V-CYCLE MULTIGRID CONVERGENCE OF SOME FINITE DIFFERENCE SCHEME FOR THE HELMHOLTZ EQUATION

DO Y. KWAK* AND JUN S. LEE†

Abstract. In this paper, we analyze V-cycle multigrid algorithm for a positive definite Helmholtz equation on a hexagonal grid. Specifically, we apply V-cycle multigrid algorithm to the numerical scheme based on mean value solutions for the Helmholtz equation on hexagonal grids introduced by [1], and show its convergence. The theory for V-cycle multigrid convergence is carried out in the framework in [6] by estimating the energy norm of the prolongation operator and proving the approximation and regularity conditions. In numerical experiments, we report the eigenvalues, condition number and contraction number.

Key words. Multigrid method, Mean Value Solution, Finite Difference Methods.

AMS(MOS) subject classifications. Primary 65N55; Secondary 65N06.

1. Introduction. The behavior of multigrid method (MG) for elliptic problems using standard discretization is fairly well known these days (see [2], [4], [12], [13] and references therein). For conforming finite element method, V-cycle MG has reduction factor independent of the number of levels [4], [5], while for nonconforming case, the W-cycle converges if the number of smoothings are sufficiently large or the energy norm of the prolongation operator is less than or equal to $\sqrt{2}$ [6]. For finite difference methods on uniform grids, similar results hold when FDM is considered as quadrature approximation of corresponding FEM.

MG consists of a sequence of finite dimensional spaces $M_1, \ldots, M_J$ and the transfer operators (prolongation and projection) connecting these spaces. When these spaces are nested and the bilinear forms are invariant, the most natural choice of the prolongation is the inclusion operator $I\colon M_j \to M_{j+1}$. In this case the energy norm of the operator does not increase and the convergence is fast. When MG is used to solve a Laplace equation on the unit square discretized by linear finite elements, experiments show that simple V-cycle with one pre-post smoothing takes at least 10 iterations for level up to 8 to achieve the accuracy of 10^{-6}. When MG is used as a preconditioner for the conjugate gradient method, the condition number for this standard problem is known to be 1.5 ($\lambda_{max} = 0.999$, $\lambda_{min} = 0.666$). But when the spaces M_j ($j = 1, \ldots, J$) are nonnested, it is most likely that the energy norm of the transfer operator is larger than 1, and the performance is often affected by the energy of the prolongation operator [3], [6], [10], [11]. For example, it is shown [11] that the energy norm is greater than $\sqrt{2}$ for P_1-nonconforming elements and is $\sqrt{2}$ for the cell-centered finite differences [3]. In the latter case V-cycle seems to converge for levels up to 8 but the reduction factor quickly grows. See [10] how the convergence behavior can be improved by designing a new prolongation where they introduced a weighted prolongation for the cell-centered example. For a V-cycle convergence of nonconforming finite element, see [8], [9] and [11].

In this paper, we report a nonstandard example for which MG converges much faster than the standard conforming finite element method. We devise a multigrid algorithm for a finite difference scheme based on mean value solutions for the Helmholtz equation, $-\Delta u + \mu^2 u = 0$ on hexagonal grids introduced in [1]. We shall call this scheme as Mean Value Scheme(MVS). The equation is different from ordinary

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Helmholtz equations in that the lower order term has positive sign, but convection diffusion equations of the form $-\Delta u + \alpha \cdot \nabla u + qu = 0$, with constant coefficients α and q can be transformed into this special form by a suitable change of variable [1]. One advantage of this scheme is that the maximum norm error is remarkably $O(h^4)$. This superconvergence is due to the symmetry of the difference formula and the special nature of the Helmholtz equation. We have more advantages when MVS is employed to solve this transformed convection-diffusion equation : complicated upwinding scheme can be avoided, the transformed symmetric problem renders the use of conjugate gradient method, etc. Moreover, when MG is used to solve the convection-diffusion equation with the transform one can choose larger coarsest grid size, which requires less time to solve the coarsest grid problem. When applying MG to the untransformed convection-diffusion equation, one has to use small coarsest grid to capture indefiniteness of the algebraic system.

The rest of the paper is as follows. In Section 2, we briefly introduce MVS and its error estimate. Multigrid algorithm for MVS is described in Section 3. Appropriate quadratic forms and the prolongation operators are defined and we derive V-cycle convergence analysis by showing (i) the energy norm of prolongation operators is less than 1 and (ii) the approximation and regularity property holds. The latter is verified through the use of finite difference method and using the theory for standard finite element method developed in [6]. Finally, in Section 4, numerical results are presented. When V-cycle multigrid is used as a preconditioner for the conjugate gradient method, the condition number between 1.091 and 1.244 is smaller than that of standard finite element. We also report contraction numbers varying μ.

2. MVS for a Helmholtz Equation. Let Ω be an open and bounded subset of $\mathbb{R}^2$ with boundary $\partial\Omega$ and $\overline{\Omega} = \Omega \cup \partial\Omega$, be the closure of Ω. The pair (x, y) will denote a point in $\overline{\Omega}$. Let $C^k(\overline{\Omega})$ be the class of functions that possess continuous partial derivatives of order k in Ω and continuous on $\overline{\Omega}$. Let us consider the following Dirichlet problem:

$$\begin{aligned} -\Delta u + \mu^2 u &= 0, \quad \text{in } \Omega \\ u &= \phi, \quad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

We assume that $u \in C^6(\overline{\Omega})$. If Ω is a domain which can be covered by regular hexagonals and if the solution u belongs $C^6(\overline{\Omega})$ then the MVS scheme yields $O(h^4)$ accuracy as shown in [1]. However, such a domain is too restrictive to be used in practice, hence we study more realistic case of a rectangular domain whose ratio of the height and width is a multiple of $\sqrt{3}$ so that it can be covered by equilateral triangles except near the boundary as in Figure 3.2. We introduce a discrete grid on $\overline{\Omega}$ by subdividing the domain by equilateral triangles(of side length h_1) and denote the set of node points by Ω_1 and for a multigrid purpose, we further subdivide the resulting grid consecutively to obtain a sequence of grids(of side length h_k) $\Omega_k, k = 1, \cdots, J$ for some integer J. We shall consider solving (2.1) on the finest grid Ω_J.

Let U_i be the approximation of u at grid point $z_i \in \Omega_J$ and U_i^l for $l = 1, \ldots, 6$ be its six neighboring values. Then the MVS is defined as follows [1]:

$$6I_0(\mu h)U_i = U_i^1 + U_i^2 + U_i^3 + U_i^4 + U_i^5 + U_i^6, \tag{2.2}$$

where $I_0 \colon \mathbb{R} \to \mathbb{R}_+$ is the modified Bessel function of zero order, represented by series

$$I_0(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^{2k}(k!)^2}, \quad x \in \mathbb{R}.$$

When some of the values lie outside of the domain, we define artificial values by reflection along the boundary. This gives, as usual, $O(h^2)$ error in maximum norm which can be deduced from the proof in [1]. In matrix form, the resulting scheme can be written as

$$\mathcal{A}^{mvs}U = F, \tag{2.3}$$

where $\mathcal{A}^{mvs}$ is the difference operator of MVS and F is the right hand side. Clearly, $\mathcal{A}^{mvs}$ is symmetric, positive definite and diagonally dominant [1] .

3. Multigrid algorithms for MVS. In order to describe multigrid algorithms, let us start with the finest grid Ω_J, and as in Figure 3.2 we coarsen this grid by coalescing four triangles into one. The dashed lines represent fine grids and solid lines represents the coarser grids. Then some triangles we left over near the boundary of Ω_J have some interior nodes (U_l^{J-1}, etc., denoted by black dot in Figure 3.2) which become boundary nodes of Ω_{J-1}. To overcome this difficulty, we introduce artificial nodes outside Ω (written as $-U_l^{J-1}$, etc.) and treat them as boundary values. Hence, vertices U_l^{J-1}, etc. are now considered as interior nodes in Ω_{J-1}. The set of all such nodes in Ω_{J-1} are denoted by B_{J-1}(These are actually interior points of the domain but near the boundary and they need special treatment in the proof of Theorem below). At these points, the center entries of the stencils are $6I_0(\mu h_{J-1})+1$, whereas the center entries are $6I_0(\mu h_{J-1})$ for other nodes. The stencils at the interior nodes and boundary nodes are given in Figure 3.1.

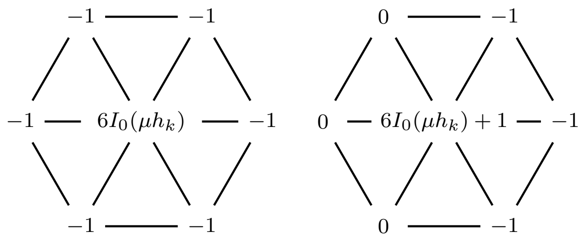

FIG. 3.1. *Interior and boundary stencils for* $k = 1, \dots, J-1$

Continue this process until we obtain coarsest grid Ω_1. By construction, $\Omega_k \subset \Omega_{k+1}$ for $k = 1, \dots, J-1$. The space M_k is defined to be the vector space of nodal values defined on Ω_k.

Now, we briefly introduce the multigrid algorithms. Let $\mathcal{A}_k^{mvs}\colon M_k \to M_k$ be an MVS operator defined by (2.3) at level k. Define linear operators $I_k\colon M_{k-1} \to M_k$ for $k = 1, \dots, J$ as follows:

(i) If z_i^k is a node on the $(k-1)$-st grid, then $(I_kV)(z_i^k) = V(z_i^k)$.

(ii) If z_i^k is a node between two adjacent nodes on the $(k-1)$-st grid, then $(I_kV)(z_i^k)$ is the average of the nodal values of V at the two nodes.

The operators I_k are "prolongation" operators and correspond to the piecewise linear interpolation on the triangulation of size $h_k = 2^{J-k}h_J$. Let $\triangle_k$ denote the set of all triangles at level k and let τ denote a triangle in $\triangle_k$. Then we write $(i, j) \ll \tau$ when i-th and j-th nodes are any two vertices of a triangle τ in $\triangle_k$. For the description of multigrid algorithm and convergence proof, we need to introduce a quadratic form

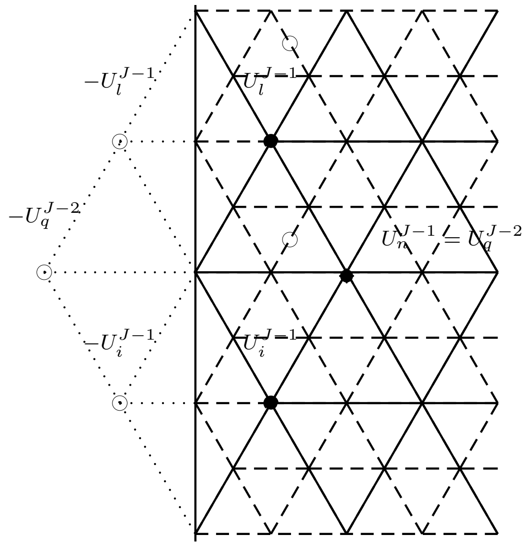

FIG. 3.2. *Coarsening the grids*

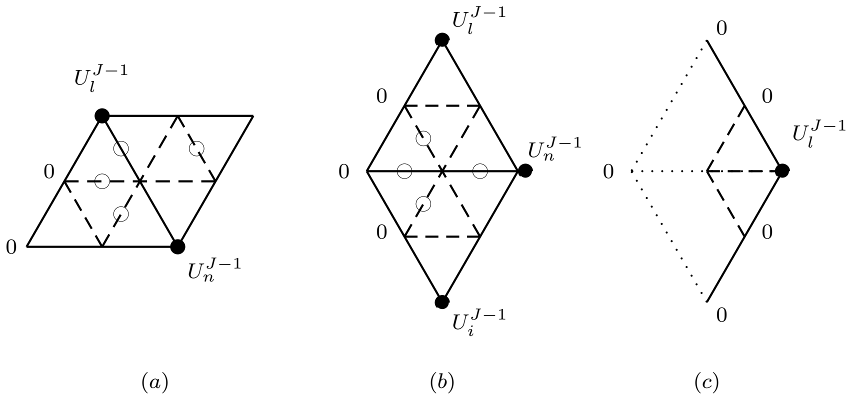

FIG. 3.3. *Generation of* A_{J-1}^{mvs}

$\mathcal{A}_k^{mvs}(\,\cdot\,,\,\cdot\,)$ induced by $\mathcal{A}_k^{mvs}$:

$$\mathcal{A}_k^{mvs}(U,V) = \sum_{z_i^k \in \Omega_k} \left[6I_0(\mu h_k)U_i - \sum_{l=1}^{6} U_i^l \right] V_i,$$

where U_i^l are six neighboring points of U_i. We also need discrete L^2-inner product $(\,\cdot\,,\,\cdot\,)_k$ defined by

$$(U,V)_k = \sum_{z_i^k \in \Omega_k} \frac{\sqrt{3}}{2}\, h_k^2\, U_i V_i.$$

Using these forms, we define the operators $P_{k-1}\colon M_k \to M_{k-1}$ and $P_{k-1}^0\colon M_k \to M_{k-1}$

by

$$\mathcal{A}_{k-1}^{mvs}(P_{k-1}U, V) = \mathcal{A}_k^{mvs}(U, I_kV), \quad \forall V \in M_{k-1}$$

and

$$(P_{k-1}^0 U, V)_{k-1} = (U, I_kV)_k, \quad \forall V \in M_{k-1}.$$

To define the smoothing processes, we require linear operators $R_k\colon M_k \to M_k$ for $k = 1, \ldots, J$ satisfying usual smoothing properties shown in [4, 6]. This operator may be symmetric or nonsymmetric with respect to the inner product $(\cdot, \cdot)_k$. If R_k is nonsymmetric, then we define R_k^t to be its adjoint and set

$$R_k^{(l)} = \begin{cases} R_k & \text{if } l \text{ is odd,} \\ R_k^t & \text{if } l \text{ is even.} \end{cases}$$

The multigrid operator $B_k\colon M_k \to M_k$ is defined by induction as follows.

Multigrid Algorithm.
Set $B_0 = (\mathcal{A}_0^{mvs})^{-1}$. Assume that B_{k-1} has been defined and define B_kg for $g \in M_k$ as follows:
(1) Set x^0=0 and q^0=0.
(2) Define x^l for $l = 1, \ldots, m(k)$ by

$$x^l = x^{l-1} + R_k^{(l+m(k))}(g - \mathcal{A}_k^{mvs}x^{l-1}).$$

(3) Define $y^{m(k)} = x^{m(k)} + I_kq^p$, where q^i for $i = 1, \ldots, p$ is defined by

$$q^i = q^{i-1} + B_{k-1}[P_{k-1}^0(g - \mathcal{A}_k^{mvs}x^{m(k)}) - \mathcal{A}_{k-1}^{mvs}q^{i-1}].$$

(4) Define y^l for $l = m(k) + 1, \ldots, 2m(k)$ by

$$y^l = y^{l-1} + R_k^{(l+m(k))}(g - \mathcal{A}_k^{mvs}y^{l-1}).$$

(5) Set $B_kg = y^{2m(k)}$.

In this algorithm, $m(k)$ is a positive integer which may vary from level to level and determines the number of smoothing iterations on that level. If R_k is symmetric and all of the $m(k)$ are the same, then the algorithm is the usual symmetric algorithm. In this algorithm, p is a positive integer. We shall study the cases $p = 1$, which corresponds to the symmetric V-cycle of multigrid algorithm. We will proceed the convergence proof following the general framework in [6]. We need two conditions. One is "regularity and approximation" condition, i.e.,

$$\begin{aligned} &|\mathcal{A}_k^{mvs}((I - I_kP_{k-1})\,U, U)| \\ &\qquad \le C_\alpha \left(\frac{\|\mathcal{A}_k^{mvs}U\|_k^2}{\lambda_k} \right)^\alpha \mathcal{A}_k^{mvs}(U, U)^{1-\alpha}, \quad \forall U \in M_k, \end{aligned} \tag{3.1}$$

where λ_k is the largest eigenvalue of $\mathcal{A}_k$, and the other is

$$\mathcal{A}_k^{mvs}(I_kU, I_kU) \le C_* \mathcal{A}_{k-1}^{mvs}(U, U), \quad \forall U \in M_{k-1} \tag{3.2}$$

for some constant C_* independent of k and U.

Remark 3.1. *In the classical multigrid theory, the condition $C_* = 1$ is enough for V-cycle convergence this condition holds for most finite difference methods or conforming finite element methods([5], [6], [10]), while for nonconforming finite element methods, $C_* > 1$([11]).*

However, in case of MVS scheme, we show below that a strict inequality in (3.2) holds with $C_* \le 1$, which is unusual. This fact seems to be related to the extraordinary fast multigrid convergence of MVS scheme.

REMARK 3.2. *Note that by definition, P_{k-1} is the adjoint of I_k and hence (3.2) holds if and only if*

$$\mathcal{A}_{k-1}^{mvs}(P_{k-1}U, P_{k-1}U) \le C_* \mathcal{A}_k^{mvs}(U,U), \quad \forall U \in M_k. \tag{3.3}$$

THEOREM 1. *Let $\mathcal{A}_k^{mvs}$ and I_k be defined above for $1 \le k \le J$. Then a strict inequality in (3.2) holds with $C_* \le 1$.*

Proof. Let δ_i^k be the i-th diagonal entry of $\mathcal{A}_k^{mvs}$. Then, as we mentioned before,

$$\delta_i^k = \begin{cases} 6I_0(\mu h_k) + 1 & \text{if } i \in B_k, \\ 6I_0(\mu h_k) & \text{otherwise.} \end{cases}$$

So we have

$$\begin{aligned}
\mathcal{A}_k^{mvs}(I_kU, I_kU) =& (I_kU)^t A_k^{mvs}(I_kU) \\
=& \sum_{i\in\Omega_k} (I_kU) \left[\delta_i^k (I_kU)_i - \sum_{l=1}^{6} (I_kU)_i^l \right] \\
=& \sum_{i\in\Omega_k} (I_kU)_i \left[6I_0(\mu h_k)(I_kU)_i - \sum_{l=1}^{6} (I_kU)_i^l \right] + \sum_{i\in B_k} (I_kU)_i^2 \\
=& \sum_{i\in\Omega_k} (I_kU)_i \sum_{l=1}^{6} \left[(I_kU)_i - (I_kU)_i^l \right] \\
&+ \sum_{i\in\Omega_k} 6(I_0(\mu h_k) - 1)(I_kU)_i^2 + \sum_{i\in B_k} (I_kU)_i^2 \\
=& 6\,(I_0(\mu h_k) - 1) \sum_{i\in\Omega_k} (I_kU)_i^2 \\
&+ \sum_{\tau\in\triangle} \sum_{(i,j)\ll\tau} [(I_kU)_i - (I_kU)_j]^2 + \sum_{i\in B_k} (I_kU)_i^2 \\
:=& I + II,
\end{aligned}$$

where

$$\begin{aligned}
I =& \ 6\,(I_0(\mu h_k) - 1) \sum_{i\in\Omega_k} (I_kU)_i^2 \\
\text{and} \quad II =& \sum_{\tau\in\triangle} \sum_{(i,j)\ll\tau} [(I_kU)_i - (I_kU)_j]^2 + \sum_{i\in B_k} (I_kU)_i^2.
\end{aligned}$$

Here the last equality was obtained by symmetry of (i,j). The proof will be complete if we give a proper bounds on I and II. These will be given in the next two Lemmas.

LEMMA 1. *We have*

$$\begin{aligned}
6\,(I_0(\mu h_k) - 1) &\sum_{i\in\Omega_k} (I_kU)_i^2 \\
&< 6\,(I_0(\mu h_{k-1}) - 1) \sum_{i\in\Omega_{k-1}} U_i^2, \quad \forall U \in M_{k-1}.
\end{aligned} \tag{3.4}$$

Proof. Let us first show the inequality

$$\sum_{i\in\Omega_k}(I_kU)_i^2 \le 4\sum_{i\in\Omega_{k-1}} U_i^2, \quad \forall U \in M_{k-1}. \tag{3.5}$$

Since each interior vertex is shared by 6 triangles, $\sum_{i\in\Omega_{k-1}} U_i^2$ can be written as

$$\frac{1}{6}\sum_{\tau_{k-1}\in\triangle_{k-1}}\sum_{j=1}^{3} U(z^j_{\tau_{k-1}})^2, \tag{3.6}$$

where $\{z^j_{\tau_{k-1}}\}_{j=1}^3$ denotes the three vertices of the triangle $\tau_{k-1}\in\triangle_{k-1}$. It clearly suffices to prove the inequality on a typical triangle of the $(k-1)$-st grid, i.e,

$$\sum_{\ell=1}^{4}\sum_{j=1}^{3}(I_kU(z^j_{\tau^\ell_k}))^2 \le 4\sum_{j=1}^{3} U(z^j_{\tau_{k-1}})^2, \quad \forall U\in M_{k-1}, \tag{3.7}$$

where $\tau^\ell_k, \ell = 1,\cdots,4$ are four subtriangles obtained from τ_{k-1}. Consider a function which takes on the values a, b and c at the nodes.(See the Figure 3.4) On the left hand side of (3.7), midpoint values are counted three times, we see that (3.7) follows from the next inequality

$$a^2+b^2+c^2+\frac{3(a+b)^2}{4}+\frac{3(b+c)^2}{4}+\frac{3(a+c)^2}{4} \le 4(a^2+b^2+c^2).$$

When this triangle is adjacent to the boundary, we have

$$a^2+b^2+0^2+\frac{3(a+b)^2}{4}+3\cdot\frac{0^2}{4}+3\cdot\frac{0^2}{4} \le 4(a^2+b^2+0^2).$$

Combining (3.7) and the fact that $4(I_0(\mu h_k)-1) < (I_0(\mu h_{k-1})-1)$, we obtain (3.4). □

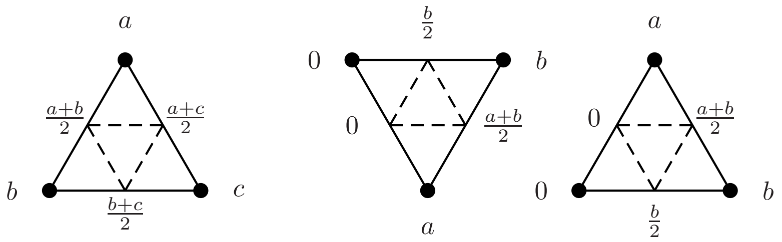

FIG. 3.4. *Interior and Boundary Elements*

LEMMA 2. *We have*

$$\begin{aligned}&\sum_{\tau\in\triangle_k}\sum_{(i,j)\ll\tau}((I_kU)_i-(I_kU)_j)^2+\sum_{i\in B_k}(I_kU_i)^2\\ &\quad\le\sum_{\tau\in\triangle_{k-1}}\sum_{(i,j)\ll\tau}(U_i-U_j)^2+\sum_{i\in B_{k-1}}(U_i)^2, \quad \forall U\in M_{k-1}.\end{aligned}\tag{3.8}$$

Proof. Let us first show the case $k = J$. Since $B_J = \varnothing$, it suffices to show that

$$\sum_{\tau\in\triangle_J}\sum_{(i,j)\ll\tau}((I_JU)_i-(I_JU)_j)^2 \\ \leq \sum_{\tau\in\triangle_{J-1}}\sum_{(i,j)\ll\tau}(U_i-U_j)^2+\sum_{i\in B_{J-1}}(U_i)^2, \quad \forall U\in M_{J-1}. \tag{3.9}$$

As in Figure 3.5 we see that the square of the diagonal difference between U_l and U_i is the square sums of differences of certain neighboring points, that is,

$$|U_l-U_i|^2 = \left|\frac{U_j+U_l}{2}-\frac{U_j+U_i}{2}\right|^2+\left|U_l-\frac{U_i+U_l}{2}\right|^2 \\ +\left|\frac{U_l+U_m}{2}-\frac{U_i+U_m}{2}\right|^2+\left|U_i-\frac{U_i+U_l}{2}\right|^2.$$

The four terms on the right hand side are marked with white circles. At the boundary, the parallelograms (a) and (b) in Figure 3.3 are typical and the square of diagonal differences are equal to the square sum of four differences of points in level J. They are also marked by white circles. On the parallelogram of type (c) in Figure 3.3, the sum of differences of lower level is equal to the sum of differences of higher level, since U_i are zero along the boundary. There are two differences left at higher level. They are marked by two white circles in Figure 3.2 . The sum of them is equal to the half of the nearest extrapolation point of lower level. For instance, two differences marked by circles in Figure 3.2 are the half of U_l^{J-1}. So we have that

$$\sum_{\tau\in\triangle_J}\sum_{(i,j)\ll\tau}((I_JU)_i-(I_JU)_j)^2 \\ \leq \sum_{\tau\in\triangle_{J-1}}\sum_{(i,j)\ll\tau}(U_i-U_j)^2+\frac{1}{2}\sum_{i\in B_{J-1}}(U_i)^2. \tag{3.10}$$

Next, let us consider the case when k is less than J. Similar inequality as (3.10) holds. Thus it suffices to consider the second term on the left hand side of (3.8). By the definition of I_k, interpolated values at level k are half of the values of level $k-1$ and it does not happen more than twice. For instance, consider the relation of U_l^{J-1} and U_i^{J-1} at level $J-1$, and U_q^{J-2} at level $J-2$. So we have the following inequality;

$$\sum_{i\in B_k}(I_kU_i)^2 \leq \frac{1}{2}\sum_{i\in B_{k-1}}(U_i)^2 \quad \text{for } k<J. \tag{3.11}$$

Combining (3.10) and (3.11), we obtain (3.8) for $k < J$. □

Returning to the proof of Theorem 1, we see that by (3.4) and (3.8), (3.2) holds with a constant $C_* \leq 1$. Hence the strict inequality holds in the assertion. □

Next, we will show the regularity and approximation property of $\mathcal{A}_k^{mvs}$. To do this, we need auxiliary numerical schemes. Expanding the representation for $u(x,y)$ by using the first $(n-1)$-st derivatives, we have

$$u(x+\alpha,y+\beta)=u(x,y)+\sum_{k=1}^{n-1}\frac{1}{k!}\left(\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}\right)^k u(x,y)+R_n, \tag{3.12}$$

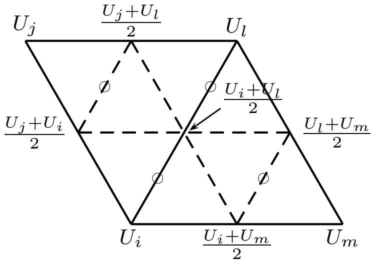

FIG. 3.5. *Linear interpolation*

where, $\alpha = \pm h_k$ and β=0, or α=$\pm h_k/2$ and β=$\pm\sqrt{3}/2$. Here, $(x+\alpha, y+\beta)$ represents the six equally spaced neighboring points of (x, y). Truncating (3.12) and summing up with respect to six neighboring points, we have the following:

$$\sum_{\alpha,\beta} u(x+\alpha, y+\beta) \approx 6u(x,y) + \frac{3h_k^2}{2!}\Delta u(x,y).$$

Based on this, we define a finite difference method (FDM) :

$$\frac{2}{3h_k^2}\left(6U_i - \sum_{l=1}^{6} U_i^l\right) + \mu^2 U_i = 0,$$

and the Dirichlet form $D_k(\cdot\,,\,\cdot)$ on M_k by

$$D_k(U,V)_k = \sum_{i\in\Omega_k} \frac{\sqrt{3}}{2}\left[6U_i - \sum_{l=1}^{6} U_i^l\right] V_i. \tag{3.13}$$

The stencil of this FDM in case of negative Laplacian operator $-\Delta$ is given as (a) in Figure 3.6. Let $\bar{A}_k$ be the corresponding finite difference matrix on Ω_k. Then the quadratic form $\bar{\mathcal{A}}_k(\cdot\,,\,\cdot)$ induced by $\bar{A}_k$ is defined as

$$\bar{\mathcal{A}}_k(U,V) = \sum_{i\in\Omega_k} \frac{\sqrt{3}}{2}\left[6(1+\frac{1}{4}\mu^2 h_k^2)U_i - \sum_{l=1}^{6} U_i^l\right] V_i.$$

and satisfies

$$\bar{\mathcal{A}}_k(U,V) = \mu^2(U,V)_k + D_k(U,V).$$

It is not difficult to see that there are positive constants α_1 and α_2, independent of k, such that

$$\alpha_1 \mathcal{A}_k^{mvs}(U,U) \le \bar{\mathcal{A}}_k(U,U) \le \alpha_2 \mathcal{A}_k^{mvs}(U,U), \quad \forall\, U \in M_k. \tag{3.14}$$

Next we shall make use of standard finite element space on triangular grid, which has the same stencil for $-\Delta$(up to scaling) as that of FDM. Together with (3.14), this fact will play a key role in the proof of regularity and approximation property as in the section 5 in [6]. Let S_k be the space of continuous, piece-wise linear functions

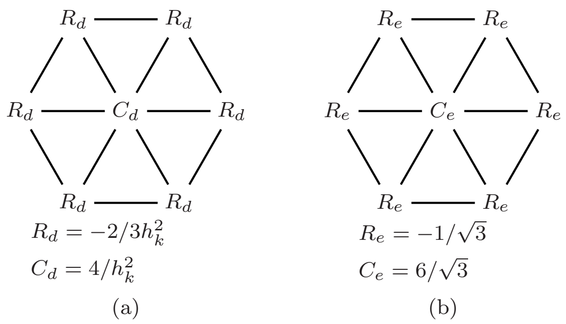

FIG. 3.6. *Stencils of FDM and FEM for* $-\Delta u$

on Ω_k vanishing on the boundary and $\tilde{\mathcal{A}}$ be the finite-element stiffness matrix for the given problem. The finite element stencil of $-\Delta$ is given in (b) Figure 3.6.

For the analysis we shall need to use various Sobolev spaces. For nonnegative integers m, the Sobolev space $H^m(\Omega)$ is defined to be the set of functions in $L^2(\Omega)$ whose distributional derivatives up to order m are in $L^2(\Omega)$. For $\sigma \in (0,1)$, the space $H^\sigma(\Omega)$ is defined to be the set of functions in $L^2(\Omega)$ for which the norm

$$\|v\|^2_{H^\sigma(\Omega)} \equiv \int_\Omega \int_\Omega \frac{(v(x) - v(y))^2}{|x-y|^{2+2\sigma}}\, dxdy$$

is finite. For $s \in (m, m+1)$, where m a positive integer, we define $H^s(\Omega)$ to be the set of all functions in $H^m(\Omega)$ for which the norm

$$\|v\|^2_{H^s(\Omega)} \equiv \|v\|^2_{H^m(\Omega)} + \sum_{|k|=m} \|D^k u\|_{H^{s-m}(\Omega)}$$

is finite. The space $H^{-1}(\Omega)$ is defined as the set of distributions on Ω for which the norm

$$\|f\|_{H^{-1}(\Omega)} = \sup_{\phi \in H^1_0(\Omega)} \frac{(f,\phi)}{\|\phi\|_{H^1(\Omega)}}$$

is finite. For $s \in (0,1)$, the spaces $H^{-s}(\Omega)$ are defined by interpolation between $L^2(\Omega)$ and $H^{-1}(\Omega)$ with norms denoted by $\|\cdot\|_{H^{-s}(\Omega)}$. It is well known that the solution u of (2.1) satisfies inequalities of the form

$$\|u\|_{H^{1+\beta}(\Omega)} \le C\|f\|_{H^{-1+\beta}(\Omega)}, \tag{3.15}$$

where $0 \le \beta \le 1$ is a constant which depends upon $\partial\Omega$ and that (3.15) holds for some $\beta \ge 1/2$ for any polygonal domain in $\mathbb{R}^2$ with interior angles less than 2π. For a nodal function $V \in M_k$, let $\tilde{V}$ denote the piecewise linear interpolant in S_k. It can be shown that

$$|(\widetilde{W}, \widetilde{V}) - (W,V)_k| \le Ch_k \|\widetilde{W}\|_{H^1(\Omega)} \|V\|_k, \quad \forall V, W \in M_k. \tag{3.16}$$

Here, $(\,\cdot\,,\,\cdot\,)$ denote the L^2 inner product on Ω and $\|\cdot\|_k = (\,\cdot\,,\,\cdot\,)_k^{1/2}$. The following norm equivalence can be found in [6].

LEMMA 3. *Let $0 \leq s \leq 1$. There are constants C_0 and C_1 , independent of k, such that*

$$C_0\|\bar{\mathcal{A}}_k^{s/2}V\|_k \leq \|\widetilde{V}\|_{H^s(\Omega)} \leq C_1\|\bar{\mathcal{A}}_k^{s/2}V\|_k, \quad \forall\, V \in M_k.$$

By (3.14) and Lemma 3, there exist constants C_2 and C_3 such that

$$C_2\|(\mathcal{A}_k^{mvs})^{s/2}V\|_k \leq \|\widetilde{V}\|_{H^s(\Omega)} \leq C_3\|(\mathcal{A}_k^{mvs})^{s/2}V\|_k, \quad \forall\, V \in M_k. \tag{3.17}$$

Now we are ready to show the regularity and approximation property:

THEOREM 2. *The bilinear form $\mathcal{A}_k^{mvs}$ satisfies*

$$\begin{aligned} &|\mathcal{A}_k^{mvs}((I-I_kP_{k-1})\,U,U)| \\ &\qquad \leq C_\alpha \left(\frac{\|\mathcal{A}_k^{mvs}U\|_k^2}{\lambda_k}\right)^\alpha \mathcal{A}_k^{mvs}(U,U)^{1-\alpha}, \quad \forall\, U \in M_k. \end{aligned} \tag{3.18}$$

Proof. Consider the following problem: Given $g \in L^2(\Omega)$, find $w \in H_0^1(\Omega)$ satisfying

$$D(w,\chi) = (g,\chi), \quad \forall\, \chi \in H_0^1(\Omega),$$

where $D(w,\chi) = (\nabla w, \nabla \chi)$ is the Dirichlet form. Then, we have

$$\bar{\mathcal{A}}_k(\,\cdot\,,\,\cdot\,) = \mu^2(\,\cdot\,,\,\cdot\,)_k + D_k(\,\cdot\,,\,\cdot\,)$$

and

$$D_k(V,W) = D(\widetilde{V},\widetilde{W}), \quad \forall\, V,W \in M_k.$$

It is straightforward to check that

$$\tilde{\mathcal{A}}(w,v) = \mu^2(w,v) + D(w,v), \quad \forall\, w,v \in H_0(\Omega).$$

Let $\tilde{P}_k$ denote the elliptic projection onto S_k with respect to $\tilde{\mathcal{A}}(\,\cdot\,,\,\cdot\,)$, i.e., $\tilde{P}_kw$ for $w \in H_0^1(\Omega)$ is the unique function in S_k satisfying

$$\tilde{\mathcal{A}}(\tilde{P}_kw,v) = \tilde{\mathcal{A}}(w,v), \quad \forall\, v \in S_k.$$

Now, if we proceed as in the proof of Proposition 5.1 in [6], we can show the regularity and approximation property of the FDM operator by using the norm equivalence in Lemma 3. Next, using (3.14) and (3.17), we can prove the regularity and approximation property of MVS. ∎

We now apply the results in [6](Theorems 2 and 8) to the above schemes to obtain the following result.

THEOREM 3. *For V-cycle algorithm for MVS with $m(k) = 1$ for all k, we have*

$$0 \leq \mathcal{A}^{mvs}((I - B_k\mathcal{A}_k^{mvs})u,u) \leq \delta_k\mathcal{A}_k^{mvs}(u,u), \quad \forall\, u \in M_k,$$

where $\delta_k = \frac{Ck}{Ck+1}$. Also, there exists η_0, η_1 such that

$$\eta_0\mathcal{A}_k^{mvs}(u,u) \leq \mathcal{A}_k^{mvs}(B_k\mathcal{A}_k^{mvs}u,u) \leq \eta_1\mathcal{A}_k^{mvs}(u,u), \quad \forall\, u \in M_k,$$

where η_1 is independent of k and $\eta_0 \leq 1 - \delta_k$.

4. Numerical Results. Consider the boundary value problem with convection term and convection vector $\alpha = (1/\varepsilon, 0)$. This example is drawn from [1]:

$$\begin{cases} -\Delta u + \alpha \cdot \nabla u = 0 \qquad\qquad 0 \le x \le 1 \text{ and } 0 \le y \le \sqrt{3}; \\ u(x,0) = 0, \\ u(x,\sqrt{3}) = \sin \pi\sqrt{3} e^{\mu x}[2e^{-\mu} \sinh \sigma x + \sinh \sigma(1-x)]/\sinh \sigma, \\ 0 \le x \le 1; \\ u(0,y) = \sin \pi y, \quad u(1,y) = 2\sin \pi y, \qquad 0 \le y \le \sqrt{3}, \end{cases} \tag{4.1}$$

where $\mu = 1/(2\varepsilon)$ and $\sigma^2 = \pi^2 + \mu^2$. Using the variable transformation

$$w = u \exp(\frac{1}{2}\alpha \cdot p) \tag{4.2}$$

with $p = (x,y)^t$ and $\mu^2 = \alpha \cdot \alpha^t/4$, the equation (4.2) is posed in the form (2.1). The exact solution is given by

$$u(x,y) = \frac{e^{\mu x}[2e^{-\mu} \sinh \sigma x + \sinh \sigma(1-x)] \sin \pi y}{\sinh \sigma}.$$

We test both FDM with MVS and compare errors and multigrid performance. We present the maximum, minimum eigenvalues, condition numbers and reduction factors and the number of V-cycles to reach steady solution. We tested various values of μ ranging from 10 to 70. For all values of μ, the numbers of iteration to achieve machine accuracy are similar for both algorithms. However, as for the accuracy, MVS is much better. For example, when $\mu = 30$, MVS gives $\|u - u_h\|_\infty = 0.211 \times 10^{-3}$ with $h_J = 1/16$. To obtain similar accuracy with FDM, one has to use finer mesh size than $h = 1/64$. The total computational complexity of FDM considering the number of V-cycle iterations (3 versus 6 in Table 4.3 and 4.4) is 8 times as large as MVS. For other values of μ, we observe similar ratios. The condition numbers of MG for MVS are ranges from 1.0912 to 1.244, which is far smaller than the case standard FEM is used on the unit square(See Table 9). We have used $m = 1$ for all test problems. As a conclusion, this new MVS scheme on hexagonal grid, though rather special, can solve elliptic equation very efficiently both in accuracy and computation speed.

TABLE 4.1
MVS($\mu = 10$)

h_J	λ_{max}	λ_{min}	K	δ_k	$V-$cycle	$\|u - u_h\|_\infty$
1/16	0.995	0.884	1.125	0.137	5	0.110×10^{-4}
1/32	1.027	0.864	1.188	0.162	6	0.200×10^{-5}
1/64	1.064	0.856	1.244	0.168	6	0.100×10^{-5}

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TABLE 4.2
FDM($\mu = 10$)

h_J	λ_{max}	λ_{min}	K	δ_k	V-*cycle*	$\|u-u_h\|_\infty$
1/16	1.000	0.862	1.159	0.139	5	0.312×10^{-2}
1/32	1.030	0.820	1.257	0.163	6	0.896×10^{-3}
1/64	1.066	0.839	1.269	0.169	6	0.239×10^{-3}

TABLE 4.3
MVS($\mu = 30$)

h_J	λ_{max}	λ_{min}	K	δ_k	V-*cycle*	$\|u-u_h\|_\infty$
1/16	0.997	0.956	1.042	0.037	3	0.211×10^{-3}
1/32	0.995	0.903	1.102	0.109	5	0.190×10^{-4}
1/64	1.003	0.874	1.147	0.152	7	0.100×10^{-5}

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Table 4.4
FDM ($\mu = 30$)

h_J	λ_{max}	λ_{min}	K	δ_k	V-*cycle*	$\|u - u_h\|_\infty$
1/16	0.999	0.968	1.039	0.045	4	0.201×10^{-1}
1/32	0.994	0.890	1.116	0.112	5	0.662×10^{-2}
1/64	1.010	0.851	1.188	0.153	6	0.206×10^{-2}

Table 4.5
MVS($\mu = 50$)

h_J	λ_{max}	λ_{min}	K	δ_k	V-*cycle*	$\|u - u_h\|_\infty$
1/16	1.000	0.992	1.007	0.007	2	0.502×10^{-3}
1/32	0.996	0.940	1.059	0.056	3	0.110×10^{-3}
1/64	0.995	0.894	1.113	0.125	4	0.900×10^{-5}

Table 4.6
FDM($\mu = 50$)

h_J	λ_{max}	λ_{min}	K	δ_k	V-*cycle*	$\|u - u_h\|_\infty$
1/16	1.000	0.994	1.005	0.014	2	0.266×10^{-1}
1/32	1.000	0.942	1.061	0.063	3	0.162×10^{-1}
1/64	0.997	0.879	1.135	0.127	4	0.494×10^{-2}

Table 4.7
MVS($\mu = 70$)

h_J	λ_{max}	λ_{min}	K	δ_k	V-*cycle*	$\|u - u_h\|_\infty$
1/16	1.000	0.998	1.001	0.001	2	0.488×10^{-3}
1/32	0.998	0.969	1.029	0.001	3	0.271×10^{-3}
1/64	0.995	0.912	1.091	0.023	4	0.300×10^{-4}

Table 4.8
FDM($\mu = 70$)

h_J	λ_{max}	λ_{min}	K	δ_k	V-*cycle*	$\|u - u_h\|_\infty$
1/16	1.000	0.998	1.001	0.005	2	0.229×10^{-1}
1/32	1.000	0.979	1.021	0.033	3	0.134×10^{-1}
1/64	0.999	0.896	1.115	0.098	3	0.902×10^{-2}

Table 4.9
Standard FEM for $-\Delta u = f$ on the unit square

h_J	λ_{max}	λ_{min}	K	δ_k
1/16	0.999	0.712	1.403	0.167
1/32	0.999	0.690	1.447	0.192
1/64	0.999	0.667	1.497	0.205