq C_7 h (||\mathbf{u}||_2 + ||p||_1). \end{aligned}$$

A simple application of the triangle inequality now proves (3.2). ☐

Remark 3.1. Note that we can symmetrize the problem (1.24)–(1.25) by replacing $(\gamma_h \mathbf{v}_h, \mathbf{w}_h)$ by $1/2[(\gamma_h \mathbf{v}_h, \mathbf{w}_h) + (\mathbf{v}_h, \gamma_h \mathbf{w}_h)]$ and still obtain the same optimal order error estimate in the $\mathbf{H}_h$ norm in the above theorem.

Remark 3.2. Estimate (3.2) is certainly not optimal in the L^2 norm. Applying the Aubin–Nitzche technique to the covolume method to prove optimal order L^2 estimate is not a trivial matter. At the present time, it is not clear whether (3.2) can be improved. The reader, however, is referred to (Chou and Li [10]) for a successful application of the Aubin–Nitzche technique to elliptic problems and related issues.

Remark 3.3. The present covolume method should be implemented as a Petrov–Galerkin method. In this paper, through the introduction of the transfer operator γ_h we showed that it is closely related to the method (3.3)–(3.4) originally proposed in [26]. When $\alpha_0 = 0$ it differs only in the right-hand sides—cf., the right-hand sides of (3.3) and (3.6). Hence, in this special case the covolume formulation can be viewed as a variant of (3.3)–(3.4) via a "variational crime." Of course, this surprising result is due to Lemma 2.3. However, when it comes to implementation, the covolume formulation we prefer to use is really (1.17)–(1.18). There are two reasons for this.

One is that in engineering applications these two equations correspond directly to conservation of momentum and mass and convention dictates the direct use of them. Note that the bilinear forms a and b involve only line integrals. The second reason is this. As pointed out in Remark 2.1 there does not exist a single reference element for the rotated bilinears, so they are somewhat contradictory to the finite element methodology as far as the implementation issue is concerned. In the evaluation of the bilinear forms for a finite element method, all integrals should be expressed in terms of a common reference element.

Remark 3.4. There is a natural network interpretation for the saddle point problem (3.6)–(3.7) using the arguments in (Chou [7]). We will not repeat it here.

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