L^p Error Estimates and Superconvergence for Covolume or Finite Volume Element Methods

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We consider convergence of the covolume or finite volume element solution to linear elliptic and parabolic problems. Error estimates and superconvergence results in the L^p norm, $2 \le p \le \infty$, are derived. We also show second-order convergence in the L^p norm between the covolume and the corresponding finite element solutions and between their gradients. The main tools used in this article are an extension of the "supercloseness" results in Chou and Li [Math Comp 69(229) (2000), 103–120] to the L^p based spaces, duality arguments, and the discrete Green's function method. Numer Methods Partial Differential Eq 19: 463–486, 2003

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1. INTRODUCTION

Let Ω be a convex polygonal or smooth domain in $\mathcal{R}^2$ and consider the elliptic problem

$$\mathcal{L}u := -\nabla \cdot (A\nabla u) = f, \qquad \text{in } \Omega, \tag{1.1}$$

$$u = 0, \qquad \text{on } \partial\Omega, \tag{1.2}$$

where $A := \{a_{ij}(x)\}$, $a_{ij} = a_{ji} \in W^{1,\infty}(\Omega)$ is a uniformly positive definite matrix, i.e., there exists a positive constant $r > 0$ such that

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$$\sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \geq r(\xi_1^2 + \xi_2^2) \qquad \forall \xi := (\xi_1, \xi_2) \in \mathcal{R}^2, x \in \bar{\Omega}. \tag{1.3}$$

The variational problem associated with (1.1)–(1.2) is to find $u \in U := H_0^1(\Omega)$ such that

$$a(u, v) = (f, v) \qquad \forall v \in U, \tag{1.4}$$

where

$$a(u, v) = \int_\Omega A\nabla u \cdot \nabla v dx, \tag{1.5}$$

$$(f, v) = \int_\Omega f v dx. \tag{1.6}$$

Since one of the main concerns in this article is to derive L^p error estimates, we make the following two regularity assumptions. The first one concerns the elliptic problem (1.1)–(1.2).

R1. There exists a constant $r_{\max} > 1$ such that a solution to problem (1.4) exists and such that

$$\|u\|_{2,p} \leq C_p \| f \|_{0,p} \qquad \forall p \in (1, r_{\max}), \tag{1.7}$$

where the constant $C_p > 0$ depends only on the domain Ω and p. Here $\| \cdot \|_{s,p}$ is the usual norm of the Sobolev space $W^{s,p}(\Omega)$.

The second regularity assumption is on the type of problems for which the right side of (1.1) is specialized as a divergence, i.e., $u \in H_0^1(\Omega)$ satisfies

$$\mathcal{L}u = \nabla \cdot F. \tag{1.8}$$

R2. There exists a constant $\gamma_{\max} > 1$ such that a solution to problem (1.8) exists and such that

$$\|u\|_{1,p} \leq C\|F\|_{0,p} \qquad \forall p \in (1, \gamma_{\max}), \tag{1.9}$$

where the constant C depends only on the domain Ω and p.

For a polygonal domain Ω,

$$r_{\max} = \begin{cases} \dfrac{2}{2-\beta}, & \text{if } \beta < 2, \\ \infty, & \text{if } \beta \geq 2, \end{cases}$$

where $\beta = 1/\alpha$, $\alpha\pi$ is the largest interior vertex angle of Ω (cf. [1]). When Ω is a $C^{1,1}$ domain and $a_{ij} \in C(\bar{\Omega})$, one can take $r_{\max} = \infty$ [2]. On the other hand, under the same assumptions one can take $\gamma_{\max} = \infty$ as well [3]. For rectangles, one also has $r_{\max} = \gamma_{\max} = \infty$ and $\gamma_{\max} > 2$ for

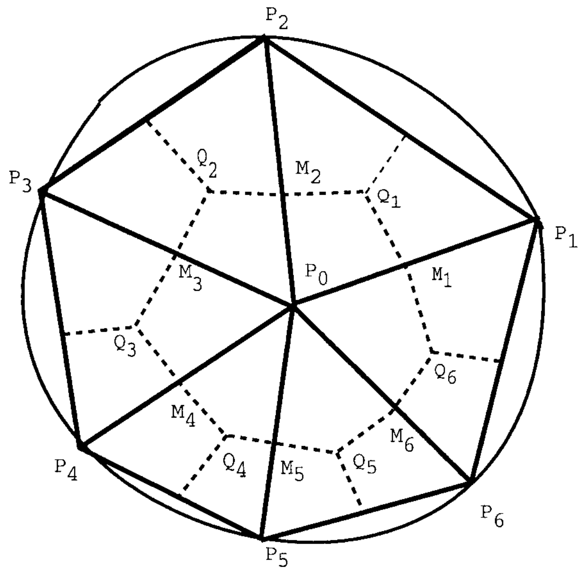

FIG. 1. Primal and dual partitions of a convex domain.

convex polygonal domains. As one may expect, our L^p estimates statements below will be more concise when both parameters are infinity.

Given a polygonal domain Ω, let $\mathcal{T}_h = \{K_Q\}$ be a regular triangulation of the domain Ω into a union of triangular elements K_Q with barycenter Q (cf. Fig. 1). Here $h := \max h_K$, the maximum of the diameters h_K over all triangles. The nodes of a triangular element are its vertices and the set of all vertices is denoted by $\mathcal{N}_h$. Associated with the primal partition $\mathcal{T}_h$ we define its dual partition $\mathcal{T}_h^*$ of Ω as follows. Let P_0 be an interior node and P_i, $i = 1, \ldots, 6$, be its adjacent nodes, and $M_i := M_{0i}$ the midpoint of $\overline{P_0P_i}$. Connect successively the points M_1, Q_1, M_2, $Q_2, \ldots, M_6$, Q_6, M_1 to obtain the dual polygonal element $K_{P_0}^*$. Its nodes are defined to be Q_i, $i = 1, \ldots, 6$, and the set of dual nodes are denoted by $\mathcal{N}_h^*$. The dual element $K_{P_2}^*$ based at a typical boundary node P_2 is $M_{12}Q_1M_2Q_2M_{23}P_2$. Let $\mathring{\mathcal{N}}_h := \mathcal{N}_h - \partial\Omega$, the set of all interior nodes in $\mathcal{T}_h$, and let S_Q and $S_{P_0}^*$ denote, respectively, the areas of triangle K_Q and polygon $K_{P_0}^*$. Throughout the article we shall assume the partitions to be quasi-uniform: There exist two positive constants C_1 and C_2 independent of h such that

$$C_1h^2 \le S_Q \le C_2h^2, \qquad \forall Q \in \mathcal{N}_h^*, \tag{1.10}$$

$$C_1h^2 \le S_{P_0}^* \le C_2h^2, \qquad \forall P_0 \in \mathcal{N}_h. \tag{1.11}$$

Corresponding to $\mathcal{T}_h$ we define the trial function space $U_h \subset H_0^1(\Omega)$ as the space of continuous functions on the closure of Ω, which vanish on $\partial\Omega$ and are linear on each triangle $K_Q \in \mathcal{T}_h$. Let $\Pi_h : U \cap C(\bar{\Omega}) \to U_h$ be the usual linear interpolation operator, and thus

$$|u - \Pi_h u|_{m,p} \le Ch^\sigma |u|_{r,p}, \qquad 0 \le m \le r \le 2,$$
$$0 \le \sigma \le r - m, \quad 1 \le p \le \infty.$$

Throughout the article C will denote a generic constant independent of h and can have different values in different places. We use $\|\cdot\|_m$ and $|\cdot|_m$, respectively, for the norm $\|\cdot\|_{m,p}$ and the seminorm of $W^{m,p}(\Omega)$ when $p = 2$.

For a convex domain Ω with a smooth boundary (cf. Fig. 1), we triangulate it as before and call the resulting polygonal domain Ω_h. It is further required that the vertices which lie on $\partial\Omega_h$ also lie on $\partial\Omega$. A trial function in U_h is now defined to be continuous piecewise linear on Ω_h and zero outside Ω_h. Since the distance

$$\mathrm{dist}(\partial\Omega_h, \partial\Omega) \leq Ch^2,$$

all the approximation properties above still hold for the smooth domain case [4, 5].

The test function space $V_h \subset L^2(\Omega)$ associated with the dual partition $\mathcal{T}_h^*$ is defined as the set of all piecewise constants. More specifically, let χ_{P_0} be the characteristic function of the set $K_{P_0}^*$, we have for $v_h \in V_h$

$$v_h = \sum_{P_0 \in \mathcal{N}_h^\circ} v_h(P_0)\chi_{P_0}. \tag{1.12}$$

Define the transfer operator $\Pi_h^* : U_h \rightarrow V_h$ connecting the trial and test spaces as

$$\Pi_h^* w := \sum_{P_0 \in \mathcal{N}_h^\circ} w_h(P_0)\chi_{P_0}. \tag{1.13}$$

Obviously, Π_h^* can be extended to $H_0^1(\Omega) \cap C(\bar{\Omega})$. By the usual interpolation theory it holds that

$$\|w - \Pi_h^* w\|_{0,p} \leq Ch^\beta |w|_{s,p},$$
$$0 \leq \beta \leq s \leq 1, \qquad 1 \leq p \leq \infty.$$

The approximate problem we consider is: Find $u_h \in U_h$ such that

$$a^*(u_h, v_h) = (f, v_h) \qquad \forall v_h \in V_h, \tag{1.14}$$

where

$$a^*(u_h, v_h) := \sum_{P_0 \in \mathcal{N}_h^\circ} v_h(P_0) a^*(u_h, \chi_{P_0}), \tag{1.15}$$

$$a^*(u_h, \chi_{P_0}) := -\int_{\partial K_{P_0}^*} (A\nabla u_h) \cdot \mathbf{n} ds, \tag{1.16}$$

where $\mathbf{n}$ is the outward unit normal to $\partial K_{P_0}^*$, and $a^*(\cdot, \cdot)$ is bilinear by construction.

Let $K_Q = \Delta P_1 P_2 P_3 \in \mathcal{T}_h$, and let M_l be midpoint of $\overline{P_{l+1} P_{l+2}}$, $1 \leq 1 \leq 3 (\mathrm{mod}\ 3)$. While (1.15) reflects a conservation law, it is more convenient for the error analysis to write it as

$$a^*(u_h, v_h) = \sum_{K \in \mathcal{T}_h} I_K(u_h, v_h), \tag{1.17}$$

where

$$I_K(u_h, v_h) = -\sum_{l=1}^{3} \int_{\partial K^*_{P_l} \cap K} A\nabla u_h \cdot \mathbf{n} ds \cdot v_l = \sum_{l=1}^{3} \int_{\overline{M_l Q}} A\nabla u_h \cdot \mathbf{n} ds \cdot (v_{l+2} - v_{l+1}). \qquad (1)18$$

Here $v_l = v_h(P_l)$ and $\mathbf{n}$ is the unit normal vector pointing to the right as one walks along $\overline{M_l Q}$ in $K^*_{P_l} \cap K$.

By (1.1) and (1.14) we have the "orthogonality" property

$$a^*(u - u_h, v_h) = 0, \qquad v_h \in V_h \tag{1.19}$$

Define the Ritz projection operator $R_h : H^1_0(\Omega) \to U_h$, so that

$$a(R_h w - w, \chi) = 0, \qquad \chi \in U_h. \tag{1.20}$$

Let us relate our work to the existing literature. The basic idea of the finite volume method for general elliptic problems is to use the divergence theorem on the elliptic operator $\mathcal{L}$ of (1.1) to convert the double integral into a boundary integral as in (1.16). The idea is old and the resulting method comes under a variety of names, e.g., the generalized difference method [6] in the early 1980s, the box method [7–10], the covolume method [11–16], and the so-called finite volume element methods [10, 17–20], among others. The term "covolume" can either mean complementary or control volume, and the term "finite volume element" seems to be coined by S. McCormick. Reference [6] contains a bulk of contributions in the early years, and the article of Bank-Rose [7] is pivotal in calling attention to the merits of variational approach in finite volume methods. Cai et al. [17, 18] give the first finite volume element analysis to some special tensor coefficient cases. The first unified approach to the analysis of finite volume element or covolume methods applied to the general anisotropic case on polygonal or smooth domains is given by Chou and Li [5]. In addition to optimal estimates in H^1, L^2, they also showed how to derive $W^{1,\infty}$ error estimate. The central idea there is to compare the covolume solution with the corresponding finite element solution via an extra-power-of-h lemma. In this article we further explore and generalize that idea to develop L^p estimates, $2 \le p \le \infty$. (Although of interest in their own right, such estimates are indispensable for nonlinear problems.) To keep the article short, results for linear elliptic and parabolic problems are given in Section 2 and Section 3, respectively. Nonlinear problem results will be given in a follow-up article.

2. ESTIMATES FOR ELLIPTIC PROBLEMS

In this section we first derive a central lemma, as in [5]. This lemma generalizes the one in [5], which shows that an "extra" power of h is possible when comparing the bilinear forms $a(\cdot,\cdot)$ and $a^*(\cdot,\cdot)$ with certain arguments. We then use it to derive convergence rates in L^p norm for covolume solutions of the elliptic problem. We also show supercloseness between the covolume and finite element solutions.

Lemma 2.1. *Let* $u_h \in U_h$, $v \in C(\bar{\Omega}) \cap H_0^1(\Omega)$, $1 \le p \le \infty$, $\frac{1}{p} + \frac{1}{p'} = 1.$

(i) If $u \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$ *and* $v \in W^{1,p'}(\Omega)$, *then*

$$|a(u - u_h, \Pi_h v) - a^*(u - u_h, \Pi_h^* v)| \le Ch[|u - u_h|_{1,p} + |u|_{2,p}]|v|_{1,p'}. \tag{2.1}$$

(ii) If $u \in W^{3,p}(\Omega) \cap H_0^1(\Omega)$ *and* $v \in W^{1,p'}(\Omega)$, *then*

$$|a(u - u_h, \Pi_h v) - a^*(u - u_h, \Pi_h^* v)| \le Ch^2[h^{-1}|u - u_h|_{1,p} + |u|_{3,p}]|v|_{1,p'}. \tag{2.2}$$

Proof. We prove the assertions only for polygonal domains. The smooth domain case only requires some additional trivial changes since all the integrals involved in the left side bilinear forms are zero in the skin layer $\Omega - \Omega_h$.

Applying Green's formula to $a(\cdot, \Pi_h \cdot)$ and $a^*(\cdot, \Pi_h^*)$

$$a(u - u_h, \Pi_h v) = \sum_{K \in \mathcal{T}_h} \int_K [A(x) - \bar{A}]\nabla(u - u_h) \cdot \nabla \Pi_h v dx - \sum_K \int_K \nabla \cdot (\bar{A} \nabla u) \Pi_h v dx$$
$$+ \sum_K \int_{\partial K} \bar{A} \nabla(u - u_h) \cdot \mathbf{n} \Pi_h v ds,$$

where $\bar{A}$ is the local L^2 projection of A on K:

$$\bar{A} = \frac{1}{\text{meas } K} \int_K A(y) dy.$$

$$a^*(u - u_h, \Pi_h^* v) = -\sum_{K \in \mathcal{T}_h} \sum_{l=1}^{3} \int_{\partial K_{p_l}^* \cap K} A \nabla(u - u_h) \cdot \mathbf{n} \Pi_h^* v ds$$
$$= -\sum_{K \in \mathcal{T}_h} \sum_{l=1}^{3} \int_{\partial K_{p_l}^* \cap K} [A(x) - \bar{A}] \nabla(u - u_h) \cdot \mathbf{n} \Pi_h^* v ds - \sum_K \int_K \nabla \cdot (\bar{A} \nabla u) \Pi_h^* v dx$$
$$+ \sum_K \int_{\partial K} \bar{A} \nabla(u - u_h) \cdot \mathbf{n} \Pi_h^* v ds.$$

Thus

$$a(u - u_h, \Pi_h v) - a^*(u - u_h, \Pi_h^* v) = \sum_{i=1}^{4} E_i(u - u_h, v), \tag{2.3}$$

where

$$E_1(u - u_h, v) = \sum_K \int_K [A(x) - \bar{A}]\nabla(u - u_h) \cdot \nabla\Pi_h v dx, \tag{2.4}$$

$$E_2(u - u_h, v) = \sum_K \sum_{l=1}^{3} \int_{\partial K_{p_l}^* \cap K} [A(x) - \bar{A}]\nabla(u - u_h) \cdot \mathbf{n}\Pi_h^* v ds, \tag{2.5}$$

$$E_3(u - u_h, v) = -\sum_K \int_K \nabla \cdot (\bar{A}\nabla u)(\Pi_h v - \Pi_h^* v) dx, \tag{2.6}$$

$$E_4(u - u_h, v) = \sum_K \int_{\partial K} \bar{A}\nabla(u - u_h) \cdot \mathbf{n}(\Pi_h v - \Pi_h^* v) ds. \tag{2.7}$$

For convenience we will estimate E_i's for $1 < p < \infty$. The $p = 1$ or $p = \infty$ cases involve only minor changes.

By the Hölder inequality,

$$|E_1(u - u_h, v)| \le Ch \sum_K |u - u_h|_{1,p,K} |\Pi_h v|_{1,p',K} \le Ch|u - u_h|_{1,p}|v|_{1,p'}. \tag{2.8}$$

By definition, (1.18) and the Hölder inequality

$$\begin{aligned} |E_2(u - u_h, v)| &= \left| \sum_K \sum_{l=1}^{3} \int_{\overline{M_l Q}} [A(x) - \bar{A}]\nabla(u - u_h)\mathbf{n} ds (v_{l+2} - v_{l+1}) \right| \\ &\le Ch^{1+(1/p')} \sum_K \sum_{l=1}^{3} \left\{ \int_{\overline{M_l Q}} (|\phi_1|^p + |\phi_2|^p) ds \right\}^{1/p} |v_{l+2} - v_{l+1}|, \end{aligned} \tag{2.9}$$

where

$$\phi_i = \frac{\partial(u - u_h)}{\partial x_i}, \qquad i = 1, 2. \tag{2.10}$$

First, on K

$$\begin{aligned} |v_{l+2} - v_{l+1}| &= \left| \frac{\partial \Pi_h v}{\partial x_1} (x_1(P_{l+2}) - x_1(P_{l+1})) + \frac{\partial \Pi_h v}{\partial x_2} (x_2(P_{l+2}) - x_2(P_{l+1})) \right| \\ &\le Ch\left(\left| \frac{\partial \Pi_h v}{\partial x_1} \right| + \left| \frac{\partial \Pi_h v}{\partial x_2} \right| \right) \end{aligned}$$

$$\le Ch2^{1/p}\left(\left|\frac{\partial \Pi_h v}{\partial x_1}\right|^{p'} + \left|\frac{\partial \Pi_h v}{\partial x_2}\right|^{p'}\right)^{1/p'}$$
$$\le Ch^{1-2/p'}\left[S_Q\left(\left|\frac{\partial \Pi_h v}{\partial x_1}\right|^{p'} + \left|\frac{\partial \Pi_h v}{\partial x_2}\right|^{p'}\right)\right]^{1/p'} \qquad (S_Q \text{ is from (1.10)})$$
$$\le Ch^{1-2/p'}|\Pi_h v|_{1,p',K}.$$

Thus we have

$$|v_{l+2} - v_{l+1}| \le Ch^{1-2/p'}|\Pi_h v|_{1,p',K}. \tag{2.11}$$

Next, introduce the usual affine transformation that maps a reference element $\hat{K}$ to K with the following correspondences: $\phi_i \to \hat{\phi}_i$, $M_l \to \hat{M}_l$, $p_l \to \hat{p}_l$, $Q \to \hat{Q}$, $l = 1, 2, 3$. Obviously,

$$\int_{\overline{M_l Q}} |\phi_i|^p ds \le Ch \int_{\overline{\hat{M}_l \hat{Q}}} |\hat{\phi}_i|^p d\hat{s}, \qquad i = 1, 2. \tag{2.12}$$

By the Trace Theorem [21], for $1 \le p \le \infty$,

$$\int_{\overline{\hat{M}_l \hat{Q}}} |\hat{\phi}_i|^p d\hat{s} \le C\|\hat{\phi}_i\|^p_{1,p,\hat{K}}. \tag{2.13}$$

By Theorem 3.1.2 of Ciarlet [22],

$$\|\hat{\phi}_i\|_{0,p,\hat{K}} \le Ch^{-2/p}\|\phi_i\|_{0,p,K},$$
$$|\hat{\phi}_i|_{1,p,\hat{K}} \le Ch^{1-2/p}|\phi_i|_{1,p,K}.$$

Thus,

$$\int_{\overline{M_l Q}} |\phi_i|^p ds \le Ch[h^{-2}\|\phi_i\|^p_{0,p,K} + h^{p-2}|\phi|^p_{1,p,K}] \le Ch[h^{-2}|u-u_h|^p_{1,p,K} + h^{p-2}|u|^p_{2,p,K}]. \tag{2.14}$$

Combining (2.11) and (2.14) with (2.9),

$$|E_2(u-u_h, v)| \le Ch^{1+(1/p')}\sum_K h^{1/p}[h^{-2}|u-u_h|^p_{1,p,K} + h^{p-2}|u|^p_{2,p,K}]^{1/p}h^{1-(2/p')}|\Pi_h v|_{1,p',K}$$
$$\le Ch^2 2^{1/p}\sum_K [h^{-(2/p)}|u-u_h|_{1,p,K} + h^{1-(2/p)}|u|_{2,p,K}]h^{1-(2/p')}|\Pi_h v|_{1,p',K}$$
$$\le Ch^2[h^{-1}|u-u_h|_{1,p} + |u|_{2,p}]|v|_{1,p'}. \tag{2.15}$$

By the definition (2.6) of E_3, the triangle inequality, and the approximation properties,

$$|E_3(u - u_h,\, v)| \le Ch|u|_{2,p}|v|_{1,p'}. \tag{2.16}$$

Among E_1 through E_4 the only term that would prevent an h^2 factor in front of a $|u|_{2,p}$ term is E_3. But if $u \in W^{3,p}$, we can proceed as follows to get an extra power of h.

First observe that for *barycentric* partitions

$$\int_K (\Pi_h v - \Pi_h^* v)dx = 0, \tag{2.17}$$

and hence

$$\begin{aligned} |E_3(u - u_h,\, v)| &= \left| \sum_K \int_K [\nabla \cdot \bar{A}\nabla(u - I_2 u)](\Pi_h v - \Pi_h^* v)dx \right| \\ &\le C \sum_K |u - I_2 u|_{2,p,K} \|\Pi_h v - \Pi_h^* v\|_{0,p',K} \\ &\le Ch^2 |u|_{3,p}|v|_{1,p'}, \end{aligned} \tag{2.18}$$

where I_2 is the quadratic interpolation operator on K.

As for E_4, let L be the common edge of two adjacent elements K_1 and K_2, and let n_1 and n_2 be unit outer normal vectors of K_1 and K_2 along L. Let E be the collection of all interior edges in $\mathcal{T}_h$. Observe that $\bar{A}\nabla u_h \cdot \mathbf{n}$ is constant along any edge L and

$$\int_L (\Pi_h v - \Pi_h^* v)ds = 0, \tag{2.19}$$

where the fact of midpoint partition was used. Thus,

$$\int_L \bar{A}\nabla u_h \cdot \mathbf{n}(\Pi_h v - \Pi_h^* v)ds = 0. \tag{2.20}$$

Obviously, on L

$$|\Pi_h v - \Pi_h^* v| \le |v_l - v_{l+1}| \tag{2.21}$$

for some $l \in \{1, 2, 3\}$. Let I_1 be the linear interpolation operator on $K_1 \cup K_2$. Now using the boundary condition and piecewise continuity of $\bar{A}\nabla u_h \cdot \mathbf{n}(\Pi_h v - \Pi_h^* v)$, we have by definition (2.7), (2.20), and (2.21)

$$\begin{aligned} |E_4(u - u_h,\, v)| &= \left| \sum_{L \in E} \int_L [\bar{A}_{K_1} - \bar{A}_{K_2}]\nabla u \cdot \mathbf{n}(\Pi_h v - \Pi_h^* v)ds \right| \\ &= \left| \sum_{L \in E} \int_L [\bar{A}_{K_1} - \bar{A}_{K_2}]\nabla(u - I_1 u) \cdot \mathbf{n}(\Pi_h v - \Pi_h^* v)ds \right| \end{aligned}$$

$$\le Ch^{1+(1/p')} \sum_L h^{1/p}[h^{-(2/p)}|u - I_1 u|_{1,p,K_1\cup K_2} + h^{1-(2/p)}|u|_{2,p,K_1\cup K_2}]h^{1-(2/p')}|\Pi_h v|_{1,p',K_1\cup K_2}$$

$$\le Ch^2|u|_{2,p}|v|_{1,p'},$$

where the last three relations were handled as in E_2. Combining all the above inequalities with (2.3) completes the proof. ∎

Remark 2.1. One should notice from (2.17) that barycentric subdivisions play a crucial role in the validity of (2.2). All the first-order results in this article are derived via (2.1) and second or near second-order results via (2.2).

Setting $u = 0$ in Lemma 2.1, one obtains the following lemma.

Lemma 2.2. *Under the same assumptions of Lemma* 2.1, *we have*

$$|a(u_h, \Pi_h v) - a^*(u_h, \Pi_h^* v)| \le Ch|u_h|_{1,p}|v|_{1,p'}. \tag{2.22}$$

Lemma 2.3. *Assume that* $u \in W^{2,p} \cap H_0^1(\Omega)$, $2 \le p \le \infty$. *Then*

$$\|u - R_h u\|_{1,p} \le Ch\|u\|_{2,p}, \qquad 2 \le p \le \infty, \tag{2.23}$$

$$\|u - R_h u\|_{0,p} \le Ch^2\|u\|_{2,p}, \qquad 2 \le p < \infty, \tag{2.24}$$

$$\|u - R_h u\|_{0,\infty} \le Ch^2 \log\frac{1}{h}\cdot\|u\|_{2,\infty}, \qquad (p = \infty). \tag{2.25}$$

Furthermore, for the conjugate index $p' = p/(p-1)$ *one has*

$$\|u - R_h u\|_{1,p'} \le Ch\|u\|_{2,p'}. \tag{2.26}$$

Inequalities (2.23)–(2.25) are well known [4, 23, 24] and (2.26) can be found in [25]. Note that they are valid for both convex polygonal and smooth domains.

We are now ready to show the main results of this section. Without loss of generality the two parameters in (1.7) and (1.9) are chosen the same.

Theorem 2.1. *Let u and* u_h *be the solutions of* (1.1) *and* (1.14), *respectively. Suppose that* $u \in W^{2,p}(\Omega)$ *and* $r_{\max} = \gamma_{\max} > 2$ [*cf.* (1.7) *and* (1.9)]. *Then for h sufficiently small*:

$$\|u - u_h\|_{1,p} \le Ch\|u\|_{2,p}, \qquad 2 \le p < r_{\max} \le \infty. \tag{2.27}$$

In particular, if the domain Ω *is either rectangular or smooth and* $a_{ij} \in C(\bar{\Omega})$, *then*

$$\|u - u_h\|_{1,p} \le Ch\|u\|_{2,p}, \qquad 2 \le p < \infty. \tag{2.28}$$

Proof. Since the conjugate index $r'_{\max} = r/(r-1) < p' \le 2$, by the regularity condition (1.9), the following the auxiliary problem is well posed, i.e., given a function $\psi \in C_0^\infty(\Omega)$, find $\boldsymbol{\Psi} \in H_0^1(\Omega)$ such that

$$a(v, \mathbf{\Psi}) = -(\psi_x, v) \qquad \forall v \in H_0^1(\Omega), \tag{2.29}$$

$$\|\mathbf{\Psi}\|_{1,p'} \le C\|\psi\|_{0,p'}, \qquad \frac{1}{p} + \frac{1}{p'} = 1, \tag{2.30}$$

where the subscript x denotes the partial derivative with respect to x. By (2.29) and (1.19),

$$\begin{aligned} ((u - u_h)_x, \psi) &= -(\psi_x, u - u_h) \\ &= a(u - u_h, \mathbf{\Psi}) \\ &= a(u - u_h, \mathbf{\Psi} - R_h\mathbf{\Psi}) + [a(u - u_h, R_h\mathbf{\Psi}) - a^*(u - u_h, \Pi_h^* R_h\mathbf{\Psi})] \\ &= I_1 + I_2 \end{aligned} \tag{2.31}$$

By (1.19), (2.23), and (2.30), we derive

$$|I_1| = |a(u - R_h u, \mathbf{\Psi} - R_h\mathbf{\Psi})| = |a(u - R_h u, \mathbf{\Psi})| \le C\|u - R_h u\|_{1,p}\|\mathbf{\Psi}\|_{1,p'} \le Ch\|u\|_{2,p}\|\psi\|_{0,p'}. \tag{2.32}$$

By (2.1) and (2.30),

$$|I_2| \le Ch(|u - u_h|_{1,p} + |u|_{2,p})\|\mathbf{\Psi}\|_{1,p'} \le Ch(|u - u_h|_{1,p} + |u|_{2,p})\|\psi\|_{0,p'}. \tag{2.33}$$

Combining (2.32), (2.33) with (2.31),

$$\|(u - u_h)_x\|_{0,p} = \sup_{\psi \in C_0^\infty(\Omega)} \frac{((u - u_h)_x, \psi)}{\|\psi\|_{0,p'}} \le Ch\|u - u_h\|_{1,p} + Ch\|u\|_{2,p}, \qquad x = x_1, x_2.$$

Hence using the Poincaré inequality, we have for h sufficiently small

$$\|u - u_h\|_{1,p} \le Ch\|u\|_{2,p}. \tag{2.34}$$

Finally, the assertion (2.28) follows from the comments following (1.9). This completes the proof. ∎

We included the rectangular domain case in the above theorem, since the finite volume method has its origin as a generalized finite difference method from rectangles to nonrectangular domains.

Theorem 2.2. *Let u and u_h be the solutions of* (1.1) *and* (1.14), *respectively. Suppose that $u \in W^{3,q}(\Omega)$ and $r_{\max} = \gamma_{\max} > 2$ [cf.* (1.7) *and* (1.9)], *then for h sufficiently small*:

$$\|u - u_h\|_{0,p} \le Ch^2\|u\|_{3,q}, \qquad 2 \le p < r_{\max} \le \infty, \tag{2.35}$$

where $q > 1$, if $p = 2$; and $q = 2p/(p + 2)$, if $p > 2$.

In particular, for rectangular or smooth domain Ω and $a_{ij} \in C(\bar{\Omega})$,

$$\|u - u_h\|_{0,p} \le Ch^2\|u\|_{3,q}, \qquad 2 \le p < \infty, \tag{2.36}$$

where $q > 1$, *if* $p = 2$; *and* $q = 2p/(p + 2)$, *if* $p > 2$.

Proof. Since $2 \le p < \infty$ and $r_{\max} > 2$, we have by (1.7) that for $1 < p' < r_{\max}$ the solution to the following problem exists: given a function ϕ, find $\mathbf{\Phi} \in H_0^1(\Omega)$ such that

$$a(v, \mathbf{\Phi}) = (\phi, v), \qquad v \in H_0^1(\Omega), \tag{2.37}$$

$$\|\mathbf{\Phi}\|_{2,p'} \le C\|\phi\|_{0,p'}, \qquad \frac{1}{p} + \frac{1}{p'} = 1. \tag{2.38}$$

Thus,

$$\begin{aligned}(u - u_h, \phi) &= a(u - u_h, \mathbf{\Phi})\\ &= a(u - u_h, \mathbf{\Phi} - R_h\mathbf{\Phi}) + [a(u - u_h, R_h\mathbf{\Phi}) - a^*(u - u_h, \Pi_h^* R_h\mathbf{\Phi})] := J_1 + J_2.\end{aligned} \tag{2.39}$$

By (2.27) and (2.26),

$$|J_1| \le c\|u - u_h\|_{1,p}\|\mathbf{\Phi} - R_h\mathbf{\Phi}\|_{1,p'} \le Ch^2\|u\|_{2,p}\|\mathbf{\Phi}\|_{2,p'}. \tag{2.40}$$

By (2.2), (2.27), and the imbedding theorems (e.g., $W_0^{2,p'} \subset W_0^{1,q'}$, $1 + 2/q' = 2/p'$, $p > 2$)

$$|J_2| \le Ch^2[h^{-1}|u - u_h|_{1,q} + |u|_{3,q}]|\mathbf{\Phi}|_{1,q'}, \frac{1}{q} + \frac{1}{q'} = 1 \le Ch^2\|u\|_{3,q}\|\mathbf{\Phi}\|_{2,p'}. \tag{2.41}$$

Combining (2.40), (2.41) with (2.39) and applying (2.38) complete the proof. ∎

Given any $z \in \bar{\Omega}$, we define $G_z^h \in U_h$ to be the discrete Green's function associated with the form $a(\cdot, \cdot)$ if

$$a(G_z^h, w_h) = w_h(z) \qquad \forall w_h \in U_h. \tag{2.42}$$

Let v be a given unit vector (direction) and let Δz be any vector parallel to v. Then we define

$$\partial_z G_z^h := \lim_{\Delta z \to 0} \frac{G_{z+\Delta z}^h - G_z^h}{|\Delta z|}. \tag{2.43}$$

Here following [26], we have used ∂_z even for nonpartials. However, in this article the reader can think of ∂_z as $v \cdot \nabla_z$, where v is either $(1, 0)^t$ or $(0, 1)^t$.

Lemma 2.4. *The derivative* $\partial_z G_z^h \in U_h$ has the following properties.

$$a(\partial_z G_z^h, v_h) = \partial_z v_h(z) \qquad \forall v_h \in U_h. \tag{2.44}$$

For $r_{\max} > 2$ [cf. (1.7)], there exists a positive constant C, independent of z and h such that

$$\|\partial_z G_z^h\|_0^2 + \|\partial_z G_z^h\|_{1,1} + \|G_z^h\|_{1,2} \le C|\ln h|. \tag{2.45}$$

For $r_{\max} > 1$ there exists a positive constant C_p, dependent on p but not on z and h, such that for $p < 2$,

$$\|G_z^h\|_{1,p} \le C_p. \tag{2.46}$$

The inequalities (2.45) and (2.46) can be found in Theorems 3.14 and 3.17 of [26], respectively.

Theorem 2.3. *Let u and u_h be the solutions of* (1.1) *and* (1.14), *respectively. Then for h sufficiently small*

$$\|u - u_h\|_{1,\infty} \le Ch(\|u\|_{2,\infty} + \|u\|_3), \tag{2.47}$$

$$\|u - u_h\|_{0,\infty} \le Ch^2 \log \frac{1}{h} (\|u\|_{2,\infty} + \|u\|_3), \tag{2.48}$$

$$\|u - u_h\|_{1,\infty} \le Ch|\log h|\|u\|_{2,\infty}. \tag{2.49}$$

provided that the solution u has the indicated smoothness.

Proof. The first two inequalities were proved in [5]. The third inequality simply says we do not need $\|u\|_3$ if we are willing to pay with a logarithmic factor.

Let us show (2.49). It is known from Lemma 2.3 that $\|u - R_h u\|_{1,\infty} \le Ch\|u\|_{2,\infty}$. Thus, by the triangle inequality it suffices to show the following. By (2.44), (1.19), (1.20), and (2.1):

$$\begin{aligned}
|\partial_z(R_h u - u_h)(z)| &= |a(R_h u - u_h, \partial_z G_z^h)| \\
&= |a(u - u_h, \partial_z G_z^h)| \\
&= |a(u - u_h, \partial_z G_z^h) - a^*(u - u_h, \Pi_h^* \partial_z G_z^h)| \\
&\le Ch(|u - u_h|_{1,\infty} + |u|_{2,\infty})\|\partial_z G_z^h\|_{1,1} \\
&\le Ch(|u - R_h u|_{1,\infty} + |R_h u - u_h|_{1,\infty} + |u|_{2,\infty})\|\partial_z G_z^h\|_{1,1}.
\end{aligned} \tag{2.50}$$

Now to complete the proof, use (2.45) and absorb the $Ch|\log h||R_h u - u_h|_{1,\infty}$ term to the left side. [Note that if we apply (2.2) instead of (2.1), we can prove (2.47).] ∎

Remark 2.2. One might ask if the technique of compensating a logarithmic factor for relaxing regularity also works for (2.48). Unfortunately, the answer is negative. This may be seen as follows. Recall from Lemma 2.3 that $\|u - R_h u\|_{0,\infty} \le h^2|\log h|\|u\|_{2,\infty}$. Now

$$\begin{aligned}
|(R_h u - u_h)(z)| &= |a(R_h u - u_h, G_z^h)| \\
&= |a(u - u_h, G_z^h)| \\
&= |a(u - u_h, G_z^h) - a^*(u - u_h, \Pi_h^* G_z^h)| \\
&\le Ch(|u - u_h|_{1,2} + |u|_{2,2})|G_z^h|_{1,2}.
\end{aligned} \tag{2.51}$$

We lose a power of h in the $W^{0,\infty}$ error if u is less smooth. In contrast, if we invoke (2.2) instead of (2.1), we actually get (2.48).

The remainder of this section deals with the issue of how close the covolume solution is to the standard finite element solution.

Theorem 2.4. *Let u and u_h be the solutions of* (1.1) *and* (1.14), *respectively. Suppose that $u \in W^{3,p}(\Omega)$ and $r_{\max} = \gamma_{\max} > 2$. Then for h sufficiently small,*

$$\|R_h u - u_h\|_{1,p} \leq Ch^2 \|u\|_{3,p}, \qquad 2 \leq p < r_{\max} \leq \infty. \tag{2.52}$$

Furthermore, for rectangular or smooth domains and $a_{ij} \in C(\bar{\Omega})$, one can take $r_{\max} = \infty$ in the above inequality.

Proof. We proceed as in Theorem 2.1. By (2.29), we have

$$\begin{aligned}
|((R_h u - u_h)_x, \psi)| &= |a(R_h u - u_h, \mathbf{\Psi})| \\
&= |a(R_h u - u_h, \mathbf{\Psi} - R_h \mathbf{\Psi}) + a(R_h u - u_h, R_h \mathbf{\Psi})| \\
&= |a(u - u_h, R_h \mathbf{\Psi}) - a^*(u - u_h, \Pi_h^* R_h \mathbf{\Psi})| \qquad \text{(by (1.19), (1.20))} \\
&\leq Ch^2 [h^{-1} |u - u_h|_{1,p} + |u|_{3,p}] |\mathbf{\Psi}|_{1,p'} \qquad \text{(by (2.2))} \\
&\leq Ch^2 \|u\|_{3,p} \|\psi\|_{0,p'}. \qquad \text{(by (2.27), (2.30))}
\end{aligned}$$

The proof is complete. ∎

Theorem 2.5. *Let u and u_h be the solutions of* (1.1) *and* (1.14), *respectively. Then for h sufficiently small*

$$\|R_h u - u_h\|_{1,\infty} \leq Ch^2 \log \frac{1}{h} \cdot \|u\|_{3,\infty}, \tag{2.53}$$

provided that the solution has the indicated smoothness.

Proof.

$$\begin{aligned}
|\partial_z (R_h u - u_h)(z)| &= |a(R_h u - u_h, \partial_z G_z^h)| \\
&= |a(u - u_h, \partial_z G_z^h)| \\
&= |a(u - u_h, \partial_z G_z^h) - a^*(u - u_h, \Pi_h^* \partial_z G_z^h)| \\
&\leq Ch^2 [h^{-1} |u - u_h|_{1,\infty} + |u|_{3,\infty}] |\partial_z G_z^h|_{1,1} \\
&\leq Ch^2 |u|_{3,\infty} \log \frac{1}{h},
\end{aligned}$$

where we have used (2.45). This completes the proof. ∎

The logarithmic factor can be removed in the $W^{0,\infty}(\Omega)$ case.

Theorem 2.6.

$$\|R_h u - u_h\|_{0,\infty} \leq Ch^2 \|u\|_{3,p} \qquad p > 2,$$

provided that the solution has the indicated smoothness.

Proof. Using Lemma 2.1, we deduce

$$\begin{aligned}|(R_h u - u_h)(z)| &= |a(R_h u - u_h, G_z^h)| \\ &= |a(u - u_h, G_z^h) - a^*(u - u_h, \Pi_h^* G_z^h)| \\ &\leq Ch^2[h^{-1}|u - u_h|_{1,p} + |u|_{3,p}]|G_z^h|_{1,p'} \\ &\leq Ch^2\|u\|_{3,p},\end{aligned}$$

where we have used (2.46). ∎

Remark 2.3. It should be noted that for smooth domains and smooth data, the assumption u $\in W^{3,p}$, $2 \leq p \leq \infty$, that appeared in the last few theorems is not stringent at all. For the polygonal domain case, the assumption is of course unrealistic. However, one can always replace the $\|u\|_{3,p}$ term by a lower order term times a logarithmic factor using the techniques in the proof of (2.49).

3. ESTIMATES FOR PARABOLIC PROBLEMS

By now one should be clear about the role of the two parameters $r_{\max} = \gamma_{\max}$ in the assumptions **R1, R2** in the statements of various theorems. Thus in the remainder of this article, we will simply deal with rectangular or smooth domains. Furthermore, we assume that $a_{ij} \in C(\bar{\Omega})$ and that the solution have required smoothness. For ease of exposition, these conditions will not be stated explicitly in the theorems below.

Consider the parabolic problem

$$u_t + \mathscr{L}u = f(x, t), \qquad (x, t) \in \Omega \times (0, T], \tag{3.1}$$

$$u = 0, \qquad (x, t) \in \partial\Omega \times [0, T], \tag{3.2}$$

$$u = u_0(x), \qquad t = 0, x \in \Omega, \tag{3.3}$$

where $\mathscr{L}u = -\nabla \cdot (A\nabla u)$ as in (1.1) and $u_t =: \partial u/\partial t$. Then

$$(u_t, v_h) + a^*(u, v_h) = (f, v_h), \qquad v_h \in V_h. \tag{3.4}$$

The approximation problem is then to find $u_h(t) : [0, T] \to U_h$ such that

$$(u_{h,t}, v_h) + a^*(u_h, v_h) = (f, v_h), \qquad v_h \in V_h, \qquad 0 < t \leq T, \tag{3.5}$$

$$u_h(0) = R_h^* u_0, \tag{3.6}$$

where $R_h^*: H_0^1(\Omega) \to U_h$ is the generalized elliptic projection operator defined by

$$a^*(R_h^* w - w,\ v_h) = 0, \qquad v \in V_h. \tag{3.7}$$

Theorem 3.1. *For h sufficiently small*

$$\|u_h - R_h^* u\|_1 \le Ch^2 \left(\int_0^t \|u_t\|_{3,r}^2 d\tau \right)^{1/2}, \qquad 0 \le t \le T,\ r > 1. \tag{3.8}$$

Proof. Let $\xi = u_h - R_h^* u$, $\eta = R_h^* u - u$, so that

$$u_h - u = \xi + \eta. \tag{3.9}$$

By (3.4)–(3.7), we derive the error equation

$$(\xi_t,\ v_h) + a^*(\xi,\ v_h) = -(\eta_t,\ v_h), \qquad v_h \in V_h. \tag{3.10}$$

Taking $v_h = \Pi_h^* \xi_t$

$$|||\, \xi_t \,|||_0^2 + \frac{1}{2}\frac{d}{dt} a^*(\xi,\ \Pi_h^* \xi) = -(\eta_t, \Pi_h^* \xi_t) + \frac{1}{2}[a^*(\xi_t, \Pi_h^* \xi) - a^*(\xi, \Pi_h^* \xi_t)], \tag{3.11}$$

where $|||\, \xi_t \,|||_0^2 = (\xi_t, \Pi_h^* \xi_t)$. Recall that

$$(w_h,\ \Pi_h^* \bar{w}_h) = (\bar{w}_h,\ \Pi_h^* w_h) \qquad \forall w_h, \bar{w}_h \in U_h,$$

and that $|||\cdot|||_0$ is an equivalent norm to the usual $|||\cdot|||_0$ norm on U_h (cf. Lemma 2.2 of Chou and Li [9]). In addition, (cf. Lemma 2.4 of [5])

$$|a^*(w_h,\ \Pi_h^* T_h) - a^*(T_h,\ \Pi_h^* w_h)| \le Ch\|w_h\|_1 \|T_h\|_1 \qquad \forall w_h, T_h \in U_h.$$

Combining these with an inverse inequality, we derive

$$|a^*(\xi_t,\ \Pi_h^* \xi) - a^*(\xi,\ \Pi_h^* \xi_t)| \le Ch\|\xi_t\|_1\|\xi\|_1 \le C\|\xi_t\|_0\|\xi\|_1 \le C\|\xi\|_1^2 + \epsilon\|\xi_t\|_0^2,$$

where we have used the ϵ-inequality: $ab \le \epsilon a^2 + \frac{1}{4\epsilon} b^2$, $a > 0$, $b > 0$. Also

$$|(\eta_t,\ \Pi_h^* \xi_t)| \le C\|\eta_t\|_0^2 + \epsilon\|\xi_t\|_0^2.$$

Taking ϵ small enough to absorb the ξ_t term into the left side of (3.11), we have

$$\frac{d}{dt} a^*(\xi,\ \Pi_h^* \xi) \le C(\|\eta_t\|_0^2 + \|\xi\|_1^2). \tag{3.12}$$

Integrating and noting $\xi(0) = 0$

$$\alpha\|\xi\|_1^2 \leq a^*(\xi, \Pi^*{}_h\xi) \leq C\int_0^t (\|\eta_t\|_0^2 + \|\xi\|_1^2)d\tau. \tag{3.13}$$

Then Gronwall's inequality and (2.35) imply that

$$\|\xi\|_1^2 \leq C\int_0^t \|\eta_t\|_0^2 d\tau \leq Ch^4\int_0^t \|u_t\|_{3,r}^2 d\tau. \tag{3.14}$$

This completes the proof. ∎

Theorem 3.2. *Let $r > 1$. For h sufficiently small,*

$$\|u_h - u\|_{0,p} \leq Ch^2\left[\|u\|_{3,q} + \left(\int_0^t \|u_t\|_{3,r}^2 d\tau\right)^{1/2}\right], \qquad 2 \leq p < \infty, \tag{3.15}$$

where $q > 1$, if $p = 2$, and $q = 2p/(p + 2)$ if $p > 2$.

Proof. By an imbedding theorem

$$\|\xi\|_{0,p} \leq C\|\xi\|_1. \tag{3.16}$$

Combining (3.14) and (2.35) with (3.9) completes the proof. ∎

Theorem 3.3. *For h sufficiently small,*

$$\|u - u_h\|_{1,p} \leq Ch\left[\|u\|_{2,p} + \|u_t(0)\|_2 + \|u_t\|_2 + \int_0^t \|u_{tt}\|_2 d\tau\right], 2 \leq p < \infty, \qquad 0 < t < T. \tag{3.17}$$

Proof. By (3.4) and (3.5)

$$a^*(u - u_h, v_h) = -((u - u_h)_t, v_h), \qquad v_h \in V_h. \tag{3.18}$$

Let $\boldsymbol{\Psi}$ be as in (2.29). Then

$$\begin{aligned}((u - u_h)_x, \psi) &= a(u - u_h, \boldsymbol{\Psi})\\ &= a(u - u_h, \boldsymbol{\Psi} - R_h\boldsymbol{\Psi}) + [a(u - u_h, R_h\boldsymbol{\Psi})\\ &\quad -a^*(u - u_h, \Pi_h^*R_h\boldsymbol{\Psi})] - ((u - u_h)_t, \Pi_h^*R_h\boldsymbol{\Psi}) \qquad \text{(by (3.18))}\\ &:= R_1 + R_2 + R_3. \end{aligned} \tag{3.19}$$

Let us estimate R_1 first.

$$|R_1| = |a(u - R_h u, \mathbf{\Psi} - R_h \mathbf{\Psi})| = |a(u - R_h u, \mathbf{\Psi})| \le C\|u - R_h u\|_{1,p}\|\mathbf{\Psi}\|_{1,p'} \le Ch\|u\|_{2,p}\|\mathbf{\Psi}\|_{1,p'}. \tag{3.20}$$

By (2.1)

$$|R_2| \le Ch[|u - u_h|_{1,p} + |u|_{2,p}]|\mathbf{\Psi}|_{1,p'}. \tag{3.21}$$

Finally, R_3 is estimated as follows. Differentiating (3.10) and setting $v_h = \Pi_h^* \xi_t$, we have

$$(\xi_{tt}, \Pi_h^* \xi_t) + a^*(\xi_t, \Pi_h^* \xi_t) = -(\eta_{tt}, \Pi_h^* \xi_t).$$

For brevity, write $\|\cdot\|_0$ as $\|\cdot\|$. Hence

$$\frac{1}{2}\frac{d}{dt} ||| \xi_t |||_0^2 \le \|\eta_{tt}\| \, \|\Pi_h^* \xi_t\|.$$

To handle the possibility of non-differentiability at $\xi_t = 0$, we rewrite it as

$$\frac{1}{2}\frac{d}{dt}(||| \xi_t |||_0^2 + \epsilon^2) \le \|\eta_{tt}\| \, \|\Pi_h^* \xi_t\|, \qquad \epsilon > 0,$$

or

$$(||| \xi_t |||_0^2 + \epsilon^2)^{1/2} \frac{d}{dt}(||| \xi_t |||_0^2 + \epsilon^2)^{1/2} \le \|\eta_{tt}\| \, \|\Pi_h^* \xi_t\|.$$

Now using the equivalence of $||| \cdot |||_0$ and $\|\cdot\|$, the fact $||| \xi_t |||_0 \le (||| \xi_t |||_0^2 + \epsilon^2)^{1/2}$, integrating and letting ϵ tend to zero, we get

$$\|\xi_t\| \le C\|\xi_t(0)\| + C\int_0^t \|\eta_{tt}\| d\tau.$$

Setting $t = 0$ and $v_h = \Pi_h^* \xi_t(0)$ in (3.10), one has $\|\xi_t(0)\| \le C\|\eta_t(0)\|$. Thus

$$\|\xi_t\| \le C\left[\|\eta_t(0)\| + \int_0^t \|\eta_{tt}\| d\tau\right] \le Ch\left[\|u_t(0)\|_2 + \int_0^t \|u_{tt}\|_2 d\tau\right], \tag{3.22}$$

where we have used Theorem 2.1. (Similarly, using Theorem 2.2 one can also derive

$$\|\xi_t\| \le Ch^2\left[\|u_t(0)\|_{3,r} + \int_0^t \|u_{tt}\|_{3,r} d\tau\right], \ r > 1, \tag{3.23}$$

which will not be used in this proof, but later in other ones.) Thus by (3.22) and (2.27)

$$|R_3| \le (\|\xi_t\| + \|\eta_t\|)\|\mathbf{\Psi}\|_0 \le Ch\left[\|u_t\|_2 + \|u_t(0)\|_2 + \int_0^t \|u_{tt}\|_2 d\tau\right]\|\mathbf{\Psi}\|_{1,p'}, \tag{3.24}$$

where we have used the fact $\|\mathbf{\Psi}\|_0 \le C\|\mathbf{\Psi}\|_{1,r}$, $r > 1$. Combining (3.20), (3.21), and (3.24) with (3.19), we have

$$|((u - u_h)_x, \psi)| \le Ch\left[|u - u_h|_{1,p} + \|u\|_{2,p} + \|u_t(0)\|_2 + \|u_t\|_2 + \int_0^t \|u_{tt}\|_2 d\tau\right]\|\mathbf{\Psi}\|_{1,p'}.$$

Then by (2.30) we have for sufficiently small h that

$$\|u - u_h\|_{1,p} \le Ch\|u - u_h\|_{1,p} + Ch\left[\|u\|_{2,p} + \|u_t(0)\|_2 + \|u_t\|_2 + \int_0^t \|u_{tt}\|_2 d\tau\right].$$

The proof is complete. ∎

Remark 3.1. The quantity $\|u_t(0)\|_2$ on the right side of (3.17) is treated as data, since we can use (3.1) with smooth initial function in $H^1(\Omega)$.

Theorem 3.4. *For h sufficiently small*,

$$\|u - u_h\|_{1,\infty} \le Ch\left[\|u\|_{2,\infty} + \|u\|_3 + \left(\int_0^t \|u_t\|_{3,q}^2 d\tau\right)^{1/2}\right],\ q > 1, \tag{3.25}$$

$$\|u - u_h\|_{0,\infty} \le Ch^2 \log\frac{1}{h} \cdot \left[\|u\|_{2,\infty} + \|u\|_3 + \left(\int_0^t \|u_t\|_{3,q}^2 d\tau\right)^{1/2}\right],\ q > 1. \tag{3.26}$$

Proof. Let $\xi = u_h - R_h^* u$, $\eta = R_h^* u - u$. By an inverse property and (3.8)

$$\|\xi\|_{1,\infty} \le Ch^{-1}\|\xi\|_1 \le Ch\left(\int_0^t \|u_t\|_{3,q}^2 d\tau\right)^{1/2}. \tag{3.27}$$

By (2.47)

$$\|\eta\|_{1,\infty} \le Ch[\|u\|_{2,\infty} + \|u\|_3], \tag{3.28}$$

which along with (3.27) derives (3.25). On the other hand, by using the asymptotic Sobolev inequality [4] and (3.8)

$$\|\xi\|_{0,\infty} \leq C\left(\log\frac{1}{h}\right)^{1/2}\|\nabla\xi\|_0 \leq Ch^2\left(\log\frac{1}{h}\right)^{1/2}\left(\int_0^t \|u_t\|_{3,q}^2 d\tau\right)^{1/2}. \tag{3.29}$$

By (2.48)

$$\|\eta\|_{0,\infty} \leq Ch^2 \log\frac{1}{h}[\|u\|_{2,\infty} + \|u\|_3], \tag{3.30}$$

which with (3.29) gives (3.26). ■

Theorem 3.5. *For h sufficiently small*

$$\|R_h u - u_h\|_{1,p} \leq Ch^2\left[\|u\|_{3,p} + \|u_t(0)\|_{3,r} + \int_0^t \|u_{tt}\|_{3,r} d\tau\right], 2 \leq p < \infty,\ r > 1. \tag{3.31}$$

Proof. By (2.29), (1.20), and (3.18),

$$\begin{aligned}((R_h u - u_h)_x, \psi) &= a(R_h u - u_h, \mathbf{\Psi} - R_h\mathbf{\Psi}) + a(R_h u - u_h, R_h\mathbf{\Psi}) \\ &= [a(u - u_h, R_h\mathbf{\Psi}) - a^*(u - u_h, \Pi_h^* R_h\mathbf{\Psi})] - ((u - u_h)_t, \Pi_h^* R_h\mathbf{\Psi}) := Q_1 + Q_2.\end{aligned} \tag{3.32}$$

On the other hand,

$$\begin{aligned}|Q_1| &\leq Ch^2[h^{-1}|u - u_h|_{1,p} + |u|_{3,p}]|\mathbf{\Psi}|_{1,p'} \\ &\leq Ch^2\left[\|u\|_{3,p} + \|u_t(0)\|_2 + \|u_t\|_2 + \int_0^t \|u_{tt}\|_2 d\tau\right)\|\mathbf{\Psi}\|_{1,p'}, \qquad \text{(by (3.17))}\end{aligned} \tag{3.33}$$

and by (3.23) and (2.36)

$$|Q_2| \leq C(\|\xi_t\| + \|\eta_t\|)|\mathbf{\Psi}\|_0 \leq Ch^2\left[\|u_t(0)\|_{3,r} + \|u_t\|_{3,r} + \int_0^t \|u_{tt3,r} d\tau\right)\|\mathbf{\Psi}\|_{1,p'}. \tag{3.34}$$

Noticing $\|g\|_2 \leq C\|g\|_{3,r}$, $r \geq 1$ and applying the duality completes the proof. ■

Theorem 3.6. *For h sufficiently small*,

$$\begin{aligned}\|R_h u - u_h\|_{1,\infty} \leq Ch^2\left[\|u\|_{2,\infty} + \|u\|_3 + \left(\int_0^t \|u_t\|_{3,r}^2 d\tau\right)^{1/2}\right]\log\frac{1}{h} \\ + Ch^2\left[\|u_t(0)\|_{3,r} + \int_0^t \|u_{tt}\|_{3,r} d\tau\right]\left(\log\frac{1}{h}\right)^{1/2}, r > 1.\end{aligned}$$

Proof. Since

$$\begin{aligned}\partial_z(R_h u - u_h)(z) &= a(R_h u - u_h, \partial_z G_z^h) \\ &= [a(u - u_h, \partial_z G_z^h) - a^*(u - u_h, \Pi_h^* \partial_z G_z^h)] - (\xi_t + \eta_t, \Pi_h^* \partial_z G_z^h),\end{aligned}$$

we have

$$|\partial_z(R_h u - u_h)(z)| \leq Ch(|u - u_h|_{1,\infty} + |u|_{2,\infty})|\partial_z G_z^h|_{1,1} + C(\|\xi_t\| + \|\eta_t\|)\|\partial_z G_z^h\|.$$

Recalling that

$$|\partial_z G_z^h|_{1,1} + \|\partial_z G_z^h\|^2 \leq C \log \frac{1}{h}$$

and using (3.23) as in (3.34) complete the proof. ■

Once again, we compare the covolume solution with the Galerkin finite element solution and demonstrate a second-order convergence.

Theorem 3.7. *Let $\tilde{u}_h$ be the finite element solution to* (3.1)–(3.3), *i.e.*,

$$(\tilde{u}_{h,t}, v) + a(\tilde{u}_h, v) = (f, v), \qquad v \in U_h, \tag{3.35}$$

$$\tilde{u}_h(\cdot, 0) = R_h u_0. \tag{3.36}$$

Then for $p > 2$ we have for h sufficiently small that

$$\|\tilde{u}_h - u_h\|_{1,p} \leq Ch^2\left[\|u\|_{3,p} + \|u_t(0)\|_{3,r} + \|u_t\|_{3,r} + \int_0^t \|u_{tt}\|_{3,r} d\tau\right], \qquad r > 1.$$

Proof. By (3.1) and (3.35),

$$((\tilde{u}_h - u)_t, v) + a(\tilde{u}_h - u, v) = 0, \qquad v \in U_h. \tag{3.37}$$

As in (2.29),

$$\begin{aligned}((\tilde{u}_h - u_h)_x, \psi) &= a(\tilde{u}_h - u_h, \mathbf{\Psi}) \\ &= a(\tilde{u}_h - u_h, \mathbf{\Psi} - R_h\mathbf{\Psi}) + a(u - u_h, R_h\mathbf{\Psi}) - a^*(u - u_h, \Pi_h^* R_h \mathbf{\Psi}) \\ &\quad - ((u - u_h)_t, \Pi_h^* R_h \mathbf{\Psi}) + a(\tilde{u}_h - u, R_h \mathbf{\Psi}) \quad \text{(by (1.20), (3.18))} \\ &= [a(u - u_h, R_h\mathbf{\Psi}) - a^*(u - u_h, \Pi_h^* R_h \mathbf{\Psi})] - ((u - u_h)_t, \Pi_h^* R_h \mathbf{\Psi}) \\ &\quad - ((\tilde{u}_h - u)_t, R_h\mathbf{\Psi}) \quad \text{(by (1.20), (3.37))} = Q_1 + Q_2 - ((\tilde{u}_h - u)_t, R_h\mathbf{\Psi}),\end{aligned}$$

where Q_1 and Q_2 are as in (3.32). We know from the known finite element estimate that

$$\begin{aligned}\|(\tilde{u}_h - u)_t\|_0 &\le C\left[h^2(\|u_t\|_2 + \|u_t(0)\|_2) + \int_0^t \|(u - R_h u)_{tt}\|_0 d\tau\right]\\ &\le Ch^2\left[\|u_t\|_2 + \|u_t(0)\|_2 + \int_0^t \|u_{tt}\|_2 d\tau\right]. \qquad (3.38)\end{aligned}$$

Combining (3.38), (3.33), and (3.34) completes the proof. ■

Theorem 3.8. *Let $r > 1$. Then for sufficiently small h,*

$$\|\tilde{u}_h - u_h\|_{1,\infty} \le Ch^2 \log\frac{1}{h}\left[\|u\|_{3,\infty} + \|u_t(0)\|_{3,r} + \|u_t\|_{3,r} + \int_0^t \|u_{tt}\|_{3,r} d\tau\right].$$

Proof. By (2.2) and the estimates similar to (3.24) and (3.34), we have

$$\begin{aligned}|\partial_z(\tilde{u}_h - u_h)| &= |a(\tilde{u}_h - u_h, \partial_z G_z^h)| = |[a(u - u_h, \partial_z G_z^h) - a^*(u - u_h, \Pi_h^*\partial_z G_z^h)]\\ &\quad - ((u - u_h)_t, \Pi_h^*\partial_z G_z^h) - ((\tilde{u}_h - u)_t, \partial_z G_z^h)|\\ &\le Ch^2[h^{-1}|u - u_h|_{1,\infty} + |u|_{3,\infty}] \cdot \|\partial_z G_z^h\|_{1,1}\\ &\quad + Ch^2\left[\|u_t(0)\|_{3,r} + \|u_t\|_{3,r} + \int_0^t \|u_{tt}\|_{3,r} d\tau\right] \cdot \|\partial_z G_z^h\|_{1,1}\\ &\quad + Ch^2\left[\|u_t(0)\|_2 + \|u_t\|_2 + \int_0^t \|u_{tt}\|_2 d\tau\right] \cdot \|\partial_z G_z^h\|_{1,1}.\end{aligned}$$

This completes the proof. ■

The above theorem corresponds to Theorem 2.5. As in Theorem 2.6, the logarithmic factor can be removed in the $W^{0,\infty}(\Omega)$ case and one obtains the following.

Theorem 3.9. *Let $r > 1$. Then for sufficiently small h,*

$$\|\tilde{u}_h - u_h\|_{0,\infty} \le Ch^2\left[\|u\|_{3,\infty} + \|u_t(0)\|_{3,r} + \|u_t\|_{3,r} + \int_0^t \|u_{tt}\|_{3,r} d\tau\right].$$

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