Superconvergence of a finite element method for linear integro-differential problems

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Abstract

We introduce a new way of approximating initial condition to the semi-discrete finite element method for integro-differential equations using any degree of elements. We obtain several superconvergence results for the error between the approximate solution and the Ritz-Volterra projection of the exact solution. For $k > 1$, we obtain first order gain in $L_p(2 \leq p \leq \infty)$ norm, second order in $W^{1,p}(2 \leq p \leq \infty)$ norm and almost second order in $W^{1,\infty}$ norm. For $k = 1$, we obtain first order gain in $W^{1,p}(2 \leq p \leq \infty)$ norms. Further, applying interpolated postprocessing technique to the approximate solution we get one order global superconvergence between the exact solution and the interpolation of the approximate solution in the L_p and $W^{1,p}$ $(2 \leq p \leq \infty)$.

Keywords: finite element method, linear integro-differential equation, superconvergence

1 Introduction

Consider the following problem with memory:

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$$
\begin{aligned}
&\text{(a)} & u_t = &\ \nabla\cdot\left\{a(x,t)\nabla u+\int_0^t b(x,t,\tau)\nabla u(x,\tau)d\tau\right\}+f(x,t),\\
& & &(x,t)\in\Omega\times(0,T],\\
&\text{(b)} & u(x,0) = &\ u_0(x),\quad x\in\Omega,\\
&\text{(c)} & u(x,t) = &\ 0,\quad x\in\partial\Omega\times[0,T],
\end{aligned}
\tag{1.1}
$$

where Ω is a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and a, b, f, u_0 are bounded together with their derivatives and

$$0 < a_0 \le a(x,t), \quad (x,t)\in\Omega\times[0,T]. \tag{1.2}$$

For the existence, uniqueness and stability of above integro-differential equations, we refer to [2], [10] and [12]. The weak form of (1.1) is to find a map $u(t) : [0,T] \to H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega)$, the following holds.

$$
\begin{aligned}
(u_t,v)+(a(t)\nabla u,\nabla v)+\left(\int_0^t b(t,\tau)\nabla u(\tau)d\tau,\nabla v\right) &= (f(t),v),\\
u(0) &= u_0.
\end{aligned}
\tag{1.3}
$$

For the semi-discrete finite element methods of such problems, various error estiamtes are known, for example, maximum norm error estimate for $k = 1$ was shown by Lin [8], Zhang and Lin [14], optimal L_p error estiamtes are by Lin, Thomée and Wahlbin [9], maximum norm estimates are shown by[4] and [13], optimal L_2 error estimates for nonlinear equations are shown by Cannon and Lin [1], while optimal L_2 estimates for Crank-Nicolson scheme are shown in [7]. In this paper, we introduce a semi-discrete finite element formulation for solving (1.3) using a new initial approximation U_0(see (2.4) below). This initial condition enables us to obtain a higher order accuracy of our finite element solution U to the Ritz-Volterra projection $V_h u$ of the exact solution u for all $k \ge 1$. Furthermore, our estimates can be used to improve the approximation accuracy of U to u by certain postprocessing as in [4], [5], [6]. For this purpose, we proceed as follows. We first adopt the rectangular partition $\mathcal{T}_h$ on the domain Ω and then introduce two interpolation operators, the "vertices-edges-element" interpolation operator i_h^k and high interpolation operator I_{2h}^{k+1}. Applying the properties of these operators and superconvergence of $U - V_h u$ we can easily establish superconvergence results of $u - I_{2h}^{k+1} U$ to gain one order comparing with the standard finite element methods of degree k in both L_p and $W^{1,p}$, $(2 \le p \le \infty)$ norms by one order.

The rest of this paper is organized as follows. In Section 2, we give the semidiscrete Galerkin approximation scheme of the problem and define the Ritz-Volterra projection. Section 3 is devoted to the superconvergence of $U - V_h u$ for $k > 1$. The superconvergence estimate of $U - V_h u$ for $k = 1$ are derived in Section 4. Finally, the interpolated postprocessing technique is discussed in Section 5. The global superconvergence results of $u - I_{2h}^{k+1} U$ are demonstrated in Theorem 5.4.

2 The approximation scheme and Ritz-Volteraa projection

First let us describe some of the notations used throughout this paper. Let $L_2(\Omega)$, $L_p(\Omega)$ and $W^{m,p}(\Omega)$, $H^m(\Omega) = W^{m,2}(\Omega)$ for any integer $m \geq 0$ and $1 \leq p \leq \infty$, denote the usual Lebesgue and Sobolev spaces on Ω, respectively. The L_2 and L_p norms are denoted by $\|\cdot\|$ and $\|\cdot\|_{0,p}$, the Sobolev norms by $\|\cdot\|_m$ and $\|\cdot\|_{m,p}$. For any $t \in [0, T]$ define

$$\|u(t)\|_{s,m,p} = \sum_{j=0}^{s} [\|D_t^j u(t)\|_{m,p} + \int_0^t \|D_t^j u(\tau)\|_{m,p} d\tau],$$

where $D_t^j = \partial^j / \partial t^j$. Let $(\cdot, \cdot)$ denote the inner product in $L_2(\Omega)$ or $L_2(\Omega)^2$. In this article, C will denote a generic positive constant independent of u and h, not necessarily the same at each occurrence. Moreover, we also use the notation p' to denote the conjugate index of p, $2 \leq p \leq \infty$ with $1/p + 1/p' = 1$.

Assume that $S_h \subset H_0^1(\Omega) \cap W^{1,\infty}(\Omega)$ is a finite element space which satisfies the following approximation properties.

$$\begin{array}{lll} \inf_{\chi \in S_h} \{\|v - \chi\|_{0,p} + h\|v - \chi\|_{1,p}\} & \leq & Ch^{k+1}\|v\|_{k+1,p}, \\ \quad v \in W^{k+1,p}(\Omega) \cap H_0^1(\Omega), & & k \geq 1,\ 2 \leq p \leq \infty. \end{array} \tag{2.1}$$

We also suppose that the standard inverse properties hold on S_h. Define the Ritz projection operator $R_h = R_h(t) : H_0^1(\Omega) \to S_h$ for $0 \leq t \leq T$ by

$$(a(t)\nabla(R_h w - w), \nabla\chi) = 0, \quad \chi \in S_h, \tag{2.2}$$

where $a(t) = a(x, t)$. Then the semi-discrete finite element approximation to (1.1) is to find a map $U(t) : (0, T] \to S_h$ such that

$$\begin{array}{ll} \text{(a)} (U_t, \chi) + (a(t)\nabla U, \nabla\chi) + & (\int_0^t b(t,\tau)\nabla U(\tau) d\tau . \nabla\chi) = (f(t), \chi), \\ & \qquad \chi \in S_h,\ 0 < t \leq T, \\ \text{(b)} & \qquad U(0) = U_0, \quad x \in \Omega, \end{array} \tag{2.3}$$

where $U_0 \in S_h$ is determined by

$$(a(0)\nabla U_0, \nabla \chi) = (f(0), \chi) - (R_h u_t(0), \chi), \quad \chi \in S_h \tag{2.4}$$

with

$$u_t(0) = \nabla \cdot (a(0)\nabla u_0) + f(0). \tag{2.5}$$

We now define the Ritz-Volterra projection operator. For any given function $w \in H_0^1(\Omega)$ define a function $V_h w \in S_h$ such that

$$(a(t)\nabla(V_h w - w), \nabla \chi) + (\int_0^t b(t,\tau)\nabla(V_h w(\tau) - w(\tau))d\tau, \nabla\chi) = 0, \chi \in S_h. \tag{2.6}$$

Obviously, when $t = 0$, V_h is the same as Ritz projection operator R_h. Let $\eta = V_h u - u$. The next lemmas concern the estimates of η which come from [9] and [11], respectively.

Lemma 2.1. *[9] For $k \geq 1$, we have*

$$\|D_t^s \eta(t)\|_{0,p} + h\|D_t^s \eta(t)\|_{1,p} \leq Ch^{k+1}\|u(t)\|_{s,k+1,p}, \quad 2 \leq p < \infty. \tag{2.7}$$

Lemma 2.2. *For $k > 1$, we have*

$$|(D_t^s \eta, \phi)| \leq Ch^{k+2}\|u\|_{s,k+1,p}\|\phi\|_{1,p'}, \phi \in W^{1,p'}(\Omega), 1 < p < \infty. \tag{2.8}$$

3 Superconvergence of $U - V_h u$ when $k > 1$

Let u and U respectively be the solutions of the problems (1.1) and (2.3) and let $\xi = U - V_h u$. In this section we study superconvergence of ξ for $k > 1$. We begin with the estimates for initial value of ξ and ξ_t.

Lemma 3.1. *We have, for $k > 1$*

$$\text{(i)} \qquad \xi_t(0) = 0, \ \textit{i.e.}, \ U_t(0) = R_h u_t(0), \tag{3.1}$$

$$\text{(ii)} \qquad \|\xi(0)\|_1 \leq Ch^{k+2}\{\|u_0\|_{k+1} + \|u_t(0)\|_{k+1}\}. \tag{3.2}$$

Proof. From (2.4) and (2.3)

$$\begin{aligned} (R_h u_t(0), \chi) &= (f(0), \chi) - (a(0)U_0, \nabla\chi) \\ &= (U_t(0), \chi), \quad \chi \in S_h. \end{aligned}$$

Hence $R_h u_t(0) = U_t(0)$. For (ii), we see from (1.3), (2.3a) and (2.6) that

$$(\xi_t, \chi) + (a(t)\nabla\xi, \nabla\chi) + (\int_0^t b(t,\tau)\nabla\xi(\tau)d\tau, \nabla\chi) = -(\eta_t, \chi), \quad \chi \in S_h. \tag{3.3}$$

Setting $t = 0$ and noting that $\xi_t(0) = 0$, we have

$$(a(0)\nabla\xi(0), \nabla\chi) = -(\eta_t(0), \chi). \tag{3.4}$$

Take $\chi = \xi(0)$ in (3.4). Then it follows from Lemma 2.2 that

$$\|\nabla\xi(0)\| \leq Ch^{k+2}[\|u(0)\|_{k+1} + \|u_t(0)\|_{k+1}].$$

Since $\|\nabla \cdot \|$ and $\| \cdot \|_1$ are equivalent in $H_0^1(\Omega)$, the proof is complete. $\square$

We will turn to the superconvergence estimates for ξ and show the following theorem.

Theorem 3.1. *We have, for $k > 1$*

$$\begin{aligned} &\|\xi_t(t)\| + \|\xi(t)\|_1 + \left(\int_0^t \|\xi_t(\tau)\|_1^2 d\tau\right)^{1/2} \\ \leq \ & Ch^{k+2}\{\|u_0\|_{k+1} + \|u_t(0)\|_{k+1} + (\sum_{j=0}^{2}\int_0^t \|D_\tau^j u\|_{k+1}^2 d\tau)^{1/2}\}. \end{aligned} \tag{3.5}$$

Proof. Setting $\chi = \xi_t$ in (3.3), we obtain by Lemma 2.2

$$\begin{aligned} \|\xi_t\|^2 \ &+ \ \frac{1}{2}\frac{d}{dt}(a(t)\nabla\xi, \nabla\xi) \\ &= \ \frac{1}{2}(a_t(t)\nabla\xi, \nabla\xi) - (\int_0^t b(t,\tau)\nabla\xi(\tau)d\tau, \nabla\xi_t) - (\eta_t, \xi_t) \\ &\leq \ C[\|\nabla\xi\|^2 + \int_0^t \|\nabla\xi\|d\tau\|\nabla\xi_t\| + h^{k+2}\|u\|_{1,k+1,2}\|\xi_t\|_1] \\ &\leq \ C[h^{2(k+2)}\|u\|_{1,k+1,2}^2 + \|\nabla\xi\|^2 + \int_0^t \|\nabla\xi\|^2 d\tau] + \frac{a_0}{4}\|\nabla\xi_t\|^2. \end{aligned} \tag{3.6}$$

Differentiating (3.3) with respect to t, we see that

$$\begin{aligned} (\xi_{tt}, \chi) + (a(t)\nabla\xi_t, \nabla\chi) \ &= \ -(\eta_{tt}, \chi) - (\alpha(t)\nabla\xi, \nabla\chi) \\ &\quad -(\int_0^t b_t(t,\tau)\nabla\xi(\tau)d\tau, \nabla\chi). \end{aligned} \tag{3.7}$$

Setting $\chi = \xi_t$ in (3.7) and using the similar technique to deriving (3.6), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\xi_t\|^2 \quad &+ \quad (a(t)\nabla\xi_t, \nabla\xi_t) \leq C[h^{2(k+2)}\|u\|^2_{2,k+1,2} + \|\nabla\xi\|^2 \\ &+ \int_0^t \|\nabla\xi\|^2 d\tau] + \frac{a_0}{4}\|\nabla\xi_t\|^2. \end{aligned} \tag{3.8}$$

Adding (3.8) onto (3.6) and using (1.2) we have

$$\begin{aligned}\frac{d}{dt}[\|\xi_t\|^2 \quad &+ \quad (a(t)\nabla\xi, \nabla\xi)] + \|\nabla\xi_t\|^2 \\ &\leq \quad C[h^{2(k+2)}\|u\|^2_{2,k+1,2} + \|\nabla\xi\|^2 + \int_0^t \|\nabla\xi\|^2 d\tau]. \end{aligned} \tag{3.9}$$

Integrating (3.9) with respect to t, we get by Lemma 3.1

$$\begin{aligned}\|\xi_t\|^2 \quad &+ \quad a_0\|\nabla\xi\|^2 + \int_0^t \|\nabla\xi_t\|^2 d\tau \leq \|\xi_t(0)\|^2 + (a(0)\nabla\xi(0), \nabla\xi(0)) \\ &+Ch^{2(k+2)}\int_0^t \|u\|^2_{2,k+1,2} d\tau + C\int_0^t \|\nabla\xi\|^2 d\tau \\ &\leq \quad Ch^{2(k+2)}[\|u_0\|^2_{k+1} + \|u_t(0)\|^2_{k+1} + \int_0^t \|u\|^2_{2,k+1,2} d\tau] \\ &+C\int_0^t \|\nabla\xi\|^2 d\tau. \end{aligned} \tag{3.10}$$

Finally, applying Gronwall's inequality, we obtain the result. □

To derive maximum norm superconvergence estimates, we introduce Green's functions. Let the discrete Green's functions $G_h^z \in S_h$ and $g_{h,i}^z \in S_h, i = 1, 2,$ for $z \in \Omega$ be defined by

$$(a(t)\nabla G_h^z, \nabla\chi) = \chi(z), \quad \chi \in S_h, \tag{3.11}$$

$$(a(t)\nabla g_{h,i}^z, \nabla\chi) = \frac{\partial}{\partial x_i}\chi(z), \quad \chi \in S_h, \tag{3.12}$$

respectively. Let the pre-Green's functions G_*^z and $g_{*,i}^z (i = 1, 2)$ be defined by

$$(a(t)\nabla G_*^z, \nabla v) = P_h v(z), \quad v \in H_0^1(\Omega), \tag{3.13}$$

$$(a(t)\nabla g_{*,i}^z, \nabla v) = P_h \frac{\partial}{\partial x_i} v(z), \quad v \in H_0^1(\Omega), \tag{3.14}$$

respectively, where $P_h : L_2(\Omega) \to S_h$ is the L_2 projection operator.

Lemma 3.2. *[15] We have*

$$\|G_h^z\| + \|G_h^z\|_{1,p'} + \|G_*^z\|_{1,1} + \|g_{h,i}^z - g_{*,i}^z\|_{1,1} + h\|g_{*,i}^z\|_{2,1} \le C, \tag{3.15}$$

for $1 \le p' < 2$ *and*

$$\|G_h^z\|'_{2,1} + \|g_{h,i}^z\|^2 + \|g_{*,i}^z\|_{1,1} \le C\log\frac{1}{h}, \tag{3.16}$$

$$\|G_h^z - G_*^z\|_{1,1} \le C\begin{cases} h, & \text{if } k>1, \\ h\log\frac{1}{h}, & \text{if } k=1. \end{cases} \tag{3.17}$$

where

$$\|G_h^z\|'_{2,1} = \sum_{\tau\in\mathcal{T}_h} \|G_h^z\|_{2,1,\tau}. \tag{3.18}$$

We can thus prove the superconvergence results for ξ in L_∞ and $W^{1,\infty}$.

Theorem 3.2. *We have, for* $k > 1$

$$\begin{aligned} &\|\xi(t)\|_{0,\infty} \le Ch^{k+2}\{\|u_0\|_{k+1} + \|u_t(0)\|_{k+1} + \|u(t)\|_{1,k+1,p} \\ &+(\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_{k+1}^2 d\tau)^{1/2}\}, \quad p>2. \end{aligned} \tag{3.19}$$

Proof. By (3.11) and (3.3), we have

$$\begin{aligned} \xi(z) &= (a(t)\nabla\xi, \nabla G_h^z) \\ &= -(\xi_t + \eta_t, G_h^z) - \left(\int_0^t b(t,\tau)\nabla\xi(\tau)d\tau, \nabla(G_h^z - G_*^z)\right) \\ &\quad - \int_0^t (b(t,\tau)\nabla\xi(\tau), \nabla G_*^z)d\tau \\ &\equiv I_1 + I_2 + I_3. \end{aligned} \tag{3.20}$$

By Lemma 2.2, Theorem 3.1 and Lemma 3.2, we get

$$\begin{aligned} |I_1| &\le |(\eta_t, G_h^z)| + |(\xi_t, G_h^z)| \\ &\le Ch^{k+2}\|u\|_{1,k+1,p}\|G_h^z\|_{1,p'} + \|\xi_t\|\|G_h^z\| \\ &\le Ch^{k+2}\{\|u\|_{1,k+1,p} + \|u_0\|_{k+1} + \|u_t(0)\|_{k+1} \\ &\quad +(\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_{k+1}^2 d\tau)^{1/2}\}, \\ |I_2| &\le C\int_0^t \|\xi\|_{1,\infty}d\tau\|G_h^z - G_*^z\|_{1,1} \\ &\le Ch\int_0^t \|\xi\|_{1,\infty}d\tau \le C\int_0^t \|\xi\|_{0,\infty}d\tau. \end{aligned}$$

By $\nabla(fg) = f\nabla g + g\nabla f$ and (3.13), we have

$$
\begin{aligned}
|I_3| &= \left| \int_0^t (a\nabla(\frac{b}{a}\xi(\tau)), \nabla G_*^z) d\tau - \int_0^t (a\xi(\tau)\nabla(\frac{b}{a}), \nabla G_*^z) d\tau \right| \\
&\le \left| \int_0^t P_h \left(\frac{b(z,t,\tau)}{a(z,\tau)} \xi(z,\tau) \right) d\tau \right| + C\int_0^t \|\xi\|_{0,\infty} d\tau \|G_*^z\|_{1,1} \\
&\le C\int_0^t \|\xi\|_{0,\infty} d\tau.
\end{aligned}
$$

Substituting estimates on I_1, I_2 and I_3 into (3.20), we obtain

$$
\begin{aligned}
|\xi(z)| \le\; & Ch^{k+2}\{\|u\|_{1,k+1,p} + \|u_0\|_{k+1} + \|u_t(0)\|_{k+1} \\
& +(\sum_{j=0}^{2} \int_0^t \|D_t^j u\|_{k+1}^2 d\tau)^{1/2}\} + C\int_0^t \|\xi\|_{0,\infty} d\tau.
\end{aligned}
$$

Applying Gronwall's lemma, we complete the proof. □

Theorem 3.3. *We have, for $k > 1$*

$$
\begin{aligned}
\|\xi(t)\|_{1,\infty} \le\; & Ch^{k+2-\epsilon}\{\|u_0\|_{k+1} + \|u_t(0)\|_{k+1} + \|u(t)\|_{1,k+1,p} \\
& +(\sum_{j=0}^{2} \int_0^t \|D_t^j u\|_{k+1}^2)^{1/2}\},
\end{aligned}
\tag{3.21}
$$

with $\epsilon > \frac{2}{p}$, p large enough.

Proof. Writing $g_h = g_{h,i}^z$ and $g_* = g_{*,i}^z$, we have, by definition (3.12) of g_h

$$
\begin{aligned}
\frac{\partial}{\partial x_i}\xi(z) &= (a(t)\nabla\xi, \nabla g_h) \\
&= -(\xi_t + \eta_t, g_h) - (\int_0^t b(t,\tau)\nabla\xi(\tau)d\tau), \nabla(g_h - g_*)) \\
&\quad -\int_0^t (b(t,\tau)\nabla\xi(\tau), \nabla g_*) d\tau \\
&\equiv J_1 + J_2 + J_3.
\end{aligned}
\tag{3.22}
$$

Using similar argument as in Theorem 3.2, we have by inverse properties and Theorems 3.1 and 3.2,

$$
\begin{aligned}
|J_1| &\le |(\eta_t, g_h)| + (\xi_t, g_h)| \\
&\le Ch^{k+2}\|u\|_{1,k+1,p}\|g_h\|_{1,p'} + \|\xi_t\|\|g_h\| \\
&\le Ch^{k+2}\{\|u\|_{1,k+1,p}h^{-\frac{2}{p}}\|g_h\|_{1,1} + [\|u_0\|_{k+1} \\
&\quad +\|u_t(0)\|_{k+1} + (\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_{k+1}^2 d\tau)^{1/2}]\|g_h\|\} \qquad (3.23)\\
&\le Ch^{k+2}\{\|u\|_{1,k+1,p}h^{-\frac{2}{p}} + \|u_0\|_{k+1} \\
&\quad +\|u_t(0)\|_{k+1} + (\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_{k+1}^2)^{1/2}\}\log\frac{1}{h}, \\
|J_2| &\le C\int_0^t \|\xi\|_{1,\infty}d\tau\|g_h - g_*\|_{1,1} \\
&\le C\int_0^t \|\xi\|_{1,\infty}d\tau \qquad (3.24)
\end{aligned}
$$

and

$$
\begin{aligned}
|J_3| &\le |\int_0^t (a\nabla(\frac{b}{a}\xi(\tau)), \nabla g_*)d\tau - \int_0^t (a\xi(\tau)\nabla(\frac{b}{a}), \nabla g_*)d\tau| \\
&\le \int_0^t |P_h\frac{\partial}{\partial x_i}(\frac{b(z,t,\tau)}{a(z,\tau)}\xi(z,\tau))|d\tau + C\int_0^t \|\xi\|_{0,\infty}d\tau\|g_*\|_{1,1} \\
&\le C\int_0^t \|\xi\|_{1,\infty}d\tau + Ch^{k+2}\{\|u_0\|_{k+1} + \|u_t(0)\|_{k+1} \qquad (3.25)\\
&\quad +\|u\|_{1,k+1,p} + (\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_{k+1}^2 d\tau)^{1/2}\}\log\frac{1}{h}.
\end{aligned}
$$

Combining (3.23) - (3.25) with (3.22) and applying Gronwall's lemma we derive the conclusion. □

Now let us turn to superconvergence of ξ in L_p and $W^{1,p}$ for $2 < p < \infty$.

Theorem 3.4. *We have, for $k > 1$*

$$
\|\xi(t)\|_{0,p} \le Ch^{k+2}\{\|u_0\|_{k+1} + \|u_t(0)\|_{k+1} + (\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_{k+1}^2 d\tau)^{1/2}\}, \quad 2 < p < \infty. \qquad (3.26)
$$

Proof. By Sobolev inequality, we have

$$\|\chi\|_{0,p} \le C\|\chi\|_1, \quad \chi \in S_h.$$

The conclusion follows from Theorem 3.1. □

Theorem 3.5. *We have, for $k > 1$*

$$\begin{aligned} \|\xi(t)\|_{1,p} \;\; &\le \;\; Ch^{k+2}\{\|u_0\|_{k+1} + \|u_t(0)\|_{k+1} + \|u_t(t)\|_{k+1,p} \\ &\quad + (\sum_{j=0}^{2} \int_0^t \|D_t^j u\|_{k+1}^2 d\tau)^{1/2}\},\ 2 < p < \infty. \end{aligned} \tag{3.27}$$

Proof. For any $\phi \in W^{1,p'}(\Omega)$ with $\|\phi\|_{0,p'} = 1$, let Φ be the solution of

$$(a(t)\nabla v, \nabla\Phi) = -(\phi_x, v), \quad v \in H_0^1(\Omega), \tag{3.28}$$

where ϕ_x is any component of $\nabla\phi$. Then

$$\|\Phi\|_{1,p'} \le C_p\|\phi\|_{0,p'} \le C_p. \tag{3.29}$$

Now by Green's formula, (3.28), (3.3), Theorem 3.1 and (3.29) we have as before

$$\begin{aligned} (\xi_x, \phi) &= -(\phi_x, \xi) \\ &= (a(t)\nabla\xi, \nabla\Phi) \\ &= (a(t)\nabla\xi, \nabla R_h\Phi) \\ &= -(\eta_t, R_h\Phi) - (\xi_t, R_h\Phi) - (\int_0^t b(t,\tau)\nabla\xi(\tau)d\tau, \nabla R_h\Phi) \\ &\le Ch^{k+2}\|u_t\|_{k+1,p}\|R_h\Phi\|_{1,p'} + \|\xi_t(t)\|\|R_h\Phi\| \\ &\quad + C\int_0^t \|\nabla\xi\|_{0,p}d\tau\|R_h\Phi\|_{1,p'} \\ &\le Ch^{k+2}\{\|u_t\|_{k+1,p} + \|u_0\|_{k+1} + \|u_t(0)\|_{k+1} \\ &\quad + (\sum_{j=0}^{2}\int_0^t \|D_t^j\|_{k+2}^2 d\tau)^{1/2}\} + C\int_0^t \|\nabla\xi\|_{0,p}d\tau. \end{aligned} \tag{3.30}$$

It follows from (3.30) that

$$\begin{aligned} \|\xi_x\|_{0,p} &= \sup_{\phi\in L^{p'}(\Omega)} (\xi_x, \phi) \\ &\le Ch^{k+2}\{\|u_t\|_{k+1,p} + \|u_0\|_{k+1} + \|u_t(0)\|_{k+1} \\ &\quad + (\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_{k+1}^2 d\tau)^{1/2}\} + C\int_0^t \|\nabla\xi\|_{0,p}d\tau. \end{aligned} \tag{3.31}$$

Summing both components of $\nabla\xi$ and using Gronwall's lemma we get (3.27). □

4 Superconvergence of $U - V_h u$ when $k = 1$

In this section we consider superconvergence when $k = 1$. Throughout this section $k = 1$. If Lemma 2.2 is replaced by Lemma 2.1 in the proof of Lemma 3.1 and Theorem 3.1, we obtain the following two results, respectively.

Lemma 4.1. *We have, for*

$$\begin{aligned} &(i) \qquad \xi_t(0) = 0, \ \ i.e, \ U_t(0) = R_h u_t(0), &(4.1)\\ &(ii) \qquad \|\xi(0)\|_1 \le Ch^2[\|u_0\|_2 + \|u_t(0)\|_2]. &(4.2) \end{aligned}$$

Theorem 4.1. *We have*

$$\begin{aligned} & \|\xi_t(t)\| + \|\xi(t)\|_1 + \left(\int_0^t \|\xi_t\|_1^2 d\tau\right)^{1/2} \\ \le\ & Ch^2\{\|u_0\|_2 + \|u_t(0)\|_2 + (\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_2^2 d\tau)^{1/2}\}. \qquad (4.3) \end{aligned}$$

Theorem 4.2. *We have*

$$\|\xi_t(t)\|_1 \ \le\ Ch^2\{\|u_0\|_2 + \|u_t(0)\|_2 + (\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_2^2 d\tau)^{1/2}\}. \qquad (4.4)$$

Proof. Taking $\chi = \xi_{tt}$ in (3.7), we have

$$\begin{aligned} \|\xi_{tt}\|^2 \ &+\ \tfrac{1}{2}[\tfrac{d}{dt}(a(t)\nabla\xi_t, \nabla\xi_t) - (a_t(t)\nabla\xi_t, \nabla\xi_t)] \\ &=\ -(\eta_{tt}, \xi_{tt}) - (\alpha(t)\nabla\xi, \nabla\xi_{tt}) - (\int_0^t b_t(t,\tau)\nabla\xi(\tau)d\tau, \nabla\xi_{tt}). \end{aligned}$$

Integrating this, we obtain by $\xi_t(0) = 0$,

$$\begin{aligned} & \int_0^t \|\xi_{tt}\|^2 d\tau + \frac{a_0}{2}\|\nabla\xi_t\|^2 \\ \le\ & \frac{1}{2}\int_0^t (a_t\nabla\xi_t, \nabla\xi_t)d\tau - \int_0^t (\eta_{tt}, \xi_{tt})d\tau \\ & - \int_0^t (\alpha\nabla\xi, \nabla\xi_{tt})d\tau - \int_0^t (\int_0^s (b_t(s,\tau)\nabla\xi(\tau)d\tau, \nabla\xi_{tt}(s))ds \qquad (4.5) \\ \equiv\ & K_1 + K_2 + K_3 + K_4. \end{aligned}$$

Obviously,

$$|K_1| \le C\int_0^t \|\nabla\xi_t\|^2 d\tau$$

and

$$|K_2| \leq C\int_0^t \|\eta_{tt}\|^2 d\tau + \int_0^t \|\xi_{tt}\|^2 d\tau.$$

By integration by parts, we have

$$K_3 = (\alpha(t)\nabla\xi, \nabla\xi_t) - \int_0^t [(\alpha_t \nabla\xi, \nabla\xi_t) + (\alpha\nabla\xi_t, \nabla\xi_t)]d\tau.$$

Noting (4.1) and

$$\|\nabla\xi(t)\|^2 \leq \|\nabla\xi(0)\|^2 + \int_0^t \|\nabla\xi_t\|^2 d\tau,$$

we see by arithematic-geometric inequality,

$$|K_3| \leq C[\|\nabla\xi(0)\|^2 + \int_0^t \|\nabla\xi_t\|^2 d\tau] + \frac{a_0}{8}\|\nabla\xi_t\|^2.$$

By integrating by part and changing order of integration, we have

$$\begin{aligned} K_4 &= -\int_0^t d\tau \int_\tau^t (b_t(s,\tau)\nabla\xi(\tau), \nabla\xi_{tt}(s))ds \\ &= \int_0^t \{(b_t(\tau,\tau)\nabla\xi(\tau), \nabla\xi_t(\tau)) - (b_t(t,\tau)\nabla\xi(\tau), \nabla\xi_t(t))\}d\tau \\ &\quad + \int_0^t d\tau \int_\tau^t (b_{tt}(s,\tau)\nabla\xi(\tau), \nabla\xi_t(s))ds, \end{aligned}$$

and hence by using similar argument as before, we have

$$|K_4| \leq C[\|\nabla\xi(0)\|^2 + \int_0^t \|\nabla\xi_t\|^2 d\tau] + \frac{a_0}{8}\|\nabla\xi_t\|.$$

Substituting estimates of K_1-K_4 into (4.5), we get

$$\|\nabla\xi_t\|^2 \leq C[\|\nabla\xi(0)\|^2 + \int_0^t \|\eta_{tt}\|^2 d\tau + \int_0^t \|\nabla\xi_t\|^2 d\tau].$$

Now Gronwall's inequality gives

$$\|\nabla\xi_t\|^2 \leq C[\|\nabla\xi(0)\|^2 + \int_0^t \|\eta_{tt}\|^2 d\tau].$$

This together with (4.2) and (2.7) completes the proof. $\square$

To derive superconvergence in $W^{1,\infty}$ we first bound $\|g^z_{h,i}\|_{0,p}$.

Lemma 4.2. *For $1 < p < 2$, we have*

$$\|g^z_{h,i}\|_{0,p} \le C, \ \text{for } i = 1,2. \tag{4.6}$$

Proof. We introduce an auxiliary problem: For any given $\psi \in L_{p'}(\Omega)$, find $\Psi \in H^1_0(\Omega)$ such that

$$(a(t)\nabla v, \nabla \Psi) = (v, \psi), \quad v \in H^1_0(\Omega). \tag{4.7}$$

Then Ψ saitisfies the elliptic regularity

$$\|\Psi\|_{2,p'} \le C\|\psi\|_{0,p'}. \tag{4.8}$$

Writing $g_h = g^z_{h,i}$, we have by (4.7) and (3.12)

$$(g_h, \psi) = (a(t)\nabla g_h, \nabla \Psi) = (a(t)\nabla g_h, \nabla R_h\Psi) = \frac{\partial}{\partial x_i} R_h\Psi(z).$$

It follows from $W^{1,\infty}$ stability of the Ritz projection operator R_h, imbedding theorem and (4.10) that

$$\begin{aligned}(g_h, \psi) &\le \|R_h\Psi\|_{1,\infty} \le C\|\Psi\|_{1,\infty} \\ &\le C\|\Psi\|_{2,p'} \le C\|\psi\|_{0,p'}, \forall \psi \in L_{p'}(\Omega).\end{aligned}$$

Thus the proof is complete. □

We will now show the following superconvergence estimates for ξ in $W^{1,\infty}$.

Theorem 4.3. *We have*

$$\begin{aligned}\|\xi(t)\|_{1,\infty} &\le Ch^2\{\|u_0\|_2 + \|u_t(0)\|_2 + \|u(t)\|_{1,2,p} \\ &\qquad +(\sum_{j=0}^{2}\int_0^t \|D^j_t u\|^2_2 d\tau)^{1/2}\}, \quad p > 2.\end{aligned} \tag{4.9}$$

Proof. Writing $g_h = g^z_{h,i}$ and $g_* = g^z_{*,i}$, we have by (3.12) and (3.3)

$$\begin{aligned}|\frac{\partial}{\partial x_i}\xi(z)| &= |(a(t)\nabla\xi, \nabla g_h)| \\ &= -(\xi_t + \eta_t, g_h) - \int_0^t (b(t,\tau)\nabla\xi(\tau), \nabla(g_h - g_*))d\tau \\ &\quad - \int_0^t (b(t,\tau)\nabla\xi(\tau), \nabla g_*)d\tau \\ &\equiv R_1 + R_2 + R_3.\end{aligned} \tag{4.10}$$

Applying similar argument as before, we see by Lemma 3.2 and Lemma 4.2 that

$$
\begin{aligned}
|R_1| &\le (\|\xi_t\|_{0,p} + \|\eta_t\|_{0,p})\|g_h\|_{0,p'} \\
&\le C(\|\xi_t\|_1 + \|\eta_t\|_{0,p}), \\
|R_2| &\le C\|\nabla\xi\|_{0,\infty} d\tau \|\nabla(g_h - g_*)\|_{0,1} \\
&\le C\int_0^t \|\nabla\xi\|_{0,\infty} d\tau, \\
|R_3| &= \left| \int_0^t (a(\nabla(\frac{b}{a}\xi(\tau)), \nabla g_*) d\tau - \int_0^t (a\xi(\tau)\nabla(\frac{b}{a}), \nabla g_*) d\tau \right| \\
&\le \left| \int_0^t P_h \frac{\partial}{\partial x_i} \left(\frac{b(z,t,\tau)}{a(z,\tau)} \xi(z,\tau) \right) d\tau \right| \\
&\quad + \left| \int_0^t \left(\nabla \cdot \left(a\xi(\tau)\nabla(\frac{b}{a}) \right), g_* \right) d\tau \right| \\
&\le C\int_0^t \|\xi\|_{1,\infty} d\tau (1 + \|g_*\|_{0,1}),
\end{aligned}
$$

and

$$\|g_*\|_{0,1} \le \|g_* - g_h\|_{1,1} + \|g_h\|_{0,p'} \le C.$$

Combining above estimate with (4.10), we get

$$\|\xi(t)\|_{1,\infty} \le C[\|\xi_t\|_1 + \|\eta_t\|_{0,p} + \int_0^t \|\xi\|_{1,\infty} d\tau].$$

This together with Lemma 2.1, Theorem 4.2 and Gronwall's inequality completes the proof. □

We next derive the superconvergence results of ξ in $W^{1,p}$.

Theorem 4.4. *We have for* $2 < p < \infty$

$$\|\xi(t)\|_{1,p} \le Ch^2 \left\{ \|u_0\|_2 + \|u_t(0)\|_2 + \|u_t(t)\|_{2,p} + (\sum_{j=0}^{2} \int_0^t \|D_t^j u\|_2^2 d\tau)^{1/2} \right\}. \tag{4.11}$$

Proof. By Lemma 2.1 and Theorem 4.1, we see that

$$\|\eta_t\|_{0,p} \le Ch^2 \|u\|_{1,2,p},$$

and

$$\|\xi_t\| \leq Ch^2 \left\{ \|u_0\|_2 + \|u_t(0)\|_2 + (\sum_{j=0}^{2} \int_0^t \|D_t^j u\|_2^2 d\tau)^{1/2} \right\}.$$

Applying the same argument as in the proof of Theorem 3.5 we get the desried results at once.

□

5 The interpolated postprocessing technique

In this section, we apply the interpolated postprocessing technique to enhance the accuracy of the approximate solution U. The global one order superconvergence result in L_p and $W^{1,p}$, $2 \leq p \leq \infty$, are established. Let T_h be a quasi-uniform rectangular partition of $\Omega \subset \mathbb{R}^2$ and let S_h be the space of continuous piecewise polynomials

$$S_h = \{v \in H_0^1(\Omega),\ v \in Q^k(\tau),\ \tau \in T_h\},$$

where

$$Q^k = \text{span}\{x_1^i x_2^j\ 0 \leq i, j \leq k\}.$$

Introduce two kinds of operators(see [4], [5],[6]), the vertices-edges-element interpolation operator i_h^k and the high interpolation operator I_{2h}^{k+l}. They satisfy the the following properties:

$$\text{(i)} \qquad \|u - I_{2h}^{k+1} u\|_{m,p} \leq Ch^{k+2-m} \|u\|_{k+2,p}, \qquad k \geq 1, m = 0, 1,\ 2 \leq p \leq \infty. \tag{5.1}$$

$$\text{(ii)} \qquad I_{2h}^{k+1} i_h^k = I_{2h}^{k+1}, k \geq 1. \tag{5.2}$$

$$\text{(iii)} \qquad \|I_{2h}^{k+1}\chi\|_{m,p} \leq C\|\chi\|_{m,p}, \forall \chi \in S_k, \qquad k \geq 1, m = 0, 1,\ 2 \leq p \leq \infty. \tag{5.3}$$

Using these properties we can improve the global convergence for the solution and its gradient. Let $\theta = V_h u - i_h^k u$. We will begin by demonstrating a lemma.

Lemma 5.1. *We have*

$$\begin{aligned} & (a(t)\nabla\theta, \nabla\chi) + \left(\int_0^t b(t,\tau)\nabla\theta(\tau)d\tau, \nabla\chi \right) \\ = \quad & O(h^{k+m})\|u(t)\|_{0,r,p}\|\chi\|_{m,p'}, \quad \chi \in S_h, \end{aligned} \tag{5.4}$$

where

$$2 \le p \le \infty, r = \begin{cases} 3 & \text{if } k = 1, \\ k+3 & \text{if } k > 1 \end{cases}, \quad m = \begin{cases} 1, & \text{if } k \ge 1, \\ 2, & \text{if } k \ge 3. \end{cases}$$

Proof. By [4] and [5], we see that for any $\alpha(x)$

$$(\alpha(x)\nabla(u - i_h^k u), \nabla\chi) = O(h^{k+m})\|u\|_{r,p}\|\chi\|_{m,p'} \tag{5.5}$$

and so that by (2.6) and (5.5)

$$\begin{aligned} & (a(t)\nabla\theta, \nabla\chi) + \int_0^t (b(t,\tau)\nabla\theta(\tau), \nabla\chi)d\tau \\ = \; & (a(t)\nabla(u - i_h^k u), \nabla\chi) + \int_0^t (b(t,\tau)\nabla(u(\tau) - i_h^k u(\tau)), \nabla\chi)d\tau \\ = \; & O(h^{k+m})[\|u\|_{r,p} + \int_0^t \|u\|_{r,p}d\tau]\|\chi\|_{m,p'}. \end{aligned}$$

The proof is completed. □

We will now aim for superconvergence of θ in L_p and $W^{1,p}$, $2 \le p \le \infty$, which are given in the following theorems.

Theorem 5.1. *For $k \ge 3$, we have*

$$\|\theta(t)\|_{0,p} \le Ch^{k+2}\|u(t)\|_{0,k+3,p}, \quad 2 \le p < \infty. \tag{5.6}$$

Proof. Again using (4.9) and (5.4), we have

$$\begin{aligned} (\theta, \psi) &= (a(t)\nabla\theta, \nabla\Psi) \\ &= (a(t)\nabla\theta, \nabla R_h\Psi) \\ &= O(h^{k+2})\|u(t)\|_{0,r,p}\|R_h\Psi\|_{2,p'} \\ &\quad + \int_0^t (b(t,\tau)\nabla\theta(\tau), \nabla(\Psi - R_h\Psi))d\tau - \int_0^t (b(t,\tau)\nabla\theta(\tau), \nabla\Psi)d\tau \\ &\le C[h^{k+2}\|u\|_{0,r,p}\|R_h\Psi\|_{2,p'} + \int_0^t \|\theta\|_{1,p}d\tau\|\Psi - R_h\Psi\|_{1,p'}] \\ &\quad + \int_0^t (\theta(\tau)\nabla\cdot(b(t,\tau)\nabla\Psi))d\tau \\ &\le C[h^{k+2}\|u\|_{0,r,p} + \int_0^t \|\theta\|_{0,p}d\tau]\|\Psi\|_{2,p'}. \end{aligned}$$

Hence by (4.10)

$$\|\theta\|_{0,p} \le \sup_{\psi \in L^{p'}(\Omega)} \frac{(\theta,\psi)}{\|\psi\|_{0,p'}} \le C[h^{k+2}\|u\|_{0,r,p} + \int_0^t \|\theta\|_{0,p} d\tau].$$

This together with Gronwall's inequality completes the proof. □

Theorem 5.2. *We have, for $k \ge 1$,*

$$\|\theta(t)\|_{1,p} \le Ch^{k+1}\|u(t)\|_{0,r,p}, \quad 2 \le p < \infty, \tag{5.7}$$

where $r = 3$ if $k = 1$, and $r = k+3$ if $k \ge 2$.

Proof. We have, by (3.28) and (3.29)

$$\begin{aligned}
(\theta_x, \phi) &= -(\phi_x, \theta) \\
&= (a(t)\nabla\theta, \nabla\Phi) \\
&= (a(t)\nabla\theta, \nabla R_h\Phi) \\
&\le Ch^{k+1}\|u\|_{0,r,p}\|R_h\Phi\|_{1,p'} - \int_0^t (b(t,\tau)\nabla\theta(\tau), \nabla R_h\Phi)d\tau \\
&\le C[h^{k+1}\|u\|_{0,r,p} + \int_0^t \|\theta\|_{1,p}d\tau]\|R_h\Phi\|_{1,p'} \\
&\le C[h^{k+1}\|u\|_{0,r,p} + \int_0^t \|\theta\|_{1,p}d\tau].
\end{aligned}$$

Thus the proof is complete. □

Theorem 5.3. *We have*

$$\|\theta(t)\|_{0,\infty} \le Ch^{k+2}\log\frac{1}{h}\|u(t)\|_{0,k+3,\infty}, \text{ if } k \ge 3, \tag{5.8}$$

$$\|\theta(t)\|_{1,\infty} \le Ch^{k+1}(\log\frac{1}{h})^{\bar\alpha}\|u(t)\|_{0,r,\infty}, \tag{5.9}$$

where $\bar\alpha = 1$ if $k = 1, 2$, $\bar\alpha = 0$ if $k \ge 3, r = 3$ if $k = 1$, and $r = k+3$ if $k \ge 2$.

Proof. By Lemma 6.1, we have

$$\begin{aligned}
&|\theta(z)| = |(a(t)\nabla\theta, \nabla G_h^z)| \\
\le\ & Ch^{k+2}\|u\|_{0,r,\infty}\|G_h^z\|_{2,1}' + \left|\int_0^t (b(t,\tau)\nabla\theta(z), \nabla G_h^z)d\tau\right|,
\end{aligned} \tag{5.10}$$

where

$$
\begin{aligned}
& \left|\int_0^t (b(t,\tau)\nabla\theta(z), \nabla G_h^z) d\tau|\right| \\
\leq \; & \left|\int_0^t (b(t,\tau)\nabla\theta(z), \nabla(G_h^z - G_*^z)) d\tau + \int_0^t (a\nabla(\frac{b}{a}\theta(z)), \nabla G_*^z) d\tau \right. \\
& \left. - \int_0^t (a\theta(\tau)\nabla(\frac{b}{a}\theta(z)), \nabla G_*^z) d\tau \right| \\
\leq \; & C\int_0^t \|\theta\|_{1,\infty} d\tau \|G_h^z - G_*^z\|_{1,1} + \left|\int_0^t P_h(\frac{b}{a}\theta(\tau)) d\tau\right| \\
& + C\int_0^t \|\theta\|_{0,\infty} d\tau \|G_*^z\|_{1,1}.
\end{aligned} \tag{5.11}
$$

Combining (5.11) with (5.10), we have by Lemma 3.2

$$
\|\theta(z)\|_{0,\infty} \leq C[h^{k+2}\log\frac{1}{h}\|u(t)\|_{0,r,\infty} + \int_0^t \|\theta\|_{0,\infty} d\tau]. \tag{5.12}
$$

The conclusion (5.8) follows from Gronwall's inequality.

(ii) Writing $g_h = g_{h,i}^z$ and $g_* = g_{*,i}^z$ again, we see by a similar argument as above, for $k = 1, 2$

$$
\begin{aligned}
|\frac{\partial}{\partial x_i}\theta(z)| \; = \; & |(a(t)\nabla\theta, \nabla g_h)| \\
\leq \; & Ch^{k+1}\|g_h\|_{1,1}\|u\|_{0,r,\infty} + \left|\int_0^t (b(t,\tau)\nabla\theta(\tau), \nabla g_h) d\tau\right| \\
\leq \; & Ch^{k+1}\log\frac{1}{h}\cdot\|u(t)\|_{0,r,\infty} + \left|\int_0^t (b(t,\tau)\nabla\theta(\tau), \nabla(g_h - g_*)) d\tau \right. \\
& \left. + \int_0^t (a\nabla(\frac{b}{a}\theta(\tau)), \nabla g_*)) d\tau - \int_0^t (a\theta\nabla(\frac{b}{a}), \nabla g_*)) d\tau \right| \\
\leq \; & Ch^{k+1}\log\frac{1}{h}\cdot\|u\|_{0,r,\infty} + C\int_0^t \|\theta\|_{1,\infty} d\tau \|g_h - g_*)\|_{1,1} \\
& + \left|\int_0^t P_h\frac{\partial}{\partial x_i}(\frac{b}{a}\theta(\tau)) d\tau + \int_0^t (\nabla\cdot(a\theta(\tau)\nabla(\frac{b}{a})), g_*)) d\tau\right| \\
\leq \; & Ch^{k+1}\log\frac{1}{h}\cdot\|u\|_{0,r,\infty} \\
& + C\int_0^t \|\theta\|_{1,\infty} d\tau (1 + \|g_h - g_*\|_{1,1} + \|g_*\|_{0,1}) \\
\leq \; & C[h^{k+1}\log\frac{1}{h}\|u\|_{0,r,\infty} + \int_0^t \|\theta\|_{1,\infty} d\tau].
\end{aligned}
$$

For $k \geq 3$, we have

$$\begin{aligned}
\frac{\partial}{\partial x_i}\theta(z) &= (a(t)\nabla\theta, \nabla g_h) \\
&= [(a(t)\nabla\theta, \nabla(g_h - \Pi_h g_*) + \int_0^t (b(t,\tau)\nabla\theta(\tau), \nabla(g_h - \Pi_h g_*))d\tau] \\
&\quad +[(a(t)\nabla\theta, \nabla\Pi_h g_*)) + \int_0^t (b(t,\tau), \nabla(\Pi_h g_*))d\tau] \\
&\quad - \int_0^t (b(t,\tau)\nabla\theta(\tau), \nabla g_h)d\tau] \equiv J_1 + J_2 + J_3, \qquad (5.13)
\end{aligned}$$

where by Lemma 5.1, inverse property and Lemma 3.2, we have

$$\begin{aligned}
|J_1| &\leq Ch^{k+2}\|u\|_{0,r,\infty}\|g_h - \Pi_h g_*\|'_{2,1} \\
&\leq Ch^{k+2}\|u\|_{0,r,\infty}[h^{-1}\|g_h - g_*\|_{1,1} + \|g_*\|_{2,1}] \\
&\leq Ch^{k+1}\|u\|_{0,r,\infty}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|J_2| &\leq Ch^{k+2}\|\Pi_h g_*\|'_{2,1}\|u\|_{0,r,\infty} \leq Ch^{k+2}\|g_*\|_{2,1}\|u\|_{0,r,\infty} \leq Ch^{k+1}\|u\|_{0,r,\infty}, \\
|J_3| &= \Big|\int_0^t (b(t,\tau)\nabla\theta(\tau), \nabla(g_h - g_*))d\tau + \int_0^t (a\nabla(\frac{b}{a}\theta(\tau), \nabla g_*)d\tau \\
&\quad - \int_0^t (a\theta(\tau)\nabla(\frac{b}{a}), \nabla g_*)d\tau\Big| \\
&\leq C\int_0^t \|\theta\|_{1,\infty}d\tau\|g_h - g_*\|_{1,1} \\
&\quad + \Big|\int_0^t P_h\frac{\partial}{\partial x_i}(\frac{b}{a}\theta(\tau))d\tau + \int_0^t (\nabla\cdot(a\theta(\tau)\nabla(\frac{b}{a})), g_*))d\tau\Big| \\
&\leq C\int_0^t \|\theta\|_{1,\infty}d\tau(1 + \|g_h - g_*\|_{1,1} + \|g_*\|_{0,1}) \\
&\leq C\int_0^t \|\theta\|_{1,\infty}d\tau.
\end{aligned}$$

Substituting $J_1 - J_3$ into (5.13) completes the proof. □

Finally, we give the main results of this paper.

Theorem 5.4. *We have the following superconvergence.*

$$\begin{aligned}\|u(t)-I_{2h}^{k+1}U(t)\|_{0,p} &\le Ch^{k+2}\{\|u_0\|_{k+1}+\|u_t(0)\|_{k+1}\\ &\quad +(\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_{k+1}^2 d\tau)^{1/2}+\|u(t)\|_{0,k+3,p}\},\\ &\quad k\ge 3,\quad 2\le p<\infty, \end{aligned} \tag{5.14}$$

$$\begin{aligned}\|u(t)-I_{2h}^{k+1}U(t)\|_{0,\infty} &\le Ch^{k+2}\{\|u_0\|_{k+1}+\|u_t(0)\|_{k+1}+\|u(t)\|_{1,k+1,p}\\ &\quad +(\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_{k+1}^2 d\tau)^{1/2}+\log\frac{1}{h}\|u(t)\|_{0,k+3,\infty}\},\\ &\quad k\ge 3,\quad p>2, \end{aligned} \tag{5.15}$$

$$\begin{aligned}\|u(t)-I_{2h}^{k+1}U(t)\|_{1} &\le Ch^{k+1}\{\|u_0\|_{k+1}+\|u_t(0)\|_{k+1}\\ &\quad +(\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_{k+1}^2 d\tau)^{1/2}+\|u(t)\|_{0,r,2}\},\\ &\quad k\ge 1, \end{aligned} \tag{5.16}$$

$$\begin{aligned}\|u(t)-I_{2h}^{k+1}U(t)\|_{1,p} &\le Ch^{k+1}\{\|u_0\|_{k+1}+\|u_t(0)\|_{k+1}+\|u(t)\|_{1,k+1,p}\\ &\quad +(\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_{k+1}^2 d\tau)^{1/2}+\|u(t)\|_{0,k+3,p}\},\\ &\quad k\ge 1,\quad 2<p<\infty, \end{aligned} \tag{5.17}$$

$$\begin{aligned}\|u(t)-I_{2h}^{k+1}U(t)\|_{1,\infty} &\le Ch^{k+1}\{\|u_0\|_{k+1}+\|u_t(0)\|_{k+1}\\ &\quad +\|u(t)\|_{1,k+1,p}+(\sum_{j=0}^{2}\int_0^t \|D_t^j u\|_{k+1}^2 d\tau)^{1/2}\\ &\quad +(\log\frac{1}{h})^{\bar\alpha}\|u(t)\|_{0,r,\infty}\},\quad k\ge 1, p>2, \end{aligned} \tag{5.18}$$

where $\bar\alpha=1$ *if* $k=1$, $\bar\alpha=0$ *if* $k\ge 3$, $r=3$ *if* $k=1$, *and* $r=k+3$ *if* $k\ge 2$.

Proof. By (5.2), we have

$$u-I_{2h}^{k+1}U=u-I_{2h}^{k+1}u+I_{2h}^{k+1}(i_h^k u-V_h u)+I_{2h}^{k+1}(V_h u-U). \tag{5.19}$$

Then by (5.3)

$$\|u - I_{2h}^{k+1}U\|_{m,p} = \|u - I_{2h}^{k+1}u\|_{m,p} + C\|i_h^k u - V_h u\|_{m,p} + C\|V_h u - U\|_{m,p}. \tag{5.20}$$

The estimate of first term is shown in (5.1), second term in Theorems 5.1, 5.2 and 5.3, third one in Theorems 3.1, 3.2, 3.3, 4.1, 4.3 and 4.4. Thus, we complete the proof.

□

Remark 5.1. *In the case Ω is a general domain we may take the piecewise "regular partition" or "most rectangular" partition to do the postprocessing(see [6]). Then we can still get half order gain for $u - I_{2h}^{k+1}U$ in L_p and $W^{1,p}, 2 \le p \le \infty$.*

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