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MIXED COVOLUME METHODS FOR QUASI-LINEAR SECOND-ORDER ELLIPTIC PROBLEMS *

DO Y. KWAK† AND KWANG Y. KIM†

Abstract. We consider covolume methods for the mixed formulations of quasi-linear second-order elliptic problems. Covolume methods for the mixed formulations of linear elliptic problem was first considered by Russell [*Rigorous Block-Centered Discretizations on Irregular Grids: Improved Simulation of Complex Reservoir Systems*, Tech. report 3, Project Report, Reservoir Simulation Research Corporation, Tulsa, OK, 1995] and tested extensively in [Cai et al., *Comput. Geosci.,* 1 (1997), pp. 289–315], [Jones, *A Mixed Finite Volume Element Method for Accurate Computation of Fluid Velocities in Porous Media,* Ph.D. thesis, University of Colorado, Denver, 1995]. The analysis was carried out by Chou and Kwak [*SIAM J. Numer. Anal.,* 37 (2000), pp. 758–771] for linear symmetric problems, where they showed optimal error estimates in L^2 norm for the pressure and in $H(\text{div})$ norm for the velocity. In this paper we extend their results to quasi-linear problems by following Milner's argument [*Math. Comp.,* 44 (1985), pp. 303–320] through an adaptation of the duality argument of Douglas and Roberts [*Math. Comp.,* 44 (1985), pp. 39–52] for mixed covolume methods.

Key words. mixed method, covolume method, quasi-linear elliptic problems

AMS subject classifications. 65N15, 65N30, 35J60

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1. Introduction.

Finite volume methods have been widely used as discretization techniques for conservation laws [18], [22], [30], [31], [32], [37]. For diffusion equations on rectangular grids, see Süli [43] or Weiser and Wheeler [45], and for triangular grids, see Cai [8], Cai et al. [9], or Heinrich [26]. See also [20], [23], [27]. Cell-centered finite differences can also be viewed as a finite volume method whose analysis for regular or general triangulation is shown in [2], [3], [19]. For cell-vertex finite volume methods, see [33], [36]. Meanwhile, mixed formulations of elliptic problems have been advocated for their accurate velocity computations [25], [38] and have been the subjects of extensive research [4], [5], [6], [17], [21], [24], [28], [34], [39].

Mixed covolume method is a natural attempt to combine two such approaches. It was first proposed by Russell [40], who applied the control-volume finite element methods [7] to the mixed formulation of linear elliptic problems. The numerical experiment on a variety of test problems was very promising [10], [29]. The optimal convergence of the mixed covolume method was given by Chou and Kwak [13] and Chou, Kwak, and Vassilevski [14], who adapted a covolume methodology used in [12] and formulated the mixed covolume method in the Galerkin framework. A variant of the mixed finite volume method has been also suggested for convection-diffusion equations with an application to semiconductor simulation in [41], [42]. Another type of the mixed finite volume method based on primal-dual formulation has been given in [44].

The goal of this paper is to study the mixed covolume method for quasi-linear elliptic problems. Although general convex polygonal domains can be treated (cf.

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http://www.siam.org/journals/sinum/38-4/35855.html
†Department of Mathematics, Korea Advanced Institute of Science and Technology, Taejon, Korea 305-701 (dykwak@math.kaist.ac.kr, kky@mathx.kaist.ac.kr).

[14]), we restrict ourselves to rectangular domains for simplicity.

We consider the quasi-linear second-order elliptic problem

$$-\nabla \cdot (a(p)\nabla p + \mathbf{b}(p)) + c(p) = f \qquad \text{in } \Omega, \tag{1.1a}$$

$$p = 0 \qquad \text{on } \partial\Omega, \tag{1.1b}$$

where Ω is a bounded, axiparallel rectangular domain in $\mathbb{R}^2$, and $\partial\Omega$ is the boundary of Ω. We assume $\partial \mathbf{b}/\partial p$ is not too large in comparison to a, i.e., we are dealing with diffusion-dominated problems. Further, we assume that the coefficients $a : \bar{\Omega}\times\mathbb{R} \to \mathbb{R}$, $\mathbf{b} : \bar{\Omega}\times\mathbb{R} \to \mathbb{R}^2$, and $c : \bar{\Omega}\times\mathbb{R} \to \mathbb{R}$ are twice continuously differentiable with bounded derivatives through second order. Moreover, we assume $a(p) \geq a_1 > 0$ and $c_p \geq 0$. Dependence of the coefficients $a, \mathbf{b}$, and c on the space variable x will be omitted throughout the paper. We also assume that for some ε, $0 < \varepsilon < 1$, there exists a unique solution $p \in H^{2+\varepsilon}(\Omega)$ to (1.1) for each given $f \in H^{\varepsilon}(\Omega)$.

For integer $s \geq 0$ and $1 \leq q \leq \infty$ we denote by $W^{s,q}(\Omega)$ the usual Sobolev space equipped with a norm $\|\cdot\|_{s,q}$ given by

$$\|v\|_{s,q} = \left(\sum_{|\alpha|\leq s} \|D^\alpha v\|_{L^q(\Omega)} \right)^{1/q},$$

with the obvious modification for $q = \infty$. When $q = 2$ we shall write $H^s(\Omega)$ and $\|\cdot\|_s$ instead of $W^{s,2}(\Omega)$ and $\|\cdot\|_{s,2}$. Also, we define $H^{-s}(\Omega)$ to be the dual space of $H^s(\Omega)$ with a norm $\|\cdot\|_{-s}$ defined by

$$\|\phi\|_{-s} = \sup_{v\in H^s(\Omega)} \frac{(\phi, v)}{\|v\|_s}.$$

By introducing the velocity variable

$$\mathbf{u} = -(a(p)\nabla p + \mathbf{b}(p)),$$

we can rewrite the problem (1.1) as a system of first-order equations

$$\alpha(p)\mathbf{u} + \nabla p + \boldsymbol{\beta}(p) = 0, \qquad \operatorname{div} \mathbf{u} + c(p) = f, \tag{1.2}$$

where $\alpha = a^{-1}$, $\boldsymbol{\beta}(p) = \alpha(p)\mathbf{b}(p)$, and the boundary condition $p = 0$ on $\partial\Omega$ is imposed. Let

$$\mathbf{V} = \mathbf{H}(\operatorname{div}; \Omega) = \{\mathbf{v} \in (L^2(\Omega))^2 : \operatorname{div} \mathbf{v} \in L^2(\Omega)\}, \tag{1.3}$$

$$\mathbf{H}^1(\operatorname{div}; \Omega) = \{\mathbf{v} \in (L^2(\Omega))^2 : \operatorname{div} \mathbf{v} \in H^1(\Omega)\}, \tag{1.4}$$

$$W = L^2(\Omega). \tag{1.5}$$

Then the associated weak formulation is to find $(\mathbf{u}, p) \in \mathbf{V} \times W$ such that

$$(\alpha(p)\mathbf{u}, \mathbf{v}) - (\operatorname{div} \mathbf{v}, p) + (\boldsymbol{\beta}(p), \mathbf{v}) = 0 \qquad \forall \mathbf{v} \in \mathbf{V}, \tag{1.6a}$$

$$(\operatorname{div} \mathbf{u}, w) + (c(p), w) = (f, w) \qquad \forall w \in W, \tag{1.6b}$$

where $(\cdot, \cdot)$ denotes the standard inner product in $L^2(\Omega)$ or $(L^2(\Omega))^2$.

Many finite element spaces based on this formulation have been developed and analyzed for linear problems (cf. [4], [5], [6], [17], [24], [38]). Meanwhile, the analyses for quasi-linear problems are relatively scarce [11], [16], [35]. Milner [35] analyzed mixed finite element spaces for the quasi-linear problem (1.1) and showed that there exists a unique solution near $(\mathbf{u}, p)$ and the optimal error estimates established in [17] for linear problems hold as well for quasi-linear problems. His argument was based on an adaptation of the method used by Douglas [16].

We will formulate the mixed covolume method in the Galerkin framework, apply to it an adaptation of the duality argument of Douglas and Roberts [17], and show that there exists a unique solution near $(\mathbf{u}, p)$ with an optimal order error.

The rest of the paper is organized as follows. In the next section we introduce some definitions and formulate the mixed covolume method as a conservative scheme. Then we show that the method can be cast into a Galerkin form by introducing a one-to one transfer operator between the Raviart–Thomas space and the space of piecewise constant functions on covolumes. In section 3 the error equations are derived and compared with the standard mixed finite element method. In section 4 we deal with the existence of a solution, and in section 5 optimal error estimates in L^2 norm are established for both velocity and pressure variables. Finally, a superconvergence result for the pressure variable is shown in section 6.

2. Preliminaries.

Let $\mathcal{Q}_h = \{Q_{i,j}\}$ be a partition of the domain Ω into rectangular elements

$$Q_{i,j} := [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}],$$

with centers $c_{i,j} = (x_i, y_j)$. Let $c_{i\pm 1/2,j} = (x_{i\pm 1/2}, y_j)$ and $c_{i,j\pm 1/2} = (x_i, y_{j\pm 1/2})$, i.e., the midpoints of four edges of $Q_{i,j}$. We assume the partition $\mathcal{Q}_h$ is quasi-regular in the sense that there exist two positive constants C_1, C_2 independent of h such that

$$C_1 h^2 \le |Q_{i,j}| \le C_2 h^2 \qquad \forall Q_{i,j} \in \mathcal{Q}_h, \tag{2.1}$$

where $h = \text{diam } Q_{i,j}$ and $|Q_{i,j}|$ denotes the area of $Q_{i,j}$.

Now we define the lowest order Raviart–Thomas space on $\mathcal{Q}_h$ to be

$$\mathbf{V}_h = \left\{ \mathbf{v} \in \mathbf{V} : \mathbf{v}(x,y) = \begin{pmatrix} a + bx \\ c + dy \end{pmatrix} \text{ on } Q_{i,j} \in \mathcal{Q}_h \right\},$$
$$W_h = \{ w \in W : w \text{ is constant over } Q_{i,j} \in \mathcal{Q}_h \}$$

(cf. [6], [17], [38]). Observe that the requirement $\mathbf{v} \in \mathbf{V}$ imposes the continuity of normal components across the edges of the elements; i.e., if e is the common edge of two elements Q_1 and Q_2, and $\mathbf{n}_i$ denotes the outer unit normal vector of Q_i, then we must have $\mathbf{v} \cdot \mathbf{n}_1|_{Q_1} + \mathbf{v} \cdot \mathbf{n}_2|_{Q_2} = 0$ on e.

The Raviart–Thomas projection $\Pi_h : \mathbf{V} \to \mathbf{V}_h$ is defined in [24], [38] so that it satisfies the orthogonality relation

$$(\text{div}(\mathbf{u} - \Pi_h \mathbf{u}), w) = 0 \qquad \forall w \in W_h. \tag{2.2}$$

Let $P_h : W \to W_h$ denote the L^2 orthogonal projection defined by

$$(P_h \chi - \chi, w) = 0 \qquad \forall \chi \in W,\ w \in W_h. \tag{2.3}$$

Then the following properties of Π_h and P_h are well known from [6], [17], [28], [38]:

$$\|\Pi_h \mathbf{u}\|_0 \le C \|\mathbf{u}\|_{1,1} \qquad \forall \mathbf{u} \in (W^{1,1}(\Omega))^2, \tag{2.4}$$

$$\|\mathbf{u} - \Pi_h \mathbf{u}\|_0 \le Ch \|\mathbf{u}\|_1 \qquad \forall \mathbf{u} \in (H^1(\Omega))^2, \tag{2.5}$$

$$\| \text{div}(\mathbf{u} - \Pi_h \mathbf{u})\|_0 \le Ch \| \text{div}\, \mathbf{u}\|_1 \qquad \forall \mathbf{u} \in \mathbf{H}^1(\text{div}; \Omega), \tag{2.6}$$

$$\|\chi - P_h \chi\|_{-1} + h \|\chi - P_h \chi\|_0 \le Ch^2 \|\chi\|_1 \qquad \forall \chi \in L^2(\Omega). \tag{2.7}$$

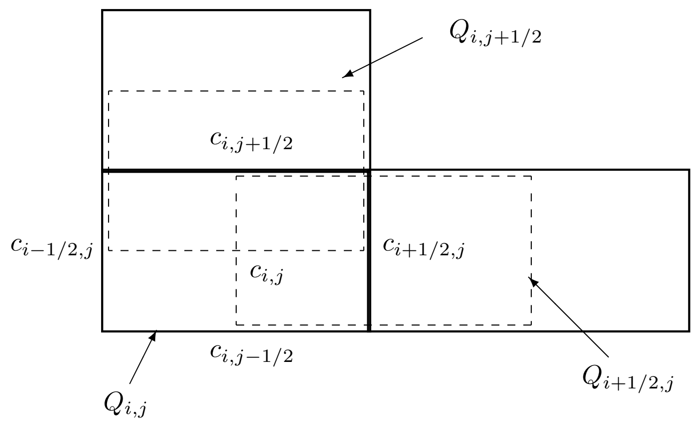

FIG. 2.1. *Primal and dual domains.*

Here and throughout the paper, C will denote a generic positive constant which is independent of h and may take on different values.

Now we are in a position to describe the main idea of the mixed covolume method on the rectangular grid $\mathcal{Q}_h$ (cf. [10], [13], [29]). First, we assign the unknowns of the approximate velocity $\mathbf{u}_h$ to the edges, and the unknowns of the approximate pressure p_h to the centers of the primal partition $\{Q_{i,j}\}$. We will denote by $p_{i,j}$ the nodal value of p_h at the center $c_{i,j}$. Next, in order to provide a finite volume around each unknown, we introduce a dual grid obtained by shifting the primal grid along x and y-axis: let

$$Q_{i+1/2,j} := [x_i, x_{i+1}] \times [y_{j-1/2}, y_{j+1/2}],$$
$$Q_{i,j+1/2} := [x_{i-1/2}, x_{i+1/2}] \times [y_j, y_{j+1}],$$

where we cut off the part outside the domain Ω. The rectangles $Q_{i+1/2,j}, Q_{i,j+1/2}$, and $Q_{i,j}$ are referred to as u-volumes, v-volumes, and p-volumes, respectively (cf. Figure 2.1). Finally, we integrate (1.2) over these volumes to obtain

$$\int_{Q_{i+1/2,j}} \left[\alpha(p)u_x + \frac{\partial p}{\partial x} + \boldsymbol{\beta}_x(p) \right] = 0, \tag{2.8}$$

$$\int_{Q_{i,j+1/2}} \left[\alpha(p)u_y + \frac{\partial p}{\partial y} + \boldsymbol{\beta}_y(p) \right] = 0, \tag{2.9}$$

$$\int_{Q_{i,j}} [\, \operatorname{div} \mathbf{u} + c(p)\,] = \int_{Q_{i,j}} f, \tag{2.10}$$

where we set $\mathbf{u} = (u_x, u_y)$ and $\boldsymbol{\beta} = (\boldsymbol{\beta}_x, \boldsymbol{\beta}_y)$.

To formulate the mixed covolume method in the Galerkin framework we need the test function space (as defined in [13])

$$\mathbf{Y}_h := \{(u_h, v_h) : u_h \in L^2(\Omega) \text{ is piecewise constant on } u\text{-volumes}, \\ v_h \in L^2(\Omega) \text{ is piecewise constant on } v\text{-volumes}\}.$$

This test space is in one-to-one correspondence with the Raviart–Thomas space $\mathbf{V}_h$ (which will be chosen as the trial function space) via the transfer map $\gamma_h : \mathbf{V}_h \to \mathbf{Y}_h$ defined by

$$\begin{aligned}\gamma_h \mathbf{w}_h &:= (\gamma_h u_h, \gamma_h v_h),\\ &:= \left(\sum_{i,j} u_h(c_{i+1/2,j})\chi_{i+1/2,j}, \sum_{i,j} v_h(c_{i,j+1/2})\chi_{i,j+1/2} \right),\end{aligned}$$

where $\mathbf{w}_h = (u_h, v_h)$ and $\chi_{i+1/2,j}$, $\chi_{i,j+1/2}$ are the characteristic functions of $Q_{i+1/2,j}$ and $Q_{i,j+1/2}$, respectively. Note that we used the same notation γ_h in the componentwise fashion.

With the help of the transfer map γ_h, we can rewrite (2.8)–(2.9) in the vector form

$$(\alpha(p)\mathbf{u} + \nabla p + \boldsymbol{\beta}(p), \gamma_h \mathbf{v}) = 0 \qquad \forall \mathbf{v} \in \mathbf{V}_h. \tag{2.11}$$

By applying Green's theorem we obtain

$$\begin{aligned}(\nabla p, \gamma_h \mathbf{v}) &= \sum_{i,j} \int_{Q_{i+1/2,j}} \frac{\partial p}{\partial x} \gamma_h v_x + \sum_{i,j} \int_{Q_{i,j+1/2}} \frac{\partial p}{\partial y} \gamma_h v_y \\ &= \sum_{i,j} \int_{\partial Q_{i+1/2,j}} p\, n_x(\gamma_h v_x)\, ds + \sum_{i,j} \int_{\partial Q_{i,j+1/2}} p\, n_y(\gamma_h v_y)\, ds \\ &\equiv b(\gamma_h \mathbf{v}, p),\end{aligned}$$

for every $\mathbf{v} = (v_x, v_y)$ in $\mathbf{V}_h$. This leads to the following equivalent form of (2.11):

$$(\alpha(p)\mathbf{u}, \gamma_h \mathbf{v}) + b(\gamma_h \mathbf{v}, p) + (\boldsymbol{\beta}(p), \gamma_h \mathbf{v}) = 0 \qquad \forall \mathbf{v} \in \mathbf{V}_h. \tag{2.12}$$

Now we define the mixed covolume method for the problem (1.1) by choosing the space $\mathbf{V}_h$ as the trial function space: find $(\mathbf{u}_h, p_h)$ in $\mathbf{V}_h \times W_h$ satisfying

$$(\alpha(p_h)\mathbf{u}_h, \gamma_h \mathbf{v}) + b(\gamma_h \mathbf{v}, p_h) + (\boldsymbol{\beta}(p_h), \gamma_h \mathbf{v}) = 0 \qquad \forall \mathbf{v} \in \mathbf{V}_h, \tag{2.13a}$$

$$(\operatorname{div} \mathbf{u}_h, w) + (c(p_h), w) = (f, w) \qquad \forall w \in W_h. \tag{2.13b}$$

This is an extension of the covolume scheme of [10], [13] to quasi-linear problems.

By simple calculations it is easy to verify the equality

$$b(\gamma_h \mathbf{v}, p_h) = -(\operatorname{div} \mathbf{v}, p_h) \qquad \forall \mathbf{v} \in \mathbf{V}_h,\ p_h \in W_h,$$

(see Lemma 2.1 of [13] for details). This implies that the mixed covolume method (2.13) can be rewritten as

$$(\alpha(p_h)\mathbf{u}_h, \gamma_h \mathbf{v}) - (\operatorname{div} \mathbf{v}, p_h) + (\boldsymbol{\beta}(p_h), \gamma_h \mathbf{v}) = 0 \qquad \forall \mathbf{v} \in \mathbf{V}_h, \tag{2.14a}$$

$$(\operatorname{div} \mathbf{u}_h, w) + (c(p_h), w) = (f, w) \qquad \forall w \in W_h. \tag{2.14b}$$

We observe that this differs from the standard mixed finite element method only in the fact that now the test function is $\gamma_h \mathbf{v}$ instead of $\mathbf{v}$. The advantage of this formulation is that we have the local conservativity (2.8)–(2.10) .

Finally, we give some properties of the operator γ_h which will be of crucial importance in establishing error estimates for (2.14).

LEMMA 2.1. *The symmetry relation*

$$(\mathbf{u}_h, \gamma_h \mathbf{v}_h) = (\gamma_h \mathbf{u}_h, \mathbf{v}_h) \qquad \forall \mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h, \tag{2.15}$$

holds, and there exist positive constants C and c_0 independent of h such that for every $\mathbf{u}_h \in \mathbf{V}_h$, we have

$$\|\gamma_h \mathbf{u}_h\|_0 \le C\,\|\mathbf{u}_h\|_0, \tag{2.16}$$

$$(\alpha \mathbf{u}_h, \gamma_h \mathbf{u}_h) \ge c_0 \|\mathbf{u}_h\|_0^2. \tag{2.17}$$

Proof. We give only the proof for the first result. The other proofs can be found in Lemmas 2.2 and 2.4 of [13].

Writing out the integrals as the sum over the rectangles $Q_{i,j} \in \mathcal{Q}_h$ and the two components, we see that it suffices to consider the integral $\int_{x_{i-1/2}}^{x_{i+1/2}} u_h(\gamma_h v_h)\,dx$ for linear polynomials u_h, v_h in x. Let $u_{i\pm1/2} = u_h(x_{i\pm1/2})$, $v_{i\pm1/2} = v_h(x_{i\pm1/2})$, and $h_i = x_{i+1/2} - x_{i-1/2}$. Then it follows that

$$\begin{aligned}
\int_{x_{i-1/2}}^{x_{i+1/2}} u_h(\gamma_h v_h)\,dx &= v_{i-1/2} \int_{x_{i-1/2}}^{x_i} u_h\,dx + v_{i+1/2} \int_{x_i}^{x_{i+1/2}} u_h\,dx \\
&= v_{i-1/2} \left(\frac{h_i}{2}\right) \frac{u_{i-1/2} + u_i}{2} + v_{i+1/2} \left(\frac{h_i}{2}\right) \frac{u_i + u_{i+1/2}}{2} \\
&= \frac{h_i}{4} \{u_{i-1/2} v_{i-1/2} + u_i(v_{i-1/2} + v_{i+1/2}) + u_{i+1/2} v_{i+1/2}\} \\
&= \frac{h_i}{4} (u_{i-1/2} v_{i-1/2} + 2u_i v_i + u_{i+1/2} v_{i+1/2}),
\end{aligned}$$

which is clearly symmetric in u_h, v_h. We remark that this is the composite trapezoidal rule for the integral $\int_{x_{i-1/2}}^{x_{i+1/2}} u_h v_h\,dx$. ∎

LEMMA 2.2. *There exists a positive constant C such that*

$$(\alpha \mathbf{u}, \mathbf{v}_h - \gamma_h \mathbf{v}_h) \le Ch\,\|\mathbf{u}\|_1 \|\mathbf{v}_h\|_0 \qquad \forall \mathbf{u} \in (H^1(\Omega))^2,\ \mathbf{v}_h \in \mathbf{V}_h, \tag{2.18}$$

$$\|\mathbf{u} - \gamma_h \Pi_h \mathbf{u}\|_{0,q} \le Ch\,\|\mathbf{u}\|_{1,q} \qquad \forall \mathbf{u} \in (W^{1,q}(\Omega))^2,\ 1 < q < \infty. \tag{2.19}$$

Proof. The first inequality is shown in Lemma 2.5 of [13]. We give a simpler proof here. We start with an observation that if $\mathbf{w}_h$ is a piecewise constant vector-valued function, then we have

$$(\mathbf{w}_h, \mathbf{v}_h - \gamma_h \mathbf{v}_h) = 0 \qquad \forall \mathbf{v}_h \in \mathbf{V}_h.$$

Let $\bar{\alpha}_h$ and $\bar{\mathbf{u}}_h$ denote piecewise constant approximations to α and $\mathbf{u}$, respectively, satisfying

$$\|\alpha - \bar{\alpha}_h\|_{0,\infty} \le Ch\|\alpha\|_{1,\infty}, \qquad \|\mathbf{u} - \bar{\mathbf{u}}_h\|_0 \le Ch\|\mathbf{u}\|_1.$$

Then it follows that

$$\begin{aligned}
(\alpha \mathbf{u}, \mathbf{v}_h - \gamma_h \mathbf{v}_h) &= ((\alpha - \bar{\alpha}_h)\mathbf{u}, \mathbf{v}_h - \gamma_h \mathbf{v}_h) + (\bar{\alpha}_h \mathbf{u}, \mathbf{v}_h - \gamma_h \mathbf{v}_h) \\
&= ((\alpha - \bar{\alpha}_h)\mathbf{u}, \mathbf{v}_h - \gamma_h \mathbf{v}_h) + (\bar{\alpha}_h(\mathbf{u} - \bar{\mathbf{u}}_h), \mathbf{v}_h - \gamma_h \mathbf{v}_h) \\
&\le \|\alpha - \bar{\alpha}_h\|_{0,\infty} \|\mathbf{u}\|_0 \|\mathbf{v}_h - \gamma_h \mathbf{v}_h\|_0 \\
&\qquad + \|\bar{\alpha}_h\|_{0,\infty} \|\mathbf{u} - \bar{\mathbf{u}}_h\|_0 \|\mathbf{v}_h - \gamma_h \mathbf{v}_h\|_0 \\
&\le Ch\|\alpha\|_{1,\infty} \|\mathbf{u}\|_1 \|\mathbf{v}_h\|_0
\end{aligned}$$

by (2.16). This proves the first result.

To prove the second inequality we note that $I - \gamma_h \Pi_h$ vanishes for constant polynomials on each rectangle, and then apply the Bramble–Hilbert lemma. ∎

3. Error equations. By subtracting (2.14) from (1.6) we obtain the error equations

$$\begin{aligned}(\alpha(p)(\mathbf{u}-\mathbf{u}_h), \gamma_h \mathbf{v}) - (\operatorname{div} \mathbf{v}, p - p_h) + (\boldsymbol{\beta}(p) - \boldsymbol{\beta}(p_h), \gamma_h \mathbf{v}) \\ = ([\alpha(p_h) - \alpha(p)]\mathbf{u}_h, \gamma_h \mathbf{v}) - (\alpha(p)\mathbf{u} + \boldsymbol{\beta}(p), \mathbf{v} - \gamma_h \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}_h,\end{aligned} \tag{3.1a}$$

$$(\operatorname{div}(\mathbf{u}-\mathbf{u}_h), w) + (c(p) - c(p_h), w) = 0 \qquad \forall w \in W_h. \tag{3.1b}$$

By the Taylor expansions

$$\alpha(p_h) - \alpha(p) = \tilde{\alpha}_p(p_h)(p_h - p) = \alpha_p(p)(p_h - p) + \tilde{\alpha}_{pp}(p_h)(p_h - p)^2,$$

where

$$\tilde{\alpha}_p(\rho) = \int_0^1 \alpha_p(\rho + t(p-\rho))\, dt,$$
$$\tilde{\alpha}_{pp}(\rho) = \int_0^1 (1-t)\alpha_{pp}(\rho + t(p-\rho))\, dt$$

are bounded functions in $\bar{\Omega}$, we can write

$$\begin{aligned}(\alpha(p_h) - \alpha(p))\mathbf{u}_h &= (\alpha(p_h) - \alpha(p))(\mathbf{u}_h - \mathbf{u}) + (\alpha(p_h) - \alpha(p))\mathbf{u} \\ &= \tilde{\alpha}_p(p_h)(p - p_h)(\mathbf{u} - \mathbf{u}_h) - \alpha_p(p)\mathbf{u}(p - p_h) \\ &\qquad + \tilde{\alpha}_{pp}(p_h)\mathbf{u}(p - p_h)^2,\end{aligned}$$

and substituting this into (3.1), together with the second-order Taylor expansions for $\boldsymbol{\beta}(p) - \boldsymbol{\beta}(p_h)$ and $c(p) - c(p_h)$, it follows that

$$\begin{aligned}(\alpha(p)(\mathbf{u}-\mathbf{u}_h), \gamma_h \mathbf{v}) - (\operatorname{div} \mathbf{v}, p - p_h) + ([\alpha_p(p)\mathbf{u} + \boldsymbol{\beta}_p(p)](p - p_h), \gamma_h \mathbf{v}) \\ = (\tilde{\alpha}_p(p_h)(p - p_h)(\mathbf{u} - \mathbf{u}_h) + [\tilde{\alpha}_{pp}(p_h)\mathbf{u} + \tilde{\boldsymbol{\beta}}_{pp}(p_h)](p - p_h)^2, \gamma_h \mathbf{v}) \\ - (\alpha(p)\mathbf{u} + \boldsymbol{\beta}(p), \mathbf{v} - \gamma_h \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}_h,\end{aligned} \tag{3.2a}$$

$$(\operatorname{div}(\mathbf{u}-\mathbf{u}_h), w) + (c_p(p)(p - p_h), w) = (\tilde{c}_{pp}(p_h)(p - p_h)^2, w) \qquad \forall w \in W_h. \tag{3.2b}$$

Note that this system differs from the standard mixed finite element method only in two respects: the test function is $\gamma_h \mathbf{v}$, and we have an additional term $(\alpha(p)\mathbf{u} + \boldsymbol{\beta}(p), \mathbf{v} - \gamma_h \mathbf{v})$. This will allow us to analyze the mixed covolume method in an analogous manner to the standard mixed finite element method.

Setting $\boldsymbol{\Gamma} = \alpha_p(p)\mathbf{u} + \boldsymbol{\beta}_p(p)$, we observe that formally the system (3.2) corresponds to the error equations of the mixed covolume method for the linear operator M given by

$$M\chi = -\nabla \cdot (a(p)\nabla\chi + a(p)\boldsymbol{\Gamma}\chi) + c_p(p)\chi.$$

Its formal adjoint M^* is

$$M^*\chi = -\nabla \cdot (a(p)\nabla\chi) + a(p)\boldsymbol{\Gamma} \cdot \nabla\chi + c_p(p)\chi.$$

By the assumptions on $a, \mathbf{b}$, and c, it follows that M and M^* are isomorphisms from $H^2(\Omega) \cap H_0^1(\Omega)$ to $L^2(\Omega)$; i.e., for any $\psi \in L^2(\Omega)$ there exists a unique $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $M\phi = \psi$ (respectively, $M^*\phi = \psi$) and $\|\phi\|_2 \le C\|\psi\|_0$.

4. Existence of solutions. We define a map $\Phi : \mathbf{V}_h \times W_h \to \mathbf{V}_h \times W_h$, letting $\Phi(\boldsymbol{\mu}, \rho) = (\mathbf{y}, z)$ be the (unique) solution of

$$\begin{aligned}(\alpha(p)(\Pi_h \mathbf{u} - \mathbf{y}), \gamma_h \mathbf{v}) &- (\operatorname{div} \mathbf{v}, P_h p - z) + (\boldsymbol{\Gamma}(P_h p - z), \gamma_h \mathbf{v}) \\ &= (\alpha(p)(\Pi_h \mathbf{u} - \mathbf{u}) + \boldsymbol{\Gamma}(P_h p - p) + \tilde{\alpha}_p(\rho)(p-\rho)(\mathbf{u} - \boldsymbol{\mu}) \\ &\quad + [\tilde{\alpha}_{pp}(\rho)\mathbf{u} + \tilde{\boldsymbol{\beta}}_{pp}(\rho)](p-\rho)^2, \gamma_h \mathbf{v}) - (\alpha(p)\mathbf{u} + \boldsymbol{\beta}(p), \mathbf{v} - \gamma_h \mathbf{v}) \\ &\qquad \forall \mathbf{v} \in \mathbf{V}_h, \end{aligned} \tag{4.1a}$$

$$\begin{aligned}(\operatorname{div}(\Pi_h \mathbf{u} - \mathbf{y}), w) &+ (c_p(p)(P_h p - z), w) \\ &= (c_p(p)(P_h p - p) + \tilde{c}_{pp}(\rho)(p-\rho)^2, w) \qquad \forall w \in W_h. \end{aligned} \tag{4.1b}$$

We recall that the left-hand side corresponds to the mixed covolume method for the linear operator M. It will be shown later in this section that this method has a unique solution for sufficiently small h. Hence Φ is well defined, at least for sufficiently small h.

It is easy to see that every solution of (2.14) is a fixed point of Φ. Thus, the existence of a solution of (2.14) follows from the Brouwer fixed point theorem if we can prove that Φ maps a ball of $\mathbf{V}_h \times W_h$ into itself.

For this sake we need the following lemma, which is the covolume version of Lemma 2.1 of [35].

LEMMA 4.1. *Let $2 \le \theta < \infty$. Let $\boldsymbol{\omega} \in \mathbf{V}$, and let $\mathbf{q}$ and r be linear functionals defined on $\mathbf{V}_h$ and W_h, respectively. If $\tau \in W_h$ satisfies*

$$\begin{aligned}(\alpha \boldsymbol{\omega}, \gamma_h \mathbf{v}) - (\operatorname{div} \mathbf{v}, \tau) + (\boldsymbol{\Gamma}\tau, \gamma_h \mathbf{v}) &= \mathbf{q}(\mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}_h, \\ (\operatorname{div} \boldsymbol{\omega}, w) + (c_p \tau, w) &= r(w) \qquad \forall w \in W_h,\end{aligned}$$

then there exists a positive constant C such that for h sufficiently small, we have

$$\|\tau\|_{0,\theta} \le C(h^{2/\theta}\|\boldsymbol{\omega}\|_0 + h\|\operatorname{div} \boldsymbol{\omega}\|_0 + \|\mathbf{q}\|_0 + \|r\|_0), \tag{4.2}$$

where $\|\mathbf{q}\|_0$ and $\|r\|_0$ are the operator norms given by

$$\|\mathbf{q}\|_0 = \sup_{\mathbf{v} \in \mathbf{V}_h} \frac{(\mathbf{q}, \mathbf{v})}{\|\mathbf{v}\|_0}, \qquad \|r\|_0 = \sup_{w \in W_h} \frac{(r, w)}{\|w\|_0}.$$

Proof. We use the duality argument of Douglas and Roberts [17]. Let $\theta' = \theta/(\theta - 1)$ be the conjugate exponent of θ $(1 < \theta' \le 2)$. For a given $\psi \in L^{\theta'}(\Omega)$, let $\phi \in W^{2,\theta'}(\Omega)$ be the solution of the adjoint problem $M^*\phi = \psi$ in Ω, $\phi = 0$ on $\partial\Omega$ satisfying the elliptic regularity estimate [1], [28]

$$\|\phi\|_{2,\theta'} \le C\,\|\psi\|_{0,\theta'}. \tag{4.3}$$

Let $\boldsymbol{\zeta} = a\nabla\phi$. Then we have

$$\begin{aligned}(\tau, \psi) &= (\tau, -\operatorname{div} \boldsymbol{\zeta} + \boldsymbol{\Gamma} \cdot \boldsymbol{\zeta} + c_p\,\phi) = (\tau, -\operatorname{div}(\Pi_h \boldsymbol{\zeta}) + \boldsymbol{\Gamma} \cdot \boldsymbol{\zeta} + c_p \phi) \\ &= \mathbf{q}(\Pi_h \boldsymbol{\zeta}) - (\alpha \boldsymbol{\omega}, \gamma_h \Pi_h \boldsymbol{\zeta}) + (\boldsymbol{\Gamma}\tau, \boldsymbol{\zeta} - \gamma_h \Pi_h \boldsymbol{\zeta}) + (c_p \tau, \phi) \\ &= \mathbf{q}(\Pi_h \boldsymbol{\zeta}) + (\alpha \boldsymbol{\omega} + \boldsymbol{\Gamma}\tau, \boldsymbol{\zeta} - \gamma_h \Pi_h \boldsymbol{\zeta}) - (\alpha \boldsymbol{\omega}, \boldsymbol{\zeta}) + (c_p \tau, \phi) \\ &= \mathbf{q}(\Pi_h \boldsymbol{\zeta}) + (\alpha \boldsymbol{\omega} + \boldsymbol{\Gamma}\tau, \boldsymbol{\zeta} - \gamma_h \Pi_h \boldsymbol{\zeta}) + (\operatorname{div} \boldsymbol{\omega}, \phi) + (c_p \tau, \phi) \\ &= \mathbf{q}(\Pi_h \boldsymbol{\zeta}) + (\alpha \boldsymbol{\omega} + \boldsymbol{\Gamma}\tau, \boldsymbol{\zeta} - \gamma_h \Pi_h \boldsymbol{\zeta}) + (\operatorname{div} \boldsymbol{\omega} + c_p \tau, \phi - P_h \phi) + r(P_h \phi).\end{aligned}$$

By (2.4) it follows that

$$\mathbf{q}(\Pi_h\boldsymbol{\zeta}) \le \|\mathbf{q}\|_0\|\Pi_h\boldsymbol{\zeta}\|_0 \le C\|\mathbf{q}\|_0\|\boldsymbol{\zeta}\|_{1,1} \le C\|\mathbf{q}\|_0\|\phi\|_{2,1} \le C\|\mathbf{q}\|_0\|\phi\|_{2,\theta'}.$$

To estimate the second term we invoke the Sobolev imbedding theorem:

$$W^{1-(2/\theta),\theta'}(\Omega) \hookrightarrow L^2(\Omega).$$

Since $I - \gamma_h\Pi_h$ vanishes for constant polynomials, we can apply the Bramble–Hilbert lemma (cf. Theorem 2.3 of [46]) to obtain

$$\|\boldsymbol{\zeta} - \gamma_h\Pi_h\boldsymbol{\zeta}\|_0 \le Ch^{2/\theta}\,\|\boldsymbol{\zeta}\|_{2/\theta}.$$

Thus, it follows that

$$\begin{aligned}(\alpha\boldsymbol{\omega}, \boldsymbol{\zeta} - \gamma_h\Pi_h\boldsymbol{\zeta}) &\le C\|\boldsymbol{\omega}\|_0\|\boldsymbol{\zeta} - \gamma_h\Pi_h\boldsymbol{\zeta}\|_0 \le Ch^{2/\theta}\,\|\boldsymbol{\omega}\|_0\|\nabla\phi\|_{2/\theta}\\ &\le Ch^{2/\theta}\,\|\boldsymbol{\omega}\|_0\|\nabla\phi\|_{1,\theta'} \le Ch^{2/\theta}\,\|\boldsymbol{\omega}\|_0\|\phi\|_{2,\theta'}.\end{aligned}$$

By (2.19) we also have

$$\begin{aligned}(\boldsymbol{\Gamma}\tau, \boldsymbol{\zeta} - \gamma_h\Pi_h\boldsymbol{\zeta}) &\le C\|\tau\|_{0,\theta}\|\boldsymbol{\zeta} - \gamma_h\Pi_h\boldsymbol{\zeta}\|_{0,\theta'} \le Ch\,\|\tau\|_{0,\theta}\|\nabla\phi\|_{1,\theta'}\\ &\le Ch\,\|\tau\|_{0,\theta}\|\phi\|_{2,\theta'},\end{aligned}$$

and

$$\begin{aligned}(\operatorname{div}\boldsymbol{\omega}, \phi - P_h\phi) &\le Ch\,\|\operatorname{div}\boldsymbol{\omega}\|_0\|\phi\|_1 \le Ch\,\|\operatorname{div}\boldsymbol{\omega}\|_0\|\phi\|_{2-2/\theta,\theta'}\\ &\le Ch\,\|\operatorname{div}\boldsymbol{\omega}\|_0\|\phi\|_{2,\theta'},\\ (c_p\tau, \phi - P_h\phi) &\le C\|\tau\|_{0,\theta}\|\phi - P_h\phi\|_{0,\theta'} \le Ch\,\|\tau\|_{0,\theta}\|\phi\|_{2,\theta'}.\end{aligned}$$

Finally, using the Sobolev imbedding $W^{2,1}(\Omega) \hookrightarrow L^2(\Omega)$, we obtain

$$r(P_h\phi) \le \|r\|_0\|P_h\phi\|_0 \le \|r\|_0\|\phi\|_0 \le C\|r\|_0\|\phi\|_{2,1} \le C\|r\|_0\|\phi\|_{2,\theta'}.$$

Combining these results and applying (4.3), we obtain

$$(\tau, \psi) \le C(h\|\tau\|_0 + h^{2/\theta}\|\boldsymbol{\omega}\|_0 + h\|\operatorname{div}\boldsymbol{\omega}\|_0 + \|\mathbf{q}\|_0 + \|r\|_0)\,\|\psi\|_{0,\theta'}.$$

Now we divide both sides by $\|\psi\|_{0,\theta'}$ and take the supremum with respect to ψ. The proof will be completed if the term $Ch\|\tau\|_0$ is absorbed into $\|\tau\|_0$ for sufficiently small h. ∎

As a corollary we obtain the following.

COROLLARY 4.2. *The mixed covolume method for the linear operator M has a unique solution, provided h is sufficiently small.*

Proof. The proof is almost the same as in [17]. For completeness we repeat it here. Since the system is linear, it suffices to prove uniqueness. Suppose that $(\hat{\mathbf{u}}_h, \hat{p}_h)$ satisfies

$$\begin{aligned}(\alpha\hat{\mathbf{u}}_h, \gamma_h\mathbf{v}) - (\operatorname{div}\mathbf{v}, \hat{p}_h) + (\boldsymbol{\Gamma}\hat{p}_h, \gamma_h\mathbf{v}) = 0 \qquad &\forall\mathbf{v} \in \mathbf{V}_h,\\ (\operatorname{div}\hat{\mathbf{u}}_h, w) + (c_p\,\hat{p}_h, w) = 0 \qquad &\forall w \in W_h.\end{aligned}$$

By taking $w = \operatorname{div}\hat{\mathbf{u}}_h$ we obtain

$$\|\operatorname{div}\hat{\mathbf{u}}_h\|_0 \le C\|\hat{p}_h\|_0.$$

Lemma 4.1 implies that

$$\|\hat{p}_h\|_0 \leq Ch(\|\hat{\mathbf{u}}_h\|_0 + \|\operatorname{div}\hat{\mathbf{u}}_h\|_0),$$

so that for sufficiently small h we have

$$\|\hat{p}_h\|_0 \leq Ch\|\hat{\mathbf{u}}_h\|_0.$$

Finally, if we take $\mathbf{v} = \hat{\mathbf{u}}_h$, then it follows from (2.17) that

$$\|\hat{\mathbf{u}}_h\|_0 \leq C\|\hat{p}_h\|_0 \leq Ch\|\hat{\mathbf{u}}_h\|_0,$$

which yields $\hat{\mathbf{u}}_h = \hat{p}_h = 0$ for sufficiently small h. This completes the proof. □

Now we are ready to prove the following theorem.

THEOREM 4.3. *For $\delta > 0$ sufficiently small (dependent on h), Φ maps a ball of radius δ of $\mathbf{V}_h \times W_h$ into itself.*

Proof. By Lemma 4.1 it is easy to see that the proof of this theorem is exactly the same as the proof of Theorem 2.1 in [35], except that the linear functional $\mathbf{q}$ is now given as

$$\begin{aligned}\mathbf{q}(\mathbf{v}) = (\alpha(p)(\Pi_h\mathbf{u} - \mathbf{u}) + \mathbf{\Gamma}(P_h p - p) + \tilde{\alpha}_p(\rho)(p-\rho)(\mathbf{u} - \boldsymbol{\mu}) \\ + [\tilde{\alpha}_{pp}(\rho)\mathbf{u} + \tilde{\boldsymbol{\beta}}_{pp}(\rho)](p-\rho)^2, \gamma_h\mathbf{v}) - (\alpha(p)\mathbf{u} + \boldsymbol{\beta}(p), \mathbf{v} - \gamma_h\mathbf{v}),\end{aligned} \tag{4.4}$$

where the test function in the first term is $\gamma_h\mathbf{v}$, and we have an additional term $(\alpha(p)\mathbf{u} + \boldsymbol{\beta}(p), \mathbf{v} - \gamma_h\mathbf{v})$.

By (2.16) it suffices to estimate this additional term. This can be done by (2.18), which implies that

$$(\alpha(p)\mathbf{u}, \mathbf{v} - \gamma_h\mathbf{v}) \leq Ch\|\mathbf{u}\|_1\|\mathbf{v}\|_0 \qquad \forall \mathbf{v} \in \mathbf{V}_h, \tag{4.5}$$

and by a similar technique we also have

$$(\boldsymbol{\beta}(p), \mathbf{v} - \gamma_h\mathbf{v}) = (\boldsymbol{\beta}(p) - \bar{\boldsymbol{\beta}}_h, \mathbf{v} - \gamma_h\mathbf{v}) \leq Ch\|\mathbf{v}\|_0 \qquad \forall \mathbf{v} \in \mathbf{V}_h, \tag{4.6}$$

where $\bar{\boldsymbol{\beta}}_h$ is a piecewise constant approximation to $\boldsymbol{\beta}(p)$ satisfying $\|\boldsymbol{\beta}(p) - \bar{\boldsymbol{\beta}}_h\|_0 \leq Ch$. Thus the proof is completed by absorbing (4.5) and (4.6) into the term $h\|\mathbf{u}\|_1$ in the proof of Theorem 2.1 in [35]. □

COROLLARY 4.4. *Let $0 < \varepsilon < 1$ and $\theta = (4+2\varepsilon)/\varepsilon$. Then there exists a sequence $\{(\mathbf{u}_h, p_h)\}_{h>0}$ satisfying*

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,2+\varepsilon} + \|p - p_h\|_{0,\theta} \leq Ch^{2/(2+\varepsilon)}. \tag{4.7}$$

Moreover, the following L^∞ bound holds:

$$\|\mathbf{u}_h\|_{0,\infty} \leq C(\|p\|_{2+\varepsilon}^2 + 1). \tag{4.8}$$

Proof. See equations (3.1) and (3.12) in [35]. □

5. L^2-error estimates. Throughout the remainder of the paper we set

$$\boldsymbol{\xi} = \mathbf{u} - \mathbf{u}_h, \quad \boldsymbol{\sigma} = \Pi_h\mathbf{u} - \mathbf{u}_h, \quad \tau = P_h p - p_h.$$

By means of the first-order Taylor expansions, we rewrite the error equations (3.1) in the form

$$(5.1a)\qquad (\alpha(p)\boldsymbol{\xi},\gamma_h\mathbf{v})-(\operatorname{div}\mathbf{v},\tau)+(\boldsymbol{\Gamma}_h\tau,\gamma_h\mathbf{v})=\mathbf{q}(\mathbf{v})\qquad\forall\mathbf{v}\in\mathbf{V}_h,$$
$$(5.1b)\qquad (\operatorname{div}\boldsymbol{\xi},w)+(\tilde{c}_p(p_h)\tau,w)=r(w)\qquad\forall w\in W_h,$$

where we set $\boldsymbol{\Gamma}_h=\tilde{\alpha}_p(p_h)\mathbf{u}_h+\tilde{\boldsymbol{\beta}}_p(p_h)$ and the linear functionals $\mathbf{q}$ and r are given by

$$\mathbf{q}(\mathbf{v})=(\boldsymbol{\Gamma}_h(P_hp-p),\gamma_h\mathbf{v})-(\alpha(p)\mathbf{u}+\boldsymbol{\beta}(p),\mathbf{v}-\gamma_h\mathbf{v})\qquad\forall\mathbf{v}\in\mathbf{V}_h,$$
$$r(w)=(\tilde{c}_p(p_h)(P_hp-p),w)\qquad\forall w\in W_h.$$

We remark that $\|\boldsymbol{\Gamma}_h\|_{0,\infty}\le C(\|p\|_{2+\varepsilon}^2+1)$ by (4.8).

Observe again that the system (5.1) corresponds to the error equations of the mixed covolume method for the linear operator $N_h: H^2(\Omega)\cap H_0^1(\Omega)\to L^2(\Omega)$ given by

$$N_h\chi=-\nabla\cdot(a(p)\nabla\chi+a(p)\boldsymbol{\Gamma}_h\chi)+\tilde{c}_p(p_h)\chi.$$

Its formal adjoint $N_h^*: H^2(\Omega)\cap H_0^1(\Omega)\to L^2(\Omega)$ is

$$N_h^*\chi=-\nabla\cdot(a(p)\nabla\chi)+a(p)\boldsymbol{\Gamma}_h\cdot\nabla\chi+\tilde{c}_p(p_h)\chi.$$

To apply the duality argument to the mixed system (5.1) we need the following technical result.

LEMMA 5.1. *There exists an $h_0>0$ such that, if $h<h_0$, N_h^* has a bounded inverse mapping from $L^2(\Omega)$ onto $H^2(\Omega)\cap H_0^1(\Omega)$.*

Proof. This is Lemma 3.1 in [35]. The basic idea of its proof is to compare N_h^* with M^*, which is independent of h. Following the proof, we easily see that the central part of the proof lies in the estimate (4.7). □

Now we can apply the duality argument to (5.1). Let $\phi\in H^2(\Omega)$ be the solution of the adjoint problem $N_h^*\phi=\psi$ in Ω, $\phi=0$ on $\partial\Omega$. Then, by Lemma 5.1, the elliptic regularity estimate $\|\phi\|_2\le C\|\psi\|_0$ holds. Proceeding in the same way as in the proof of Lemma 4.1, with $\boldsymbol{\Gamma}$ and c_p replaced by $\boldsymbol{\Gamma}_h$ and $\tilde{c}_p(p_h)$ and with $\theta=2$, we arrive at the same result:

$$(5.2)\qquad \|\tau\|_0\le C(h\|\boldsymbol{\xi}\|_0+h\|\operatorname{div}\boldsymbol{\xi}\|_0+\|\mathbf{q}\|_0+\|r\|_0),$$

since $\boldsymbol{\Gamma}_h$ and $\tilde{c}_p(p_h)$ are bounded functions in Ω.

Estimation of $\|\mathbf{q}\|_0$ and $\|r\|_0$ can be done in a straightforward way by using (2.16), (4.5), (4.6):

$$\begin{aligned}\mathbf{q}(\mathbf{v})&\le\|\boldsymbol{\Gamma}_h\|_{0,\infty}\|p-P_hp\|_0\|\gamma_h\mathbf{v}\|_0+Ch(\|\mathbf{u}\|_1+1)\|\mathbf{v}\|_0\\&\le C(\|p\|_{2+\varepsilon}^2+1)h(\|p\|_1+\|\mathbf{u}\|_1)\|\mathbf{v}\|_0,\end{aligned}$$

which yields

$$(5.3)\qquad \|\mathbf{q}\|_0\le C(\|p\|_{2+\varepsilon}^2+1)h\|p\|_2,$$

and

$$r(w)\le C\|p-P_hp\|_0\|w\|_0\le Ch\|p\|_1\|w\|_0,$$

which yields

$$\|r\|_0 \le Ch\|p\|_1. \tag{5.4}$$

Substituting (5.3), (5.4) into (5.2) we obtain

$$\|\tau\|_0 \le C(h\|\boldsymbol{\xi}\|_0 + h\|\operatorname{div}\boldsymbol{\xi}\|_0 + h\|p\|_2), \tag{5.5}$$

where C depends quadratically on $\|p\|_{2+\varepsilon}$.

To estimate $\|\boldsymbol{\xi}\|_0$ and $\|\operatorname{div}\boldsymbol{\xi}\|_0$ we write the system (5.1) as

$$(\alpha(p)\boldsymbol{\sigma}, \gamma_h \mathbf{v}) - (\operatorname{div}\mathbf{v}, \tau) + (\boldsymbol{\Gamma}_h\tau, \gamma_h\mathbf{v}) = (\alpha(p)(\Pi_h\mathbf{u} - \mathbf{u}), \gamma_h\mathbf{v}) + \mathbf{q}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_h, \tag{5.6a}$$

$$(\operatorname{div}\boldsymbol{\sigma}, w) + (\tilde{c}_p(p_h)\tau, w) = r(w) \qquad \forall w \in W_h. \tag{5.6b}$$

Taking $w = \operatorname{div}\boldsymbol{\sigma}$, we obtain by (5.4)

$$\|\operatorname{div}\boldsymbol{\sigma}\|_0 \le C(\|\tau\|_0 + \|r\|_0) \le C(\|\tau\|_0 + h\|p\|_1), \tag{5.7}$$

and then, taking $\mathbf{v} = \boldsymbol{\sigma}$, we obtain

$$\begin{aligned}(\alpha(p)\boldsymbol{\sigma}, \gamma_h\boldsymbol{\sigma}) &\le C(\|\operatorname{div}\boldsymbol{\sigma}\|_0\|\tau\|_0 + \|\tau\|_0\|\boldsymbol{\sigma}\|_0 + h\|\mathbf{u}\|_1\|\boldsymbol{\sigma}\|_0 + \|\mathbf{q}\|_0\|\boldsymbol{\sigma}\|_0) \\ &\le C(\|\tau\|_0^2 + h\|p\|_1\|\tau\|_0 + \|\tau\|_0\|\boldsymbol{\sigma}\|_0 + h\|\mathbf{u}\|_1\|\boldsymbol{\sigma}\|_0 + \|\mathbf{q}\|_0\|\boldsymbol{\sigma}\|_0).\end{aligned}$$

By applying the arithmetic-geometric inequality, (2.17), and (5.3) it follows that

$$\|\boldsymbol{\sigma}\|_0 \le C(\|\tau\|_0 + h\|p\|_1 + h\|\mathbf{u}\|_1). \tag{5.8}$$

From (5.7), (5.8) it is immediate that

$$\|\boldsymbol{\xi}\|_0 \le \|\mathbf{u} - \Pi_h\mathbf{u}\|_0 + \|\boldsymbol{\sigma}\|_0 \le C(\|\tau\|_0 + h\|p\|_2), \tag{5.9}$$

and for $s = 0, 1$

$$\begin{aligned}\|\operatorname{div}\boldsymbol{\xi}\|_0 &\le \|\operatorname{div}\mathbf{u} - \operatorname{div}(\Pi_h\mathbf{u})\|_0 + \|\operatorname{div}\boldsymbol{\sigma}\|_0 \\ &\le C(h^s\|\operatorname{div}\mathbf{u}\|_s + \|\tau\|_0 + h\|p\|_1),\end{aligned} \tag{5.10}$$

which, when substituted into (5.5) with $s = 0$, gives

$$\|\tau\|_0 \le C(h\|\tau\|_0 + h\|p\|_2).$$

Thus we obtain for sufficiently small h

$$\|\tau\|_0 \le Ch\|p\|_2 \tag{5.11}$$

or

$$\|p - p_h\|_0 \le \|p - P_hp\|_0 + \|\tau\|_0 \le Ch\|p\|_2. \tag{5.12}$$

Substituting (5.11) back into (5.9), (5.10) yields

$$\|\boldsymbol{\xi}\|_0 \le Ch\|p\|_2, \tag{5.13}$$

$$\|\operatorname{div}\boldsymbol{\xi}\|_0 \le Ch(\|p\|_2 + \|\operatorname{div}\mathbf{u}\|_1). \tag{5.14}$$

Our results can be summarized as follows.

THEOREM 5.2. *For sufficiently small h there is a positive constant C, depending on $\|p\|_{2+\varepsilon}$ quadratically such that*

$$\|p - p_h\|_0 \le Ch\|p\|_2, \tag{5.15}$$

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \le Ch\|p\|_2, \tag{5.16}$$

$$\|\operatorname{div}(\mathbf{u} - \mathbf{u}_h)\|_0 \le Ch(\|p\|_2 + \|\operatorname{div}\mathbf{u}\|_1). \tag{5.17}$$

6. Superconvergence for the pressure. With the help of Theorem 5.2 we can obtain the following superconvergence result for τ.

THEOREM 6.1. *For sufficiently small h there is a positive constant C, depending on* $\|p\|_{2+\varepsilon}^4$ *such that*

$$\|\tau\|_0 \le Ch^2(\|p\|_2 + \|\operatorname{div}\mathbf{u}\|_1 + 1). \tag{6.1}$$

Proof. The result is obtained by examining closely the duality argument applied to the system (5.1). We start with the formula

(6.2)

$$(\tau,\psi) = (\alpha\boldsymbol{\xi} + \boldsymbol{\Gamma}_h\tau, \boldsymbol{\zeta} - \gamma_h\Pi_h\boldsymbol{\zeta}) + (\operatorname{div}\boldsymbol{\xi} + \tilde{c}_p(p_h)\tau, \phi - P_h\phi) + \mathbf{q}(\Pi_h\boldsymbol{\zeta}) + r(P_h\phi),$$

which was given in the proof of Lemma 4.1. Recall that

$$\mathbf{q}(\Pi_h\boldsymbol{\zeta}) = (\boldsymbol{\Gamma}_h(P_hp - p), \gamma_h\Pi_h\boldsymbol{\zeta}) - (\alpha(p)\mathbf{u} + \boldsymbol{\beta}(p), \Pi_h\boldsymbol{\zeta} - \gamma_h\Pi_h\boldsymbol{\zeta}),$$
$$r(P_h\phi) = (\tilde{c}_p(p_h)(P_hp - p), P_h\phi),$$

where

$$\boldsymbol{\Gamma}_h = \tilde{\alpha}_p(p_h)\mathbf{u}_h + \tilde{\boldsymbol{\beta}}_p(p_h), \qquad \tilde{\alpha}_p(p_h) = \int_0^1 \alpha_p(p_h + t(p - p_h))\,dt,$$

and there are similar expressions for $\tilde{\boldsymbol{\beta}}_p(p_h)$ and $\tilde{c}_p(p_h)$.

The first two terms are estimated in the same way:

$$(\alpha\boldsymbol{\xi} + \boldsymbol{\Gamma}_h\tau, \boldsymbol{\zeta} - \gamma_h\Pi_h\boldsymbol{\zeta}) \le Ch(\|\boldsymbol{\xi}\|_0 + \|\tau\|_0)\|\psi\|_0, \tag{6.3}$$

$$(\operatorname{div}\boldsymbol{\xi} + \tilde{c}_p(p_h)\tau, \phi - P_h\phi) \le Ch(\|\operatorname{div}\boldsymbol{\xi}\|_0 + \|\tau\|_0)\|\psi\|_0. \tag{6.4}$$

Thus we need to examine the terms $\mathbf{q}(\Pi_h\boldsymbol{\zeta})$ and $r(P_h\phi)$.

Observe first that

$$\begin{aligned}\alpha_p(p) - \tilde{\alpha}_p(p_h) &= \int_0^1 [\alpha_p(p) - \alpha_p(p_h + t(p - p_h))]\,dt \\ &= (p - p_h)\int_0^1 (1 - t)\alpha_{pp}(p^*(t))\,dt \\ &= \bar{\alpha}_{pp}(p - p_h),\end{aligned}$$

and similarly

$$\boldsymbol{\beta}_p(p) - \tilde{\boldsymbol{\beta}}_p(p_h) = \bar{\boldsymbol{\beta}}_{pp}(p - p_h), \qquad c_p(p) - \tilde{c}_p(p_h) = \bar{c}_{pp}(p - p_h),$$

which implies that

$$\begin{aligned}\boldsymbol{\Gamma} - \boldsymbol{\Gamma}_h &= \alpha_p(p)\mathbf{u} - \tilde{\alpha}_p(p_h)\mathbf{u}_h + \boldsymbol{\beta}_p(p) - \tilde{\boldsymbol{\beta}}_p(p_h) \\ &= (\alpha_p(p) - \tilde{\alpha}_p(p_h))\mathbf{u} + \tilde{\alpha}_p(p_h)(\mathbf{u} - \mathbf{u}_h) + \boldsymbol{\beta}_p(p) - \tilde{\boldsymbol{\beta}}_p(p_h) \\ &= (\bar{\alpha}_{pp}\mathbf{u} + \bar{\boldsymbol{\beta}}_{pp})(p - p_h) + \tilde{\alpha}_p(p_h)(\mathbf{u} - \mathbf{u}_h).\end{aligned}$$

Then we obtain by Theorem 5.2

$$\begin{aligned}(\boldsymbol{\Gamma}_h(P_hp - p), \gamma_h\Pi_h\boldsymbol{\zeta}) &= ((\boldsymbol{\Gamma}_h - \boldsymbol{\Gamma})(P_hp - p), \gamma_h\Pi_h\boldsymbol{\zeta}) + (\boldsymbol{\Gamma}(P_hp - p), \gamma_h\Pi_h\boldsymbol{\zeta}) \\ &= (\{[\bar{\alpha}_{pp}\mathbf{u} + \bar{\boldsymbol{\beta}}_{pp}](p_h - p) + \tilde{\alpha}_p(p_h)(\mathbf{u}_h - \mathbf{u})\}(P_hp - p), \gamma_h\Pi_h\boldsymbol{\zeta}) \\ &\quad + (\boldsymbol{\Gamma}(P_hp - p), \gamma_h\Pi_h\boldsymbol{\zeta} - \boldsymbol{\zeta}) + (\boldsymbol{\Gamma}(P_hp - p), \boldsymbol{\zeta}) \\ &\le C(\|p - p_h\|_0 + \|\mathbf{u} - \mathbf{u}_h\|_0)\|p - P_hp\|_{0,\infty}\|\gamma_h\Pi_h\boldsymbol{\zeta}\|_0 \\ &\quad + C\|p - P_hp\|_0\|\boldsymbol{\zeta} - \gamma_h\Pi_h\boldsymbol{\zeta}\|_0 + C\|p - P_hp\|_{-1}\|\boldsymbol{\zeta}\|_1 \\ &\le Ch^2(\|p\|_1 + \|\mathbf{u}\|_1)\|\boldsymbol{\zeta}\|_1 \le Ch^2\|p\|_2\|\psi\|_0,\end{aligned}$$

and for any $w \in W_h$,

$$\begin{aligned}(\tilde{c}_p(p_h)(P_hp - p), P_h\phi) &= ([\tilde{c}_p(p_h) - c_p(p)](P_hp - p), P_h\phi) + (c_p(p)(P_hp - p), P_h\phi)\\ &= (\bar{c}_{pp}(p_h - p)(P_hp - p), P_h\phi) + ([c_p(p) - w](P_hp - p), P_h\phi)\\ &\le C\|p - p_h\|_0\|p - P_hp\|_{0,\infty}\|P_h\phi\|_0\\ &\qquad + \|c_p(p) - w\|_{0,\infty}\|p - P_hp\|_0\|P_h\phi\|_0\\ &\le Ch^2\|p\|_2\|\phi\|_0,\end{aligned}$$

where we take the infimum over $w \in W_h$. Here the constant C depends on the product $\|\mathbf{u}\|_{0,\infty}\|p\|_{0,\infty}\|p\|^2_{2+\varepsilon}$, or $\|p\|^4_{2+\varepsilon}$ by the Sobolev imbedding theorem.

Finally, we need to estimate the remaining term $(\alpha(p)\mathbf{u} + \boldsymbol{\beta}(p), \Pi_h\boldsymbol{\zeta} - \gamma_h\Pi_h\boldsymbol{\zeta})$. Letting $\mathbf{w} = \alpha(p)\mathbf{u} + \boldsymbol{\beta}(p)$ and $\bar{\mathbf{w}}_h$ be a piecewise constant approximation to $\mathbf{w}$ which satisfies $\|\mathbf{w} - \bar{\mathbf{w}}_h\|_0 \le C\|\mathbf{w}\|_1$, we obtain

$$\begin{aligned}(\alpha(p)\mathbf{u} + \boldsymbol{\beta}(p), \Pi_h\boldsymbol{\zeta} - \gamma_h\Pi_h\boldsymbol{\zeta}) &= (\mathbf{w}, \Pi_h\boldsymbol{\zeta} - \gamma_h\Pi_h\boldsymbol{\zeta}) = (\mathbf{w} - \bar{\mathbf{w}}_h, \Pi_h\boldsymbol{\zeta} - \gamma_h\Pi_h\boldsymbol{\zeta})\\ &= (\mathbf{w} - \bar{\mathbf{w}}_h, \Pi_h\boldsymbol{\zeta} - \boldsymbol{\zeta}) + (\mathbf{w} - \bar{\mathbf{w}}_h, \boldsymbol{\zeta} - \gamma_h\Pi_h\boldsymbol{\zeta})\\ &\le Ch^2\,\|\mathbf{w}\|_1\|\boldsymbol{\zeta}\|_1 \le Ch^2\,(\|\mathbf{u}\|_1 + 1)\|\phi\|_2.\end{aligned}$$

Consequently, we arrive at

$$\mathbf{q}(\Pi_h\boldsymbol{\zeta}) \le Ch^2(\|p\|_2 + 1)\|\psi\|_0, \tag{6.5}$$

$$r(P_h\phi) \le Ch^2\|p\|_2\|\psi\|_0. \tag{6.6}$$

Now combining (6.3)–(6.6) and taking the supremum with respect to ψ give

$$\|\tau\|_0 \le C(h\|\boldsymbol{\xi}\|_0 + h\|\operatorname{div}\boldsymbol{\xi}\|_0 + h^2(\|p\|_2 + 1)),$$

and when substituting (5.9) and (5.10) with $s = 1$ into this, we obtain for sufficiently small h

$$\|\tau\|_0 \le Ch^2(\|p\|_2 + \|\operatorname{div}\mathbf{u}\|_1 + 1). \tag{6.7}$$

This completes the proof. ∎

COROLLARY 6.2. *For $2 < q \le \infty$ the following optimal L^q-error estimate for the pressure variable holds:*

$$\|p - p_h\|_{0,q} \le Ch(\|p\|_{1,q} + \|p\|_2 + \|\operatorname{div}\mathbf{u}\|_1 + 1). \tag{6.8}$$

Proof. The result can be derived in a straightforward manner by using the inverse inequality. For $2 < q < \infty$, we obtain

$$\begin{aligned}\|p - p_h\|_{0,q} &\le \|p - P_hp\|_{0,q} + \|\tau\|_{0,q} \le Ch\|p\|_{1,q} + Ch^{-(q-2)/q}\|\tau\|_0\\ &\le Ch\|p\|_{1,q} + Ch^{-(q-2)/q}h^2(\|p\|_2 + \|\operatorname{div}\mathbf{u}\|_1 + 1)\\ &\le Ch(\|p\|_{1,q} + \|p\|_2 + \|\operatorname{div}\mathbf{u}\|_1 + 1),\end{aligned}$$

and for $q = \infty$,

$$\begin{aligned}\|p - p_h\|_{0,\infty} &\le \|p - P_hp\|_{0,\infty} + \|\tau\|_{0,\infty} \le Ch\|p\|_{1,\infty} + Ch^{-1}\|\tau\|_0\\ &\le Ch\|p\|_{1,\infty} + Ch^{-1}h^2(\|p\|_2 + \|\operatorname{div}\mathbf{u}\|_1 + 1)\\ &\le Ch(\|p\|_{1,\infty} + \|p\|_2 + \|\operatorname{div}\mathbf{u}\|_1 + 1),\end{aligned}$$

which implies the result. □

REMARK 6.1. *Negative-norm error estimates and uniqueness of a solution near* $(\mathbf{u}, p)$ *can be established by a similar technique in* [35].

REMARK 6.2. *When* $\partial \mathbf{b}/\partial p$ *is large, we have a convection-dominated problem and one should employ special discretizations such as upwinding schemes in* [15] *or* [41]. *This will be the subject of our future research.*

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