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Multigrid algorithms for a vertex–centered covolume method for elliptic problems

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Summary. We analyze V–cycle multigrid algorithms for a class of perturbed problems whose perturbation in the bilinear form preserves the convergence properties of the multigrid algorithm of the original problem. As an application, we study the convergence of multigrid algorithms for a covolume method or a vertex–centered finite volume element method for variable coefficient elliptic problems on polygonal domains. As in standard finite element methods, the V–cycle algorithm with one pre-smoothing converges with a rate independent of the number of levels. Various types of smoothers including point or line Jacobi, and Gauss-Seidel relaxation are considered.

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1 Introduction

The purpose of this paper is to analyze some multigrid algorithms for solving perturbed equations arising from discretizing second order elliptic problems by a nonstandard method. One such method is the covolume method. Covolume methods are efficient and popular finite volume methods for the discretization of PDEs governing fluid flows in the CFD circle [10, 21, 15, 16] due mainly to its simplicity and local conservation properties. The covolume method used in this paper can also be viewed as a vertex centered finite volume element method or a generalized finite difference method in some old literature.

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In [12], a general theory is developed for analyzing covolume methods as a finite element method resulting from variational crimes. In other words, each covolume method can be obtained from a nearby finite element method by adding small perturbation terms to the left side bilinear form and the right side linear functional corresponding to the weak formulation of the underlying second order elliptic problem. This also opens up the possibility of analyzing and design multigrid algorithms along the same line of approach. We carry out this program below. The mathematical analyses of covolume methods applied to other problems such as the Stokes problems and convection-diffusion equations can be found in [11, 13, 14]. An efficient way of solving algebraic systems in covolume methods for Navier-Stokes equations is the dual variable method (DVM) introduced by Porsching and studied in [15, 16].

For general elliptic problems the multigrid method has proven to be robust and effective in conjunction with the finite element method. Although the convergence behavior of multigrid algorithms for standard finite element methods is by now well understood [1, 17, 2, 5, 6], much less is known for the behavior of multigrid algorithms for covolume or finite volume methods. For cell centered finite difference methods on rectangular or triangular meshes, see [3, 19, 20]. In this paper we present a rather general theory for the convergence of multigrid algorithms for perturbed problems(even when the underlying bilinear form is not variational) and show that covolume method fits into the general framework. Thus, for the present covolume method (a vertex-centered finite volume method) [12], the simplest V–cycle multigrid algorithm converges with a rate independent of the number of levels, as long as the coarsest grid is sufficiently small. The requirement for small coarse grid is necessary for the covolume method to make sense. Yet, it does not affect the rate of convergence of the multigrid algorithm. Our framework can be used to show the convergence of other perturbed problems such as diffusion dominated convection-diffusion problem. The rest of our paper is organized as follows: In Sect. 2, we present a general framework which can handle a perturbation of standard variational problems and prove the V–cycle multigrid convergence of the perturbed problem under certain reasonable assumptions. In Sect. 3, we introduce a covolume method for elliptic problems and show that the assumptions of Sect. 2 are satisfied, thus establishing the V–cycle convergence. Two most important classes of smoothers, Jacobi type and Gauss–Seidel type, are analyzed.

2 Multigrid algorithms

Let $U_1 \subset U_2 \subset \cdots \subset U_J$ be a sequence of nested finite dimensional subspaces of a Hilbert space H and let $a(\cdot, \cdot)$ be a symmetric, coercive bilinear form on $H \times H$. Consider the following problem: Find $u \in H$ such that

$$a(u, v) = (f, v), \quad \forall v \in H \tag{2.1}$$

where f is a bounded linear functional on H. The discrete problem corresponding to (2.1) is: Find $u_J \in U_J$ such that

$$a(u_J, v) = (f, v), \quad \forall v \in U_J. \tag{2.2}$$

Let $A_k (k = 1, \cdots, J)$ be the matrix representations of the form $a(\cdot, \cdot)$ on $U_k \times U_k$ with respect to a certain discrete inner product $(\cdot, \cdot)_k$. Define $P_{k-1} : U_k \to U_{k-1}$by

$$(A_{k-1} P_{k-1} w, v)_{k-1} = (A_k w, I_k v)_k, \text{ for all } v, w \in U_{k-1}, \tag{2.3}$$

where $I_k : U_{k-1} \to U_k$ is the natural injection operator. The restriction operator $P^0_{k-1} : U_k \to U_{k-1}$ is defined by

$$(P^0_{k-1} w, v)_{k-1} = (w, I_k v)_k, \text{ for all } v, w \in U_{k-1}.$$

Now the discretized equation (2.1) can be rewritten in the above notation as

$$A_J u_J = f_J, \tag{2.4}$$

where f_J is the vector representation of f.

We now describe the V–cycle multigrid algorithm for iteratively computing the solution u_J of (2.4). Let R_k be any smoothing operator such as Jacobi or Gauss-Seidel. Then the V–cycle algorithm is defined as follows:

Multigrid Algorithm 2.1. Set $B_1 = A_1^{-1}$. For $k \geq 2$ define $B_k : U_k \to U_k$ in terms of B_{k-1} as follows. Let $g \in U_k$.

1. Set $x^0 = 0$.
2. Define x^l for $l = 1, \ldots, m$ by
$$x^l = x^{l-1} + R_k(g - A_k x^{l-1}).$$
3. Set $B_k g = x^m + q$ where q is defined by
$$q = B_{k-1} P^0_{k-1}(g - A_k x^m).$$

Here we smooth as we go down the coarser levels. Also, the number of smoothings m can vary from level to level, although $m = 1$ for all k suffices in our analysis. The case $m1$ or the case with post–smoothing can be analyzed similarly.

From the definition of P_{k-1}, it is straightforward to check that

$$P^0_{k-1} A_k = A_{k-1} P_{k-1}. \tag{2.5}$$

Let $K_k = I - R_k A_k$. Then $K_k x = x - x^1$. Now for $x \in U_k$, $k = 1, \cdots, J$, we have

$$\begin{aligned}(I - B_k A_k)x &= x - x^1 - q \\ &= K_k x - B_{k-1} A_{k-1} P_{k-1} K_k x \\ &= [I - B_{k-1} A_{k-1} P_{k-1}] K_k x \\ &= [(I - P_{k-1}) + (I - B_{k-1} A_{k-1}) P_{k-1}] K_k x.\end{aligned}$$

The convergence results of the multigrid method will be expressed in terms of the error operators $E_k := I - B_k A_k$ and $E := E_J$. Throughout the paper, C denotes a generic constant independent of k and can have different values in different places, unless otherwise stated.

The convergence theory of multigrid algorithms associated with self-adjoint elliptic problems is well-established in [1,17,2,5,7]. In particular, for the V–cycle convergence which is closely related to our presentation, see [2,5,7,4]. Also, when the matrix A_k is sparse, symmetric positive definite, the convergence analysis can be carried out algebraically under mild assumptions[7]. For the convergence analysis of the multigrid algorithm, one needs to impose some conditions on the smoothers, i.e.,

1. There is a constant C_R such that

$$(C.1) \qquad \frac{(u,u)_k}{\lambda_k} \le C_R(\bar{R}_k u, u)_k, \quad \text{for all } u \in U_k,$$

   where $\bar{R}_k = (I - K_k^a K_k) A_k^{-1}$ and λ_k is the largest eigenvalue of A_k. Here the superscript "a" denotes the adjoint with respect to the inner product $a(\cdot,\cdot)$.
2. Let $T_k = (I - K_k) P_k$. There exists a positive constant $\theta < 2$ independent of k such that

$$(C.2) \qquad a(T_k v, T_k v) \le \theta a(T_k v, v) \quad \forall v \in U_k.$$

   Under these assumptions, the following result holds [6,7].

Theorem 2.1. *[7] Let U_k be the usual conforming finite element space and let R_k be any smoother satisfying (C.1) and (C.2). Then there exists a $\delta < 1$ such that the following estimate holds.*

$$(2.6) \qquad \|Ew\|_a \le \delta \|w\|_a \quad \forall w \in U_J,$$

where the energy norm is defined by $\|v\|_a^2 := a(v,v)$.

V–cycles for a perturbed system of the standard Galerkin FEM. In anticipation of the fact (cf. Section three) that the covolume method can be viewed as committing a variational crime on the standard Galerkin method (3.7), let us consider its perturbed problem: Find $u_J^* \in U_J$ such that

$$a_J^*(u_J^*, v) = f_J(v), \text{ for all } v \in U_J, \tag{2.7}$$

where for each level $k = 1, \cdots, J$, $a_k^*(\cdot, \cdot)$ is a possibly non–symmetric bilinear form defined on $U_k \times U_k$ and f_k is a bounded linear form on U_k. Let A_k^* be the matrix representation of a_k^* with respect to $(\cdot, \cdot)_k$. Then, in parallel with $a(\cdot, \cdot)$ form, we define R_k^* and K_k^* using A_k^* in place of A_k. Also, define $P_{k-1}^* : U_k \to U_{k-1}$ by

$$a_{k-1}^*(P_{k-1}^* w, v) = a_k^*(w, I_k v) \quad \text{for all } w \in U_k, v \in U_{k-1}. \tag{2.8}$$

Then the following relation holds.

$$P_{k-1}^0 A_k^* = A_{k-1}^* P_{k-1}^*. \tag{2.9}$$

The a_k^*'s are considered as departures of a_k's in the sense that the two conditions $(P.1)$ and $(P.2)$ below are satisfied. Denote the difference between the two bilinear forms as

$$d_k(w, v) = a(w, v) - a_k^*(w, v) \qquad w, v \in U_k$$

and assume the perturbation condition

$$|d_k(w, v)| \le C h_k \|w\|_a \|v\|_a, \quad w, v \in U_k, \tag{$P.1$}$$

where $h_k = 2^{1-k} h_1$ and h_1 is certain parameter to be specified later.

The next condition we assume is that

$$|a((K_k - K_k^*) w, v)| \le C h_k \|w\|_a \|v\|_a, \quad w, v \in U_k. \tag{$P.2$}$$

The multigrid algorithm for (2.7) is the same as Algorithm 2.1 except that A_k, B_k and R_k are replaced by their "*" counterpart. Hence we shall not repeat it here.

Now for a linear operator $T : U_k \to U_k$, let $\|T\|_a$ denote the operator norm induced by the bilinear form $a(\cdot, \cdot)$:

$$\|T\|_a := \sup_{w, v \in U_k} \frac{a(Tw, v)}{\|w\|_a \|v\|_a}. \tag{2.10}$$

Let $E_k^* = I - B_k^* A_k^*$ and $E^* = E_J^*$, the error operators corresponding to the above modified V–cycle scheme. It is reasonable to expect under the conditions $(P.1)$ and $(P.2)$ that $\|E_k^* - E_k\|_a \le C h_1$ and hence for sufficiently small h_1 the corresponding modified V–cycle converges uniformly in the level k. The remainder of this section will make this rigorous (cf. Theorem 2.2 below).

Note that condition $(P.1)$ implies the coercivity of a_k^* with the coercivity constant independent of k if the parameter h_k's are small enough. Hence the operator P_{k-1}^* is well defined. Also, if we let $v = P_{k-1}^* w$, then by the coercivity of a_{k-1}^*, we have

$$\begin{aligned}\alpha\|v\|_a^2 &\le a_{k-1}^*(v,v) = a_k^*(w, I_k v)\\ &\le M\|w\|_a\|v\|_a.\end{aligned}$$

Thus, P_{k-1}^* is stable in the energy norm, a fact that will be used to obtain a more accurate bound for $\|P_{k-1}^*\|_a$.

Lemma 2.1. *If $(P.1)$ holds, then for $w \in U_k, v \in U_{k-1}$*

$$|a(P_{k-1}^* w, v) - a(P_{k-1} w, v)| \le Ch_k\|w\|_a\|v\|_a, \tag{2.11}$$

Proof. We have by $(P.1)$ and the stability of P_{k-1}^*,

$$\begin{aligned}|a(P_{k-1}^* w, v) - a(P_{k-1} w, v)| &= |a(P_{k-1}^* w, v) - a(w, I_k v)|\\ &= |a(P_{k-1}^* w, v) - a_k^*(w, I_k v)\\ &\quad + a_k^*(w, I_k v) - a(w, I_k v)|\\ &= |a(P_{k-1}^* w, v) - a_{k-1}^*(P_{k-1}^* w, v)\\ &\quad + a_k^*(w, I_k v) - a(w, I_k v)|\\ &= |d_{k-1}(P_{k-1}^* w, v) - d_k(w, I_k v)|\\ &\le Ch_k\|w\|_a\|v\|_a, \end{aligned}\tag{2.12}$$

□

Thus

$$\|P_{k-1}^* - P_{k-1}\|_a \le Ch_k \tag{2.13}$$

and hence using the fact $\|P_{k-1}\|_a \le 1$ we have

$$\begin{aligned}\|P_{k-1}^*\|_a &\le \|P_{k-1}\|_a + \|P_{k-1}^* - P_{k-1}\|_a\\ &\le 1 + Ch_k.\end{aligned}\tag{2.14}$$

We now prove the main theorem of this section.

Theorem 2.2. *Assume the multigrid algorithm for the original problem (2.1) has the convergence property given in Theorem 2.1. Suppose that $(P.1)$ holds and that R_k^* is a smoother satisfying $(P.2)$. Then there exists an h_0 such that for all $h_1 < h_0$,*

$$\|E^* w\|_a \le \delta^*\|w\|_a \quad \forall w \in U_J, \tag{2.15}$$

where $\delta^* = \delta + ch_1 < 1$ and δ is as in Theorem 2.1.

Proof. We compare the error operators and write their difference as

$$E_k^* - E_k = (I - B_{k-1}^* A_{k-1}^* P_{k-1}^*)(K_k^* - K_k)$$
$$-B_{k-1}^* A_{k-1}^* (P_{k-1}^* - P_{k-1}) K_k - (E_{k-1}^* - E_{k-1}) P_{k-1} K_k.$$

Thus in terms of the operator norm in (2.10) we have

$$\|E_k^* - E_k\|_a \le \|I - B_{k-1}^* A_{k-1}^* P_{k-1}^*\|_a \|K_k^* - K_k\|_a \tag{2.16}$$
$$+\|B_{k-1}^* A_{k-1}^*\|_a \|P_{k-1}^* - P_{k-1}\|_a \|K_k\|_a \tag{2.17}$$
$$+\|E_{k-1}^* - E_{k-1}\|_a \|P_{k-1} K_k\|_a. \tag{2.18}$$

We shall show that $\|E_k^* - E_k\|_a \le c_k h_1$. For this purpose, let us make the induction hypothesis: $\|E_{k-1}^* - E_{k-1}\|_a \le c_{k-1} h_1$, where c_{k-1} a constant independent of k to be defined below. By the triangle inequality and Theorem 2.1

$$\|E_{k-1}^*\|_a \le \delta + c_{k-1} h_1 \tag{2.19}$$

and

$$\|B_{k-1}^* A_{k-1}^*\|_a \le 1 + \delta + c_{k-1} h_1. \tag{2.20}$$

We see by the induction hypothesis, (2.13), and (2.14)

$$\|I - B_{k-1}^* A_{k-1}^* P_{k-1}^*\|_a \le \|I - P_{k-1}^*\|_a + \|I - B_{k-1}^* A_{k-1}^*\|_a \|P_{k-1}^*\|_a$$
$$\le (1 + C h_{k-1}) + \|E_{k-1}^*\|_a (1 + C h_{k-1})$$
$$\le (1 + C h_{k-1})(1 + \delta + c_{k-1} h_1). \tag{2.21}$$

Collecting (2.16) through (2.18), and using $(P.2)$, (2.13), (2.21), we see that

$$\begin{aligned}\|E_k^* - E_k\|_a &\le C(1 + C h_{k-1})(1 + \delta + c_{k-1} h_1) h_k \\ &\quad + C(1 + \delta + c_{k-1} h_1) h_{k-1} + c_{k-1} h_1 \\ &\le C h_{k-1} (1 + \delta + c_{k-1} h_1)[\frac{(1 + C h_{k-1})}{2} + 1] + c_{k-1} h_1 \\ &\le C M h_{k-1} (1 + \delta + c_{k-1} h_1) + c_{k-1} h_1,\end{aligned}$$

where M is a constant such that $[\frac{(1+Ch_{k-1})}{2} + 1] \le M$ for all k. Let $\hat{C} := CM$ and in view of the above all we have to do is define

$$c_k := c_{k-1} + \hat{C} h_1^{-1} h_{k-1} (1 + \delta + c_{k-1} h_1). \tag{2.22}$$

To see that the sequence c_k is uniformly bounded in k, one notes that $c_j \le c_k$ for $j \le k$ and hence

$$\begin{aligned}c_k &= c_{k-1} + \hat{C} h_1^{-1} (1 + \delta + c_{k-1} h_1) h_{k-1} \\ &= c_1 + \hat{C} h_1^{-1} \sum_{j=2}^{k} (1 + \delta + c_{j-1} h_1) h_{j-1}\end{aligned}$$

$$\leq c_1 + \hat{C}h_1^{-1} \sum_{j=2}^{k} (1 + \delta + c_k h_1) h_{j-1}$$
$$\leq c_1 + 2\hat{C}(1+\delta) + 2\hat{C}h_1 c_k.$$

Now move the c_k term to the left to get

$$c_k \leq (c_1 + 2\hat{C}(1+\delta))/(1 - 2\hat{C}h_1)$$

provided that h_1 is small enough or equivalently the coarsest grid is sufficiently fine.

Now the theorem follows from Theorem 2.1 by the triangle inequality. □

Remark 2.1. Note that $a_k^*(\cdot,\cdot)$ in the above theorem is only defined on $U_k \times U_k$. If, for all k, $a_k^*(\cdot,\cdot) = a^*(\cdot,\cdot)$ for some common a^* as in the Galerkin approach, in reminiscence of [4] one might impose, instead of the condition $(P.1)$, the weaker conditions:

$$(P.1') \qquad \begin{array}{ll} |a(w,v) - a^*(w,v)| \leq C\|w\|_0\|v\|_a, & w, v \in U_k \\ |a(w,v) - a^*(w,v)| \leq C\|w\|_a\|v\|_0, & w, v \in U_k. \end{array}$$

In this case, we have

$$d_k(\cdot,\cdot) = d(\cdot,\cdot), \quad k = 1, \cdots, J$$

and if we can show that (2.11) holds, we would obtain convergence analysis again. But it is obvious that condition $(P.1')$ alone will not even guarantee that the operator P_k^* is well defined. In view of (2.8), one needs to impose that

$$(2.23) \qquad a_k^*(v,w) = 0 \quad \forall w \in U_k \text{ implies that } v = 0.$$

Under this condition, we have

$$\begin{aligned} |a(P_{k-1}^*w, v) - a(P_{k-1}w, v)| &= |d(P_{k-1}^*w, v) - d(w,v)| \\ &= |d((I - P_{k-1}^*)w, v)| \\ &\leq C\|(I - P_{k-1}^*)w\|_0\|v\|_a \\ &\leq Ch_k\|w\|_a\|v\|_a, \end{aligned}$$

provided that in the last step one can have a first order projection error estimate for P_{k-1}^*. This whole framework is very similar to one in [4]. However, for our main result to be presented below, we need the stronger assumption $(P.1)$ because the difference operators $d_k(\cdot,\cdot)$ at different levels are incomparable.

3 Application of perturbation analysis to covolume methods

In this section we apply the result obtained in the previous section to the covolume method. Now let us first describe standard finite element method for an elliptic problem.

Let Ω be a convex polygonal domain in R^2 with boundary $\partial\Omega$ and consider the general self-adjoint second order elliptic problem

$$Lu := -\sum_{i,j}^{2} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j}) + qu = f \qquad x \in \Omega, \tag{3.1}$$

$$u = 0, \qquad x \in \partial\Omega, \tag{3.2}$$

where $q \in L^\infty$ is nonnegative, $f \in L^2(\Omega)$, and the matrix of coefficients $A := (a_{ij})$, $a_{ij} = a_{ji} \in W^{1,\infty}(\Omega)$ is uniformly elliptic, i.e., there exists a positive constant $r0$ such that

$$\sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \ \geq \ r(\xi_1^2 + \xi_2^2)$$

$$\forall \xi := (\xi_1, \xi_2) \in R^2 \qquad a.e. \text{ in } \Omega. \tag{3.3}$$

Let $H^m = W^{m,2}$ be the usual Sobolev space of order m and let $\|\cdot\|$ and $|\cdot|$ denote the associated norm and the semi-norm.

The natural variational problem associated with (3.1)-(3.2) is: Find $u \in U := H_0^1(\Omega)$ such that

$$a(u, v) = (f, v) \quad \forall v \in U, \tag{3.4}$$

where

$$a(u, v) := \int_\Omega (\sum_{i,j}^{2} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + quv)dx, \tag{3.5}$$

$$(f, v) = \int_\Omega fvdx. \tag{3.6}$$

Under the above assumptions of the problem data, the exact solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ [18].

To apply the multigrid algorithms for the finite element equations associated with the approximation of the above elliptic problem, we first construct a sequence of nested triangulations of Ω as follows. Suppose that a coarse triangulation $\mathcal{T}_1$ of Ω is given, we define finer triangulations $\mathcal{T}_k$ for $k \geq 2$ by subdividing a triangle in $\mathcal{T}_{k-1}$ into four subtriangles by connecting the midpoints of the edges. The maximum mesh size of $\mathcal{T}_k$ is denoted by h_k. Let J be an integer greater than or equal to one, and for $k = 1, \ldots, J$, let

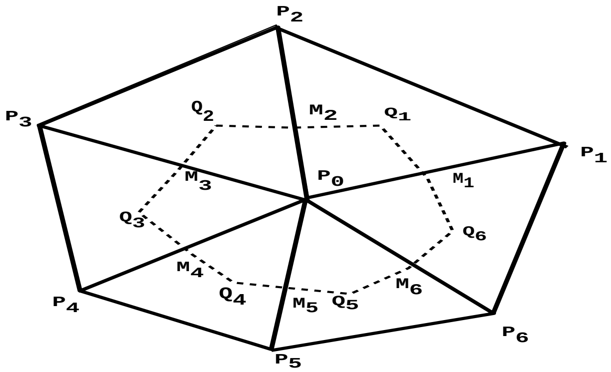

Fig. 1. Primal and dual domains

U_{h_k} be the set of continuous piecewise linear functions with respect to the triangulation $\mathcal{T}_k$ that vanish on $\partial\Omega$. Since the triangulations are nested, it follows that

$$U_{h_1} \subset U_{h_2} \subset \ldots \subset U_{h_J}.$$

The space U_{h_k} has the meshsize $h_k = 2^{1-k}h_1$. For simplicity, we write $U_k = U_{h_k}$. Then the standard Galerkin finite element formulation of (3.4) is: Find $u_J \in U_J$ such that

$$a(u_J, v) = (f, v), \quad \forall v \in U_J. \tag{3.7}$$

Next, we describe covolume method. Let $1 \leq k \leq J$. Referring to Fig. 1, let K_Q stand for the triangle with barycenter Q. The nodes of a triangular element are its vertices. Associated with the primal partition $\mathcal{T}_k$ we define its dual partition $\mathcal{T}_k^*$ of Ω as follows. Let P_0 be an interior node and $P_i, i = 1, \ldots, 6$ be its adjacent nodes, and $M_i := M_{0i}$ the midpoint of $\overline{P_0P_i}$. Connect successively the points $M_1, Q_1, M_2, Q_2, \cdots, M_6, Q_6, M_1$ to obtain the dual polygonal element $K_{P_0}^*$. The dual element $K_{P_2}^*$ based at a typical boundary node P_2 is defined by restricting the covolume to the interior of Ω. Let S_Q and $S_{P_0}^*$ denote respectively the areas of triangle K_Q and polygon $K_{P_0}^*$. We shall assume the partitions to be quasi-uniform, i.e., there exist two positive constants C_1, C_2 independent of h such that

$$C_1 h_k^2 \leq S_Q \leq C_2 h_k^2, \quad C_1 h_k^2 \leq S_{P_0}^* \leq C_2 h_k^2, \tag{3.8}$$

for all barycenters Q and all internal nodes P_0. Corresponding to $\mathcal{T}_k$ we define the trial function space $U_k \subset H_0^1(\Omega)$ as the space of continuous functions on the closure of Ω which vanish on the boundary $\partial\Omega$ and are linear on each triangle $K_Q \in \mathcal{T}_k$. Let $\Pi_k : C(\bar{\Omega}) \to U_k$ be the usual linear interpolator, and thus if $w \in H^2(\Omega)$,

$$|w - \Pi_k w|_m \leq C h_k^{2-m} |w|_2, \qquad m = 0, 1. \tag{3.9}$$

The test function space $V_k \subset L^2(\Omega)$ associated with the dual partition $\mathcal{T}_k^*$ is defined as the set of all piecewise constants over the dual volumes (covolumes) that vanish on $\partial\Omega$. More specifically, let χ_{P_0} be the characteristic function of the set $K_{P_0}^*$ we have for $v_k \in V_k$

$$v_k = \sum_{P_0 \in \Omega_k^\circ} v_k(P_0)\chi_{P_0}, \tag{3.10}$$

where Ω_k° is the set of interior nodes of $\mathcal{T}_k$.

The vertex–centered covolume method we consider is: Find $u_k^* \in U_k$ such that

$$b_k^*(u_k^*, y_k) = (f, y_k) \qquad \forall y_k \in V_k, \tag{3.11}$$

where

$$b_k^*(u_k, y_k) := \sum_{P_0 \in \Omega_k^\circ} y_k(P_0) b_k^*(u_k, \chi_{P_0}), \tag{3.12}$$

and

$$b_k^*(u_k, \chi_{P_0}) := -\int_{\partial K_{P_0}^*} (A\nabla u_k) \cdot \mathbf{n} ds + \int_{K_{P_0}^*} q u_k dx, \tag{3.13}$$

where $\mathbf{n}$ is outward unit normal to $\partial K_{P_0}^*$, and $b_k^*(\cdot,\cdot)$ is bilinear by construction. It should be noted that the above formulation is just another way of stating that we have an integral conservation form on dual domains using the divergence theorem. One can turn this Petrov-Galerkin method into a standard Galerkin method by introducing a one to one transfer operator $\Pi_k^* : U_k \to V_k$ connecting the trial and test spaces as

$$\Pi_k^* w := \sum_{P_0 \in \Omega_k^\circ} w_k(P_0)\chi_{P_0}, \tag{3.14}$$

which has the approximation property

$$||w - \Pi_k^* w||_0 \le C h_k |w|_1. \tag{3.15}$$

The error estimation along this approach has been carried out in [12]. We sketch it here as it is also relevant to the convergence analysis of our multigrid schemes. Set

$$a_k^*(u_k, v_k) = b_k^*(u_k, \Pi_k^* v_k), \text{ for } u_k, v_k \in U_k.$$

The basic idea is to observe that the standard finite element method for (3.1) is to find $u_k \in U_k$ such that

$$a(u_k, v_k) = (f, v_k), \qquad \text{for all } v_k \in U_k, \tag{3.16}$$

whereas the above covolume method is to find $u_k^* \in U_k$ such that

$$a_k^*(u_k^*, v_k) = (f, \Pi_k^* v_k), \qquad \text{for all } v_k \in U_k. \tag{3.17}$$

By (3.15), We anticipate the two problems are close, and indeed the following estimate of the difference $d_k = a - a_k^*$ between the above two bilinear forms have been established in [12].

Lemma 3.1. *There exists a constant C independent of v_k, w_k and h_k such that for $v_k, w_k \in U_k$,*

$$|d_k(v_k, w_k)| \leq Ch_k \|v_k\|_1 \|w_k\|_1. \tag{3.18}$$

Thus the bilinear form a_k^* is a perturbation of the form a. Note that if $v_k, w_k \in U_k$ have support confined to a subdomain Ω_k^i, then we also have (3.18) where in the right hand side, the norm is taken over the set Ω_k^i. Frequently, Ω_k^i is the support of a single basis function in the case of point relaxation, or the union of supports of functions along a line in case of line relaxation.

As a corollary of Lemma 3.1, we have coerciveness of the form a_k^* (h_k small enough) and its boundedness. Furthermore, we have the following error estimate.

Lemma 3.2. *The solution of u_k of the problem* (3.11) *and the exact solution u of* (3.1) *satisfy*

$$||u - u_k||_1 \leq Ch_k ||u||_2. \tag{3.19}$$

Remark 3.1. The above error estimates for the covolume method was derived in [12] where the domain was required to have a smooth boundary. But the smoothness requirement on the domain was necessary only for the max-norm and L^2 norm error estimates. For the convergence of multigrid algorithm in this paper, we only need H^1 norm estimates and the polygonal domain assumption shall do for our purpose.

Note that the algebraic equation derived from the covolume scheme is nonsymmetric even if the underlying PDE is symmetric. This makes the analysis more difficult. Nevertheless, Lemma 3.1 allows us to use the framework of Section two.

Let us now describe the multigrid algorithm for the covolume method. For each primal triangulation $\mathcal{T}_k$, we define dual triangulation $\mathcal{T}_k^*$ by connecting the barycenter of a triangle and its midpoint of the edges, as described in Sect. 2. The dual spaces $V_k, k = 1, \cdots, J$, are defined as the set of piecewise constant functions on dual triangulations. We note that the dual spaces are not nested because a dual element $K_k^* \in \mathcal{T}_k^*$ is not a subset of a dual element in $K_{k-1}^* \in \mathcal{T}_{k-1}^*$. In other words, refinements of a dual element at $k-1$

level do not result in dual elements at k level. Instead, the dual elements in $\mathcal{T}_k^*$ arise as the dual of refinements of primal triangulation of $\mathcal{T}_{k-1}^*$. Thus

$$V_{k-1} \not\subset V_k, \text{ for } k = 1, \cdots, J.$$

In general, the multigrid convergence theory is presented in terms of the bilinear forms a_k involved. When the spaces U_k are nested and the associated bilinear forms are inherited(variational), i.e, when

$$a_k(I_k w, I_k w) = a_{k-1}(w, w), \quad \forall w \in U_{k-1}, \tag{3.20}$$

where $I_k : U_{k-1} \to U_k$ is the injection operator, the accompanying multigrid algorithms are well analyzed. For example, V–cycle convergence is established in [2,5,7]. In our case, the associated bilinear form is nonvariational in nature, because the test function spaces are nonnested, i.e,

$$a_k^*(I_k w, I_k w) \neq a_{k-1}^*(w, w), \quad w \in U_{k-1}.$$

Hence one has to resort to the type of perturbation analysis demonstrated in Sect. 2.

As usual with multigrid algorithms, we consider two types of smoothers: the Jacobi type(additive) and Gauss-Seidel type(multiplicative). We shall present these smoothers in terms of subspace decompositions. Specifically, we write

$$U_k = \sum_{i=1}^{l} U_k^i$$

where U_k^i is the one dimensional subspace spanned by the nodal basis function ϕ_k^i or the subspace spanned by the nodal basis functions along a line. Let $A_{k,i}^* : U_k^i \to U_k^i$ be defined by

$$(A_{k,i}^* w, \chi)_k = a_k^*(w, \chi) \quad \text{for all } \chi, w \in U_k^i$$

and $Q_{k,i} : U_k \to U_k^i$ be the projection onto U_k^i with respect to the inner product $(\cdot, \cdot)_k$. Also, we let $P_k^{*i} : U_k \to U_k^i$ be defined by

$$(A_k^* P_k^{*i} w, \chi)_k = a_k^*(w, \chi), \quad \chi \in U_k^i, w \in U_k.$$

The corresponding operator P_k^i for finite element method is defined by

$$(A_k P_k^{\mathrm{i}} w, \chi)_k = a(w, \chi), \quad \chi \in U_k^i, w \in U_k.$$

We note that

$$a((I - P_k^i) w, (P_k^i - P_k^{*i}) v) = 0 \quad \forall w, v \in U_k. \tag{3.21}$$

Lemma 3.3. *We have, for $w \in U_k$,*

(3.22) $$\|P_k^{*i} w\|_{1,\Omega_k^i} \le C\|w\|_{1,\Omega_k^i},$$

(3.23) $$\|P_k^{\mathrm{i}} w\|_{1,\Omega_k^i} \le C\|w\|_{1,\Omega_k^i},$$

and

(3.24) $$\|(P_k^i - P_k^{*i})w\|_1 \le Ch_k\|w\|_{1,\Omega_k^i}.$$

Proof. By the coerciveness of the form a_k^*, we have

$$\begin{aligned}\alpha\|P_k^{*i}w\|_{1,\Omega_k^i} &\le a_k^*(P_k^{*i}w, P_k^{*i}w)\\ &= a_k^*(w, P_k^{*i}w)\\ &\le C\|w\|_{1,\Omega_k^i}\|P_k^{*i}w\|_{1,\Omega_k^i},\end{aligned}$$

which is (3.22). The estimate (3.23) can be obtained similarly.

By (3.22) and (3.23),

$$\begin{aligned}a((P_k^i - P_k^{*i})w, v) &= a(P_k^i w, v) - a(P_k^{*i}w, P_k^i v)\\ &= a(P_k^i w, v) - a_k^*(P_k^{*i}w, P_k^i v)\\ &\qquad + a_k^*(P_k^{*i}w, P_k^i v) - a(P_k^{*i}w, P_k^i v)\\ &= a(w, P_k^i v) - a_k^*(w, P_k^i v)\\ &\qquad + a_k^*(P_k^{*i}w, P_k^i v) - a(P_k^{*i}w, P_k^i v)\\ &= d_k(w, P_k^i v) - d_k(P_k^{*i}w, P_k^i v).\end{aligned}$$

Taking absolute values and using Lemma 3.1, (3.22) and (3.23), we obtain (3.24). □

Example 1. We consider the additive smoother defined by

$$R_k^* = \gamma \sum_{i=1}^{l} A_{k,i}^{*-1} Q_{k,i}.$$

The constant γ is a scaling factor which is chosen to ensure that (C.2) is satisfied for the corresponding operator R_k for the finite element method [9, 6]. In this case, we have

(3.25) $$K_k^* = I - R_k^* A_k^* = \gamma \sum_{i=1}^{l} P_k^{\mathrm{i}*}.$$

Example 2. We next consider the multiplicative smoother. Given $f \in U_k$, we define R_k^* by

1. Set $v_0 = 0 \in U_k$.

2. Define v_i, for $i = 1, \ldots, l$, by

$$v_i = v_{i-1} + A_{k,i}^{*-1} Q_{k,i}(f - A_k^* v_{i-1}).$$

3. Set $R_k^* f = v_l$.

In this case, we have

$$K_k^* = I - R_k^* A_k^* = \prod_{i=1}^{l} (I - P_k^{*i}). \tag{3.26}$$

Theorem 3.1. *Let R_k^* be defined by Examples 1 or 2. Then there exists an h_0 such that for all $h_1 < h_0$,*

$$\|E^* w\|_a \le \delta^* \|w\|_a, \quad \forall w \in U_J, \tag{3.27}$$

where $\delta^* = \delta + ch_1 < 1$ and δ is as in Theorem 2.1.

Proof. By Lemma 3.1 and the discussions in Sect. 2, it suffices to show $(P.2)$. Let us first consider Example 1. For $k1$, by definition of P_k^{*i} and (3.25) we have

$$K_k^* - K_k = \gamma \sum_{i=1}^{l} (P_k^{*i} - P_k^i).$$

Summing over i then using (3.24) and the Cauchy-Schwarz inequality, we get

$$|a((K_k^* - K_k)w, v)| \le Ch_k \sum_{i=1}^{l} \|w\|_{1,\Omega_k^i} \|v\|_{1,\Omega_k^i} \le Ch_k \|w\|_1 \|v\|_1,$$

Thus we have

$$\|K_k^* - K_k\|_a \le Ch_k. \tag{3.28}$$

Now consider Example 2. The perturbation operator for this example is

$$K_k - K_k^* = \mathcal{E}_l - \mathcal{E}_l^*.$$

where $\mathcal{E}_l^*$ is given by (3.26) and

$$\mathcal{E}_i^* = (I - P_k^{*i})(I - P_k^{*i-1}) \cdots (I - P_k^{*1}) = (I - P_k^{*i}) \mathcal{E}_{i-1}^*$$

with $\mathcal{E}_0^* = I$. Likewise $\mathcal{E}_i$ is defined. Note that

$$I - \mathcal{E}_{i-1}^* = \sum_{m=1}^{\mathrm{i}-1} P_k^{*m} \mathcal{E}_{m-1}^* \tag{3.29}$$

Since

$$\mathcal{E}_i - \mathcal{E}_i^* = (I - P_k^i)(\mathcal{E}_{i-1} - \mathcal{E}_{i-1}^*) - (P_k^i - P_k^{*i}) \mathcal{E}_{i-1}^*,$$

we have by (3.21)

$$||(\mathcal{E}_i - \mathcal{E}_i^*)w||_a^2 = ||(I - P_k^i)(\mathcal{E}_{i-1} - \mathcal{E}_{i-1}^*)w||_a^2 + ||(P_k^i - P_k^{*i})\mathcal{E}_{i-1}^* w||_a^2.$$

By (3.24) and the fact that the operator norm of $(I - P_k^i)$ is bounded by one, it follows that

$$||(\mathcal{E}_i - \mathcal{E}_i^*)w||_a^2 \leq ||(\mathcal{E}_{i-1} - \mathcal{E}_{i-1}^*)w||_a^2 + Ch_k^2 ||\mathcal{E}_{i-1}^* w||_{1,\Omega_k^i}^2.$$

Summing over i, since $\mathcal{E}_0 = \mathcal{E}_0^* = I$, we obtain

$$||(\mathcal{E}_l - \mathcal{E}_l^*)w||_a^2 \leq Ch_k^2 \sum_{i=1}^{\ell} ||\mathcal{E}_{i-1}^* w||_{1,\Omega_k^i}^2. \tag{3.30}$$

We shall show that

$$\sum_{i=1}^{\ell} ||\mathcal{E}_{i-1}^* w||_{1,\Omega_k^i}^2 \leq C||w||_a^2. \tag{3.31}$$

By the arithmetic–geometric mean inequality, the definition $\mathcal{E}_i^*$, (3.29) and the limited interaction property [6], it follows that

$$\begin{aligned}
\sum_{i=1}^{\ell} ||\mathcal{E}_{i-1}^* w||_{1,\Omega_k^i}^2 &\leq 2\sum_{i=1}^{\ell} ||w||_{1,\Omega_k^i}^2 + 2\sum_{i=1}^{\ell} ||w - \mathcal{E}_{i-1}^* w||_{1,\Omega_k^i}^2 \\
&\leq C||w||_a^2 + 2\sum_{i=1}^{\ell} || \sum_{m=1}^{\mathrm{i}-1} P_k^{*m} \mathcal{E}_{m-1}^* w||_{1,\Omega_k^i}^2 \\
&\leq C(||w||_a^2 + \sum_{m=1}^{\ell} \sum_{i=1}^{\ell} ||P_k^{*m} \mathcal{E}_{m-1}^* w||_{1,\Omega_k^i}^2) \\
&\leq C(||w||_a^2 + \sum_{m=1}^{\ell} ||P_k^{*m} \mathcal{E}_{m-1}^* w||_a^2).
\end{aligned} \tag{3.32}$$

In order to estimate the last term on the right of (3.32) we write

$$\begin{aligned}
||P_k^{*m} \mathcal{E}_{m-1}^* w||_a^2 &= a(P_k^{*m} \mathcal{E}_{m-1}^* w, P_k^{*m} \mathcal{E}_{m-1}^* w) \\
&= a((\mathcal{E}_{m-1}^* - \mathcal{E}_m^*)w, (\mathcal{E}_{m-1}^* - \mathcal{E}_m^*)w) \\
&= a((\mathcal{E}_{m-1}^* - \mathcal{E}_m^*)w, (\mathcal{E}_{m-1}^* + \mathcal{E}_m^*)w) \\
&\quad -2a(P_k^{*m} \mathcal{E}_{m-1}^* w, \mathcal{E}_m^* w) \\
&= a(\mathcal{E}_{m-1}^* w, \mathcal{E}_{m-1}^* w) - a(\mathcal{E}_m^* w, \mathcal{E}_m^* w) \\
&\quad -2a(P_k^{*m} \mathcal{E}_{m-1}^* w, (I - P_k^{*m})\mathcal{E}_{m-1}^*)w).
\end{aligned} \tag{3.33}$$

Now by (3.24)

$$
\begin{aligned}
&a(P_k^{*m}\mathcal{E}_{m-1}^{*}w,(I-P_k^{*m})\mathcal{E}_{m-1}^{*}w)\\
&\quad = a(P_k^{*m}\mathcal{E}_{m-1}^{*}w,(P_k^{m}-P_k^{*m})\mathcal{E}_{m-1}^{*})w)\\
&\quad \le Ch_k||P_k^{*m}\mathcal{E}_{m-1}^{*}w||_a||\mathcal{E}_{m-1}^{*}w||_{1,\Omega_k^m}. \qquad (3.34)
\end{aligned}
$$

Hence, combining (3.33) and (3.34), we have

$$
\begin{aligned}
||P_k^{*m}\mathcal{E}_{m-1}^{*}w||_a^2 \le{}& C[a(\mathcal{E}_{m-1}^{*}w,\mathcal{E}_{m-1}^{*}w)-a(\mathcal{E}_{m}^{*}w,\mathcal{E}_{m}^{*}w)]\\
&+Ch_k^2||\mathcal{E}_{m-1}^{*}w||_{1,\Omega_k^m}^2.
\end{aligned}
$$

Summing over m we conclude that

$$
\sum_{m=1}^{\ell}||P_k^{*m}\mathcal{E}_{m-1}^{*}w||_a^2 \le C||w||_a^2+Ch_k^2\sum_{m=1}^{\ell}||\mathcal{E}_{m-1}^{*}w||_{1,\Omega_k^m}^2.
$$

This together with (3.32) yields (3.31) when h_k is small enough. Finally, we obtain from (3.31) and (3.30) that for $k1$,

$$
||K_k^{*}-K_k||_a \le Ch_k.
$$

□

Remark 3.2. It is possible to analyze other types of smoother. For example, smoother based on A_k not A_k^*(Examples 2,3 of [4]) or smoothers based on normal equation(Example 4 of [4]) can be analyzed, but these are not practical, especially in a covolume method because A_k is not readily available.

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