Multigrid Algorithm for the Cell-Centered Finite Difference Method II: Discontinuous Coefficient Case

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Received 28 November 2002; accepted 9 December 2003

DOI 10.1002/num.20001

We consider a multigrid algorithm for the cell centered finite difference scheme with a prolongation operator depending on the diffusion coefficient. This prolongation operator is designed mainly for solving diffusion equations with strong varying or discontinuous coefficient and it reduces to the usual bilinear interpolation for Laplace equation. For simple interface problem, we show that the energy norm of this operator is uniformly bounded by 11/8, no matter how large the jump is, from which one can prove that *W*-cycle with one smoothing converges with reduction factor independent of the size of jump using the theory developed by Bramble et al. (Math Comp 56 (1991), 1–34). For general interface problem, we show that the energy norm is bounded by some constant C_* (independent of the jumps of the coefficient). In this case, we can conclude *W*-cycle converges with sufficiently many smoothings. Numerical experiment shows that even *V*-cycle multigrid algorithm with our prolongation works well for various interface problems. © 2004 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 20: 000–000, 2004

Keywords: discontinuous coefficient; cell-centered finite difference methods; finite volume methods; multigrid methods

I. INTRODUCTION

Elliptic differential equations with discontinuous coefficients arise from many areas of applications and are difficult to handle, numerically or analytically. Of many such problems, a few examples are flows through porous media with different porosity, electric currents through material of different conductivities and heat flows through heterogeneous materials, etc. [1, 2]. The numerical methods treating such problems are important areas of research. Among them, the cell-centered finite difference (CCFD) is a finite volume type of method and has been used by many engineers because of its simplicity and local conservation. It also arises from the saddle

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Contract grant sponsor: KOSEF; contract grant number: R01-2000-000-00009-0 (to D.Y.K.)
Contract grant sponsor: NCTS of Taiwan (to D.Y.K.)

point formulation of the Raviart-Thomas mixed method by taking certain quadrature. On the other hand, the multigrid algorithms have proven to be very effective for a large class of problems [3, 4] and have been the subject of extensive research [5–10], whose convergence analysis was carried out mostly for finite element methods [5, 7–9].

In this article, we consider multigrid algorithm for the cell centered finite difference (CCFD) scheme for problems where the coefficient is discontinuous. The multigrid algorithm for CCFD was considered by a few authors [6, 10–13] and its W-cycle convergence for Laplace equation was first proved by Bramble et al. [6]. However, their use of natural injection as a prolongation operator is not an optimal choice; the V-cycle convergence is slow. As is shown by Kwak [10], certain weighted prolongation works much better and guarantees V-cycle convergence. Still, it is restricted to problems with smooth coefficients. It does not work well when the jump of the diffusion coefficient is severe. In early years, there have been some efforts to handle discontinuous coefficient problem using Galerkin coarse grid approximation [12, 13]. In the finite difference case, Alcouffe et al. [14] and Kettler [15] suggested a prolongation operator based on the continuity of flux at finer grid points. However, in this method stiffness matrices of the coarse grids no longer have 5-point structure and the prolongation is nontrivial. Hence extra cost is needed to generate stiffness matrices and its implementation is difficult. There is an effort to overcome this difficulty. Liu et al. [16] generated coarse grids matrices using flux continuity, thus preserving 5-point stencil. Instead, they designed a new prolongation operator depending on the jump of the diffusion coefficient. Their numerical result shows that $V(2, 2)$-cycle is a good reducer but there is neither any report for $V(1, 1)$-cycle nor any kind of convergence proof. In this article, we introduce a new multigrid algorithm with a prolongation depending on the diffusion coefficient. We use standard 5-point stencil for every level so that the implementation is straightforward. We first consider a prolongation based on the bilinear interpolation and show the V-cycle convergence for model problems. We then modify it using the coefficients as weights for the problems with nonsmooth coefficients. The motivation lies in physical meaning: the flux continuity. As for the behavior of multigrid algorithm, it seems that smaller energy norm of a prolongation yields better convergence behavior. We estimate the energy norm of our prolongation operator and show it is bounded by 11/8 at least for a simple discontinuity. The other ingredient in showing multigrid convergence is "regularity and approximation property," which is not known to hold for problems with discontinuous coefficient. However, we can use this property of a nearby smooth problem and prove W-cycle convergence with one smoothing (see Section 2). For more general discontinuities, we show the operator norm is bounded by a constant, from which one can deduce W-cycle convergence with sufficiently many smoothings. Numerical experiment shows that even V-cycle multigrid with 1 smoothing works very well for problems with discontinuous coefficients.

The rest of this article is as follows. In Section 2, we briefly describe CCFD for elliptic problem and multigrid algorithm convergence together with some general theory. In Section 3, we introduce a prolongation operator based on the bilinear interpolation and show the energy norm is bounded by 1 for the constant diffusion case, thus show the V-cycle convergence. We modify it to handle the discontinuous coefficients. It is designed so that the resulting function has continuous flux at grid cell points in some sense. We estimate the energy norm and show W-cycle convergence of multigrid algorithms for problems with junction discontinuity. Finally, we report some numerical results in Section 4. For some problems, V-cycle with one smoothing converges while for some other problems, W-cycle is required. Variable V-cycle works well for all our test problems.

II. MULTIGRID ALGORITHM FOR THE CELL-CENTERED METHOD

In this section, we briefly describe CCFD and multigrid algorithm to solve the resulting system of linear equations. We first consider the following model problem:

$$-\nabla \cdot p\nabla \tilde{u} = f \qquad \text{in } \Omega, \tag{2.1}$$

$$\tilde{u} = 0 \qquad \text{on } \partial\Omega, \tag{2.2}$$

where Ω is the unit square. We assume that $\tilde{u} \in H^{1+\alpha}(\Omega)$ for some $0 < \alpha \leq 1$. For $k = 1, 2, \ldots, J$, divide Ω into $n \times n$ axis-parallel subsquares, where $n = 2^k$. Such triangulations are denoted by $\{\mathscr{E}_k\}$. Each subsquare in $\mathscr{E}_k$ is called a cell and denoted by E^k_{ij}, $i, j = 1, \ldots, 2^k$ and has $u_{i,j}$ as its value at center. For $k = 1, 2, \ldots, J$, let V_k denote the space of functions that are piecewise constant on each cell. Integrating equation (2.1) by parts on each cell and replacing the normal derivative $\partial\tilde{u}/\partial n$ on the edges by difference quotient of function u in V_k, we have the finite difference equations as follows:

$$-p_{i-1/2,j}(u_{i-1,j} - u_{i,j}) - p_{i+1/2,j}(u_{i+1,j} - u_{i,j})$$

$$-p_{i,j-1/2}(u_{i,j-1} - u_{i,j}) - p_{i,j+1/2}(u_{i,j+1} - u_{i,j}) = f_{i,j}h^2,$$

where $h = h_k = 1/2^k$ and $p_{i-1/2,j} = p(x_{i-1/2}, y_j)$, etc. When p is discontinuous along the interface, we take $p_{i-1/2,j}$ as

$$p_{i-1/2,j} = \frac{2p_{i-1,j}p_{i,j}}{p_{i-1,j} + p_{i,j}}. \tag{2.3}$$

Similarly, $p_{i+1/2,j}$, $p_{i,j-1/2}$ and $p_{i,j+1/2}$ are defined. When one of the edge coincides with the boundary of Ω, we assume a fictitious value by reflection. For example, $u_{0,j}$ is taken as $-u_{1,j}$ and thus $\partial\tilde{u}/\partial n$ at $x = 0$ is approximated by $-2u_{1,j}/h$. Similar rules apply to the other parts of the boundary of Ω. After dividing the resulting equation by h^2, we obtain a system of linear equation of the form

$$A_k u = f, \tag{2.4}$$

where A_k is the typical sparse, $n^2 \times n^2$ symmetric, positive definite matrix similar to those arising in the vertex finite difference method and u is the vector whose entries are $u_{i,j}$ and f is the vector whose entries are $f(x_i, y_j)$. We identify the vector u, v in V_k with their matrix representation in R^{2^k}. For analysis, we define a quadratic form $A_k(\cdot, \cdot)$ on $V_k \times V_k$ in an obvious manner by

$$A_k(u, v) = h_k^2 \sum_{i,j} (A_k u)_{i,j} v_{i,j}, \qquad \forall u, v \in V_k. \tag{2.5}$$

Then (2.4) is equivalent to the following problem: Find $u \in V_k$ satisfying

$$A_k(u, \phi) = (f, \phi), \qquad \forall \phi \in V_k, \tag{2.6}$$

where $(\cdot, \cdot)$ is the L^2 inner product. The error analysis of the cell centered finite difference method is well known (cf. [17–19]). Let $Q_k : L^2(\Omega) \to V_k$ denote the usual $L^2(\Omega)$ projection. If u is the solution of (2.6), then

$$A_k(u - Q_k\tilde{u}, u - Q_k\tilde{u}) \le Ch_k^2\|f\|^2,$$

where $\|\cdot\|$ is the usual L^2 norm. Given a certain prolongation operator $I_{k-1}^k : V_{k-1} \to V_k$, we define the restriction operator $I_k^{k-1} : V_k \to V_{k-1}$ as its adjoint with respect to $(\cdot, \cdot)_k$:

$$(I_k^{k-1}u, v) = (u, I_{k-1}^k v), \qquad \forall u \in V_k, \qquad \forall v \in V_{k-1}.$$

Because the space V_k can be viewed as the space of vectors having entries $u_{i,j}$, we also use I_{k-1}^k and I_k^{k-1} to denote their matrix representations. Now multigrid algorithm for solving (2.4) is defined as follows.

Multigrid Algorithm

Set $B_0 = A_0^{-1}$. For $1 < k \le J$, assume that B_{k-1} has been defined and define $B_k f$ for $f \in V_k$ as follows:

1. Set $x^0 = 0$ and $q^0 = 0$.
2. Define x^l for $l = 1, \ldots, m(k)$ by
$$x^l = x^{l-1} + R_k^{(l+m(k))}(f - A_k x^{l-1}).$$
3. Define $y^{m(k)} = x^{m(k)} + I_{k-1}^k q^s$, where q^i for $i = 1, \ldots, s$ is defined by
$$q^i = q^{i-1} + B_{k-1}[P_{k-1}^0(f - A_k x^{m(k)}) - A_{k-1}q^{i-1}],$$
where $P_{k-1}^0 : V_k \to V_{k-1}$ is the L^2-projection.
4. Define y^l for $l = m(k) + 1, \ldots, 2m(k)$ by
$$y^l = y^{l-1} + R_k^{(l+m(k))}(f - A_k y^{l-1}).$$
5. Set $B_k f = y^{2m(k)}$.

If $s = 1$, we obtain $V(m(k), m(k))$-cycle and if $s = 2$, we obtain $W(m(k), m(k))$-cycle. If $m(k)$ varies with k, we call it variable V-cycle. Here $R_k^{(l)}$ is a smoother on V_k, which alternates between R_k and its adjoint R_k^t, and $m(k)$ is the number of smoothings that can vary depending on k. The smoother R_k, as usual, can be taken as the Jacobi or Gauss-Seidel relaxation. There are several approaches in showing the convergence of multigrid algorithms. In this article, we will adopt the framework in [20] and briefly describe it for our purpose. According to this framework, there are two main conditions for multigrid convergence. The first one is for the operator I_{k-1}^k: There exists a constant $C_* > 0$ such that

$$A_k(I_{k-1}^k v, I_{k-1}^k v) \le C_* A_{k-1}(v, v), \qquad \forall v \in V_{k-1}. \tag{2.7}$$

The second one is a "regularity and approximation property" assumption of the form

$$A_k((I - I_{k-1}^k P_{k-1})u, u) \leq C_\alpha \left(\frac{\|A_k u\|^2}{\lambda_k} \right)^\alpha A_k(u, u)^{1-\alpha}, \qquad \forall u \in V_k. \tag{2.8}$$

Here, λ_k is the largest eigenvalue of A_k and P_{k-1} is the elliptic projection defined by

$$A_{k-1}(P_{k-1}u, v) = A_k(u, I_{k-1}^k v), \qquad \forall u \in V_k, v \in V_{k-1}. \tag{2.9}$$

Under these conditions, we have the following results from Theorems 2, 8, 6, and of [20]. Let $E_k = I - B_k A_k$ in multigrid algorithm.

Theorem 2.1. *Let $s = 1$(V-cycle). Then there is $0 < \delta_k < 1$ such that*

(i) when $C_* \leq 1$ in (2.7), we have with $m(k) = m$

$$A_k(E_k u, E_k u) \leq \delta_k A_k(u, u), \qquad \forall\ u \in V_k, \tag{2.10}$$

with

$$\delta_k = \frac{Mk^{(1-\alpha)/\alpha}}{Mk^{(1-\alpha)/\alpha} + m^\alpha},$$

where M is a constant depending on C_* and C_α but independent of p and k;

(ii) when $C_* \leq 1 + Ch$, then V-cycle multigrid with $m(k) = m$ is a good preconditioner in the sense that

$$\eta_0 A_k(u, u) \leq A_k(B_k A_k u, u) \leq \eta_1 A_k(u, u), \qquad \forall\ u \in V_k, \tag{2.11}$$

where η_1 is independent of k and $\eta_0 \leq 1 - \delta_k$;

(iii) for general constant C_*, the variable V-cycle multigrid with $\beta_0 m(k) \leq m(k-1) \leq \beta_1 m(k)$ satisfies (2.11) with

$$\eta_0 \geq \frac{m(k)^\alpha}{M + m(k)^\alpha} \text{ and } \eta_1 \leq \frac{M + m(k)^\alpha}{m(k)^\alpha}.$$

Here, β_0 and β_1 are constants which are greater than one and independent of k.

Theorem 2.2. *Let $s = 2$(W-cycle) and C_* is constant (independent of k). Then, for sufficiently large m, (2.10) holds with $\delta_k = M/(M + m^\alpha)$. Moreover, if $C_* \leq 2$, then W-cycle converges with $m = 1$.*

III. *W*-CYCLE CONVERGENCE

We first consider bilinear prolongation for a smooth problem and show *V*-cycle converges for Laplace problem and is a good preconditioner for problems with smooth coefficient. In Section 3.2, we modify the prolongation using the coefficient as weight and consider a simple discontinuity. We show the convergence of *W*-cycle with one smoothing by estimating its energy norm and proving “regularity and approximation property.” In Section 3.3, we deal with more general discontinuous problem, where one or more junction points exist.

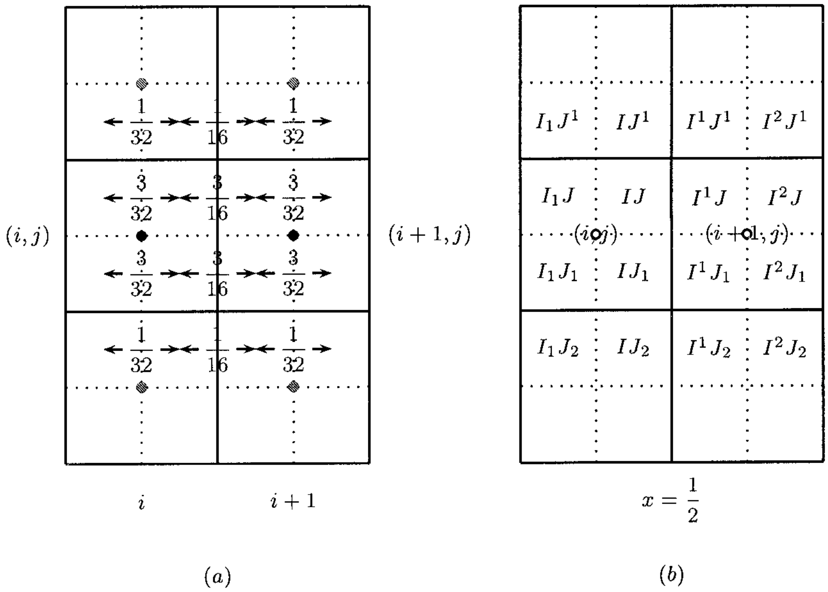

FIG. 1. Bilinear contributions and indices ($I_1 = I - 1$, $J^1 = J + 1$).

A. Smooth Coefficient

We choose a prolongation based on the bilinear interpolation. Although well known, we inspect its energy norm in some detail because the same technique can be used in the nonsmooth case.

Let $E_{i,j}^{k-1}$ be a cell at level $k - 1$, and $v_{i,j}$ be the values at the center of each cells. Let $u_{I,J}$ be the value of upper right subcell of $E_{i,j}^{k-1}$ and u_{I,J_1}, $u_{I_1,J}$ and u_{I_1,J_1} be those of lower right, upper left, lower left subcells of $E_{i,j}^{k-1}$ [see Fig. 1(b)]. Here we use the notation $I^1 = I + 1$, $J_1 = J - 1$, etc. For $v \in V_{k-1}$, define $u = I_{k-1}^k v$ as follows:

$$
\begin{aligned}
u_{I,J} &= \frac{9v_{i,j} + 3v_{i,j+1} + 3v_{i+1,j} + v_{i+1,j+1}}{16}, \\
u_{I^1,J} &= \frac{9v_{i+1,j} + 3v_{i,j} + 3v_{i+1,j+1} + v_{i,j+1}}{16}, \\
u_{I,J^1} &= \frac{9v_{i,j+1} + 3v_{i,j} + 3v_{i+1,j+1} + v_{i+1,j}}{16}, \\
u_{I^1,J^1} &= \frac{9v_{i+1,j+1} + 3v_{i+1,j} + 3v_{i,j+1} + v_{i,j}}{16}.
\end{aligned} \tag{3.1}
$$

We show the energy norm of this prolongation is bounded by $1 + Ch$, $C = C(p)$ depending on p. It is easy to see that $A_{k-1}(v, v)$ is the square sum of differences of the function values between neighboring cells in $\mathscr{E}^{k-1}$. Writing out $A_{k-1}(v, v)$ and collecting the terms corresponding to the normal derivatives along the left walls and the top walls of each cell $E_{i,j}^{k-1}$, we see that

$$A_{k-1}(v, v) = \sum_{i,j}^{n} v_{i,j}[p_{i,j-1/2}(v_{i,j} - v_{i,j-1}) + p_{i-1/2,j}(v_{i,j} - v_{i-1,j})$$
$$+ p_{i+1/2,j}(v_{i,j} - v_{i+1,j}) + p_{i,j+1/2}(v_{i,j} - v_{i,j+1})]$$
$$= \sum_{i=2,j=1}^{n,n} p_{i-1/2,j}(v_{i,j} - v_{i-1,j})^2 + \sum_{i=1,j=1}^{n,n-1} p_{i,j+1/2}(v_{i,j+1} - v_{i,j})^2 + 4\sum_{l=1}^{n} p_{l,1/2}v_{l,1}^2 + 4\sum_{l=1}^{n} p_{l,n+1/2}v_{l,n}^2$$
$$+ 4\sum_{m=1}^{n} p_{1/2,m}v_{1,m}^2 + 4\sum_{m=1}^{n} p_{n+1/2,m}v_{n,m}^2, \quad (3.2)$$

where $n = 2^{k-1}$. Similarly, we have

$$A_k(u, u) = \sum_{I,J}^{2n} [p_{I_{1/2},J_1}(u_{I,J_1} - u_{I_1,J_1})^2 + p_{I_{1/2},J}(u_{I,J} - u_{I_1,J})^2 + p_{I^{1/2},J_1}(u_{I^1,J_1} - u_{I,J_1})^2$$
$$+ p_{I^{1/2},J}(u_{I^1,J} - u_{I,J})^2 + p_{I_1,J_{1/2}}(u_{I_1,J} - u_{I_1,J_1})^2 + p_{I,J_{1/2}}(u_{I,J} - u_{I,J_1})^2$$
$$+ p_{I_1,J^{1/2}}(u_{I_1,J^1} - u_{I_1,J})^2 + p_{I,J^{1/2}}(u_{I,J^1} - u_{I,J})^2] + \text{boundary terms.} \quad (3.3)$$

In the above equation, the first two terms come from the normal derivatives along the two left edges of E_{I,J_1}^k and $E_{I,J}^k$, whereas the second two come from the normal derivatives along two left edges of E_{I^1,J_1}^k and $E_{I^1,J}^k$. Similarly, the next four come by rotation. Substituting (3.1) into (3.3) we shall count the coefficient of $(v_{i+1,j} - v_{i,j})^2$. We can see such term will appear from 12 different terms of the form $(u_{I,J} - u_{I_1,J})^2$ as shown below [see Fig. 1(a)]. We shall use the following general Cauchy-Schwarz inequality throughout the estimate.

$$\left(\sum_{i=1}^{s} a_i x_i\right)^2 \le \left(\sum_{i=1}^{s} a_i\right)\left(\sum_{i=1}^{s} a_i x_i^2\right), \qquad a_i \ge 0,\ x_i \in \mathbb{R}. \quad (3.4)$$

By applying (3.4), we have

$$(u_{I,J} - u_{I_1,J})^2 = \left[\frac{3v_{i+1,j} - 3v_{i-1,j} + v_{i+1,j+1} - v_{i-1,j+1}}{16}\right]^2$$
$$= \frac{[3(v_{i+1,j} - v_{i,j}) + 3(v_{i,j} - v_{i-1,j}) + (v_{i+1,j+1} - v_{i,j+1}) + (v_{i,j+1} - v_{i-1,j+1})]^2}{256}$$
$$\le [3(v_{i+1,j} - v_{i,j})^2 + 3(v_{i,j} - v_{i-1,j})^2 + (v_{i+1,j+1} - v_{i,j+1})^2 + (v_{i,j+1} - v_{i-1,j+1})^2]/32. \quad (3.5)$$

Let us consider the term $(v_{i+1,j} - v_{i,j})^2$ at level $k - 1$. Its coefficient is 3/32. By shifting index $I \to I + 2(i \to i + 1)$, $J \to J - 2(j \to j - 1)$ and $(I, J) \to (I + 2, J - 2)((i, j) \to (i + 1, j - 1))$, the same term appears. Thus the sum of coefficients of the term $(v_{i+1,j} - v_{i,j})^2$ in (3.5), multiplied by the diffusion coefficient is

$$\frac{3}{32}(p_{I_{1/2},J} + p_{I^{3/2},J}) + \frac{1}{32}(p_{I_{1/2},J_2} + p_{I^{3/2},J_2}). \quad (3.6)$$

Similarly, we see

$$
\begin{aligned}
(u_{I^1,J} - u_{I,J})^2 &= [6(v_{i+1,j} - v_{i,j}) + 2(v_{i+1,j+1} - v_{i,j+1})]^2/256 \\
&\le \frac{3}{16}(v_{i+1,j} - v_{i,j})^2 + \frac{1}{16}(v_{i+1,j+1} - v_{i,j+1})^2. \qquad (3.7)
\end{aligned}
$$

From this, we get $\frac{3}{16}p_{I_{1/2},J}$ and by shifting $J \to J - 2$ we obtain $\frac{1}{16}p_{I_{1/2},J_2}$. By considering $(u_{I,J_1} - u_{I_1,J_1})^2$ and $(u_{I^1,J_1} - u_{I,J_1})^2$, we can see similar terms appear. Hence the coefficient of $(v_{i+1,j} - v_{i,j})^2$ in $A_k(u, u)$ is bounded by

$$
\begin{aligned}
&\frac{3}{32}(p_{I_{1/2},J} + p_{I^{3/2},J} + p_{I_{1/2},J_1} + p_{I^{3/2},J_1}) + \frac{1}{32}(p_{I_{1/2},J_2} + p_{I^{3/2},J_2} + p_{I_{1/2},J^1} + p_{I^{3/2},J^1}) \\
&\qquad + \frac{3}{16}(p_{I_{1/2},J} + p_{I_{1/2},J_1}) + \frac{1}{16}(p_{I_{1/2},J_2} + p_{I^{1/2},J^1}). \qquad (3.8)
\end{aligned}
$$

The locations of the nodes of these 12 contributions are depicted in Fig. 1(a). We see that when p is constant, the sum of the coefficients is p. Because this is true for every term, we see that the energy norm of I_{k-1}^k is bounded by 1. For Lipschitz continuous p, we can easily see the norm is bounded by $1 + Ch_k$, $C = C(p)$ depending on p. Thus we have proved.

Proposition 3.1. *We have*

$$
A_k(I_{k-1}^k v, I_{k-1}^k v) \le C_* A_{k-1}(v, v), \qquad \text{for } v \in V_{k-1}. \qquad (3.9)
$$

where $C_* = 1$ *if* p *is constant, and* $1 + C(p)h_k$ *if* p *is Lipschitz continuous.*

This result together with the regularity and approximation property, which can be shown exactly the same way as in [10], we can deduce from Theorem 2.1 that V-cycle multigrid algorithm converges when p is constant and is a good preconditioner when p is smooth.

B. Simple Discontinuity

It is well known that when the function p is discontinuous, the multigrid scheme with the standard prolongation does not work well. The reason seems to lie in the large energy norm of the prolongation operator. We give a simple example. Consider the unit square with a line interface from (1/2, 0) to (1/2, 1), where the diffusion coefficient p is p_1 on the left half and p_2 on the right half. As usual, we define the diffusion on the interface by the harmonic average of p_1 and p_2, imposing the flux continuity across the interface. Let v be defined as follows:

$$
\begin{cases} v_{i,j} = v_0 & \text{for } i \le n/2 \\ v_{i,j} = 0 & \text{otherwise.} \end{cases} \qquad (3.10)
$$

Assume, for simplicity, homogeneous Neumann boundary condition. Then by cancellation of the terms except near the interface, $A_{k-1}(v, v)$ is written as follows:

$$A_{k-1}(v,\ v) = \sum_{j=1}^{n} \frac{2p_1p_2}{p_1 + p_2} (v_{n/2,j} - v_{n/2+1,j})^2 = n \frac{2p_1p_2}{p_1 + p_2} v_0^2. \tag{3.11}$$

Now consider $A_k(I_{k-1}^k v,\ I_{k-1}^k v)$. Let I be the index so that the cell boundary between $(I,\ J)$ and $(I^1,\ J)$ aligns with the domain interface $x = 1/2$. Then for $J = 1, \ldots, 2n$,

$$(u_{I,J} - u_{I_1,J})^2 = \frac{v_0^2}{16},$$

$$(u_{I^2,J} - u_{I^1,J})^2 = \frac{v_0^2}{16},$$

$$(u_{I^1,J} - u_{I,J})^2 = \frac{v_0^2}{4}.$$

Hence, we see

$$A_k(I_{k-1}^k v,\ I_{k-1}^k v) = 2n\left(\frac{p_1 + p_2}{16} + \frac{1}{4}\frac{2p_1p_2}{p_1 + p_2}\right) v_0^2 = C(p_1,\ p_2)A_{k-1}(v,\ v), \tag{3.12}$$

where

$$C(p_1,\ p_2) = \frac{(p_1 + p_2)^2}{16p_1p_2} + \frac{1}{2}.$$

Assume, $p_2 > p_1$ and let $\gamma = p_1/p_2$. Then

$$C(p_1,\ p_2) = C(\gamma) = \frac{(\gamma + 1)^2}{16\gamma} + \frac{1}{2},$$

which approaches infinity as $\gamma \to 0$.

This phenomenon explains why the multigrid algorithm with bilinear prolongation behaves poorly when p has a strong discontinuity. We want to devise a new prolongation whose energy norm remains bounded even if the jump ratio of the coefficients approaches ∞. For a motivation, we consider the following one-dimensional diffusion equation on [0, 1].

$$\begin{cases} -\dfrac{d}{dx}\left(p\dfrac{du(x)}{dx}\right) = f & \text{in } (0,\ 1), \\ u(0) = u(1) = 0. \end{cases}$$

Let x_i, $i = 1, 2, \ldots, n$ be the center of equally spaced grid on [0, 1] and h be the mesh size. When this grid is refined, new center will have locations at $x_{i+1/4}$, $x_{i+3/4}$, etc. Now, we consider how to define the values at $x_{i+1/4}$. A natural way is to impose the flux continuity at $x_{i+1/4}$, so that the following equality holds:

$$p_i \frac{v_{i+1/4} - v_i}{\frac{h}{4}} = p_{i+1} \frac{v_{i+1} - v_{i+1/4}}{\frac{3h}{4}}. \tag{3.13}$$

Solving, we get

$$v_{i+1/4} = \frac{3p_i v_i + p_{i+1} v_{i+1}}{3p_i + p_{i+1}}. \tag{3.14}$$

This can be viewed as the linear interpolation of neighboring point with diffusion weights. Motivated by this, we define a new prolongation I_{k-1}^k for two-dimensional problem as follows:

$$\begin{aligned}
u_{I,J} &= \frac{9p_{i,j}v_{i,j} + 3p_{i,j+1}v_{i,j+1} + 3p_{i+1,j}v_{i+1,j} + p_{i+1,j+1}v_{i+1,j+1}}{9p_{i,j} + 3p_{i+1,j} + 3p_{i,j+1} + p_{i+1,j+1}},\\
u_{I^1,J} &= \frac{9p_{i+1,j}v_{i+1,j} + 3p_{i,j}v_{i,j} + 3p_{i+1,j+1}v_{i+1,j+1} + p_{i,j+1}v_{i,j+1}}{9p_{i+1,j} + 3p_{i,j} + 3p_{i+1,j+1} + p_{i,j+1}},\\
u_{I,J^1} &= \frac{9p_{i,j+1}v_{i,j+1} + 3p_{i,j}v_{i,j} + 3p_{i+1,j+1}v_{i+1,j+1} + p_{i+1,j}v_{i+1,j}}{9p_{i,j+1} + 3p_{i,j} + 3p_{i+1,j+1} + p_{i+1,j}},\\
u_{I^1,J^1} &= \frac{9p_{i+1,j+1}v_{i+1,j+1} + 3p_{i+1,j}v_{i+1,j} + 3p_{i,j+1}v_{i,j+1} + p_{i,j}v_{i,j}}{9p_{i+1,j+1} + 3p_{i+1,j} + 3p_{i,j+1} + p_{i,j}}.
\end{aligned} \tag{3.15}$$

It is easy to check that this prolongation operator reduces to the bilinear one when the diffusion p is constant.

Proposition 3.2. *Let $p = p_1$ for $x < 1/2$ and $p = p_2$ for $x > 1/2$. Then for any pair of positive constants p_1 and p_2, we have*

$$A_k(I_{k-1}^k v, I_{k-1}^k v) \le \frac{11}{8} A_{k-1}(v, v), \qquad \forall v \in V_{k-1}.$$

The proof is very long and technical. We need some Lemmas.

Lemma 3.3. *The coefficient of $(v_{i+1,j} - v_{i,j})^2$ appearing in the expansion of $p_{I_{1/2},J}(u_{I,J} - u_{I_1,J})^2$, $p_{I_{1/2},J_2}(u_{I,J_2} - u_{I_1,J_2})^2$, $p_{I^{3/2},J}(u_{I^1,J} - u_{I^2,J})^2$, and $p_{I^{3/2},J_2}(u_{I^2,J_2} - u_{I^1,J_2})^2$ is bounded by*

$$\frac{(3p_1 + 5p_2)p_2}{4(3p_1 + p_2)^2} p_1 + \frac{(5p_1 + 3p_2)p_1}{4(p_1 + 3p_2)^2} p_2. \tag{3.16}$$

Proof. Fix $v \in V_{k-1}$ and set $u = I_{k-1}^k v$. Then we have the same expansion for $A_{k-1}(v, v)$ and $A_k(u, u)$ as in (3.2), (3.3). By definition of $u_{I,J}$ and $u_{I_1,J}$, we have

$$\begin{aligned}
u_{I,J} - u_{I_1,J} = &\frac{9p_{i,j}v_{i,j} + 3p_{i,j+1}v_{i,j+1} + 3p_{i+1,j}v_{i+1,j} + p_{i+1,j+1}v_{i+1,j+1}}{9p_{i,j} + 3p_{i+1,j} + 3p_{i,j+1} + p_{i+1,j+1}}\\
&- \frac{9p_{i,j}v_{i,j} + 3p_{i,j+1}v_{i,j+1} + 3p_{i-1,j}v_{i-1,j} + p_{i-1,j+1}v_{i-1,j+1}}{9p_{i,j} + 3p_{i-1,j} + 3p_{i,j+1} + p_{i-1,j+1}}
\end{aligned}$$

$$= \frac{9p_1v_{i,j} + 3p_1v_{i,j+1} + 3p_2v_{i+1,j} + p_2v_{i+1,j+1}}{12p_1 + 4p_2} - \frac{9v_{i,j} + 3v_{i,j+1} + 3v_{i-1,j} + v_{i-1,j+1}}{16}$$

$$= \frac{12p_2(v_{i+1,j} - v_{i,j}) + 3(3p_1 + p_2)(v_{i,j} - v_{i-1,j})}{16(3p_1 + p_2)} + \frac{4p_2(v_{i+1,j+1} - v_{i,j+1}) + (3p_1 + p_2)(v_{i,j+1} - v_{i-1,j+1})}{16(3p_1 + p_2)}. \quad (3.17)$$

By Cauchy-Schwarz inequality, we have

$$(u_{I,J} - u_{I_1,J})^2 = [12p_2(v_{i+1,j} - v_{i,j}) + 3(3p_1 + p_2)(v_{i,j} - v_{i-1,j}) + 4p_2(v_{i+1,j+1} - v_{i,j+1}) + (3p_1 + p_2)(v_{i,j+1} - v_{i-1,j+1})]^2 \leq (12p_1 + 20p_2) \times \left[\frac{12p_2(v_{i+1,j} - v_{i,j})^2 + 3(3p_1 + p_2)(v_{i,j} - v_{i-1,j})^2}{256(3p_1 + p_2)^2} + \frac{4p_2(v_{i+1,j+1} - v_{i,j+1})^2 + (3p_1 + p_2)(v_{i,j+1} - v_{i-1,j+1})^2}{256(3p_1 + p_2)^2} \right]. \quad (3.18)$$

Similarly, by shifting $j \to j - 1$, we get

$$(u_{I,J_2} - u_{I_1,J_2})^2 \leq (12p_1 + 20p_2) \left[\frac{12p_2(v_{i+1,j-1} - v_{i,j-1})^2 + 3(3p_1 + p_2)(v_{i,j-1} - v_{i-1,j-1})^2}{256(3p_1 + p_2)^2} + \frac{4p_2(v_{i+1,j} - v_{i,j})^2 + (3p_1 + p_2)(v_{i,j} - v_{i-1,j})^2}{256(3p_1 + p_2)^2} \right]. \quad (3.19)$$

By reflection with respect to $x = 1/2(i \leftrightarrow i + 1,\ i - 1 \leftrightarrow i + 2,\ p_1 \leftrightarrow p_2)$,

$$(u_{I^1,J} - u_{I^2,J})^2 \leq (20p_1 + 12p_2) \left[\frac{12p_1(v_{i,j} - v_{i+1,j})^2 + 3(p_1 + 3p_2)(v_{i+1,j} - v_{i+2,j})^2}{256(p_1 + 3p_2)^2} + \frac{4p_1(v_{i,j+1} - v_{i+1,j+1})^2 + (p_1 + 3p_2)(v_{i+1,j+1} - v_{i+2,j+1})^2}{256(p_1 + 3p_2)^2} \right]. \quad (3.20)$$

By reflection with respect to $x = 1/2$ and shifting $j \to j - 1$,

$$(u_{I^2,J_2} - u_{I^1,J_2})^2 \leq (20p_1 + 12p_2) \left[\frac{12p_1(v_{i,j-1} - v_{i+1,j-1})^2 + 3(p_1 + 3p_2)(v_{i+1,j-1} - v_{i+2,j-1})^2}{256(p_1 + 3p_2)^2} + \frac{4p_1(v_{i,j} - v_{i+1,j})^2 + (p_1 + 3p_2)(v_{i+1,j} - v_{i+2,j})^2}{256(p_1 + 3p_2)^2} \right]. \quad (3.21)$$

Let us consider the term $(v_{i+1,j} - v_{i,j})^2$. The sum of coefficients from (3.18)–(3.21) (accounting for the coefficients in A_k form) is

$$\frac{(3p_1 + 5p_2)p_2}{4(3p_1 + p_2)^2}\, p_1 + \frac{(5p_1 + 3p_2)p_1}{4(p_1 + 3p_2)^2}\, p_2, \quad (3.22)$$

and this completes the proof of Lemma. ■

Lemma 3.4. *The coefficient of* $(v_{i+1,j} - v_{i,j})^2$ *from* $p_{I^{1/2},J}(u_{I^1,J} - u_{I,J})^2$, $p_{I^{1/2},J_2}(u_{I^1,J_2} - u_{I,J_2})^2$ *is bounded by*

$$\frac{(16 \times 3 + 16)p_1^2 p_2^2}{(3p_1 + p_2)^2(p_1 + 3p_2)^2} \cdot \frac{2p_1p_2}{p_1 + p_2}. \tag{3.23}$$

Proof. Since

$$u_{I^1,J} - u_{I,J} = \frac{9p_2v_{i+1,j} + 3p_2v_{i+1,j+1} + 3p_1v_{i,j} + p_1v_{i,j+1}}{9p_2 + 3p_2 + 3p_1 + p_1}$$
$$- \frac{9p_1v_{i,j} + 3p_1v_{i,j+1} + 3p_2v_{i+1,j} + p_2v_{i+1,j+1}}{9p_1 + 3p_1 + 3p_2 + p_2} = \frac{6p_1p_2(v_{i+1,j} - v_{i,j}) + 2p_1p_2(v_{i+1,j+1} - v_{i,j+1})}{(3p_1 + p_2)(3p_2 + p_1)},$$

we have, by Cauchy-Schwarz inequality,

$$(u_{I^1,J} - u_{I,J})^2 \le \frac{16p_1^2p_2^2(3(v_{i+1,j} - v_{i,j})^2 + (v_{i+1,j+1} - v_{i,j+1})^2)}{(3p_1 + p_2)^2(p_1 + 3p_2)^2}. \tag{3.24}$$

By shifting the index $j \to j - 1$, we see

$$(u_{I^1,J_2} - u_{I,J_2})^2 \le \frac{16p_1^2p_2^2(3(v_{i+1,j-1} - v_{i,j-1})^2 + (v_{i+1,j} - v_{i,j})^2)}{(3p_1 + p_2)^2(p_1 + 3p_2)^2}. \tag{3.25}$$

Thus the sum of the coefficient of $(v_{i+1,j} - v_{i,j})^2$ is

$$\frac{(16 \times 3 + 16)p_1^2 p_2^2}{(3p_1 + p_2)^2(p_1 + 3p_2)^2} \cdot \frac{2p_1p_2}{p_1 + p_2}. \tag{3.26}$$

■

Proof (of Proposition). It suffices to consider near the interface. Let $(i, j), j = 0, \ldots, 2^k$ be the indices of cells adjacent left to the interface $x = 1/2$. We compare the terms $(v_{i+1,j} - v_{i,j})^2$, $(v_{i,j} - v_{i-1,j})^2$ and $(v_{i+2,j} - v_{i+1,j})^2$ in $A_k(u, u)$ and $A_{k-1}(v, v)$, respectively. By Lemma 3.3 and 3.4, the coefficient of $(v_{i+1,j} - v_{i,j})^2$ from 6 contributions are bounded by

$$\frac{(3p_1 + 5p_2)p_2}{4(3p_1 + p_2)^2} \cdot p_1 + \frac{(5p_1 + 3p_2)p_1}{4(p_1 + 3p_2)^2} \cdot p_2 + \frac{64p_1^2p_2^2}{(3p_1 + p_2)^2(p_1 + 3p_2)^2} \cdot \frac{2p_1p_2}{p_1 + p_2}. \tag{3.27}$$

The additional contribution to this term can be estimated by reflection ($J \to J_1$, $J_2 \to J^1$) in Lemmas 3.3 and 3.4. Because p is constant along y-direction, total sum is twice of (3.27).Because the diffusion coefficient of $(v_{i+1,j} - v_{i,j})^2$ in $A_{k-1}(v, v)$ is $2p_1p_2/(p_1 + p_2)$, dividing by $2p_1p_2/(p_1 + p_2)$, the ratio between the coefficients of the two forms is

$$X_c := \frac{(3p_1 + 5p_2)(p_1 + p_2)}{4(3p_1 + p_2)^2} + \frac{(5p_1 + 3p_2)(p_1 + p_2)}{4(p_1 + 3p_2)^2} + \frac{128p_1^2p_2^2}{(3p_1 + p_2)^2(p_1 + 3p_2)^2}. \tag{3.28}$$

Let $\gamma = p_1/p_2$. Then we can write (3.28) as

$$X_c(\gamma) = \frac{12\gamma^4 + 32\gamma^3 + 168\gamma^2 + 32\gamma + 12}{(\gamma + 3)^2(3\gamma + 1)^2}. \quad (3.29)$$

Now consider $(v_{i,j} - v_{i-1,j})^2$. From (3.18) and (3.19), we see the coefficient in $(u_{I,J} - u_{I_1,J})^2$ and $(u_{I,J_2} - u_{I_1,J_2})^2$ is, after multiplying the corresponding coefficient p_1 in $A_k(u, u)$,

$$\frac{(3p_1 + 5p_2)}{16(3p_1 + p_2)} p_1. \quad (3.30)$$

Contributions from other terms such as $(u_{I_1,J} - u_{I_2,J})^2$, $(u_{I_2,J} - u_{I_3,J})^2$, $(u_{I_1,J_2} - u_{I_2,J_2})^2$ and $(u_{I_2,J_2} - u_{I_3,J_2})^2$ are exactly the same as constant coefficient case, that is,

$$\left(\frac{3}{16} + \frac{1}{16} + \frac{3}{32} + \frac{1}{32}\right)p_1 = \frac{3}{8} p_1. \quad (3.31)$$

Adding (3.30) and (3.31), we get

$$p_1 \frac{(21p_1 + 11p_2)}{16(3p_1 + p_2)}. \quad (3.32)$$

Shifting J by -1, the total coefficient of $(v_{i,j} - v_{i-1,j})^2$ in $A_k(u, u)$ is twice of (3.32), whereas that of $A_{k-1}(v, v)$ is p_1. Hence in this case, the ratio is

$$X_l(\gamma) := \frac{21\gamma + 11}{8(3\gamma + 1)}. \quad (3.33)$$

Similarly, the ratio in the term $(v_{i+2,j} - v_{i+1,j})^2$ is

$$X_r(\gamma) := \frac{11\gamma + 21}{8(\gamma + 3)}. \quad (3.34)$$

Hence we are finished for the terms along the x direction. We can proceed to check the ratios of difference along the y direction in the same way, which can be shown to be

$$Y_l(\gamma) := \frac{3\gamma^2 + 11\gamma + 2}{(3\gamma + 1)(\gamma + 3)}, \quad (3.35)$$

and

$$Y_r(\gamma) := \frac{2\gamma^2 + 11\gamma + 3}{(3\gamma + 1)(\gamma + 3)}, \quad (3.36)$$

for the terms $(v_{i,j+1} - v_{i,j})^2$ and $(v_{i+1,j+1} - v_{i+1,j})^2$, respectively. We shall estimate the maximum of the quantities (3.29), (3.33), (3.34), (3.35), and (3.36). It is easy to check that $X_c(\gamma) = X_c(1/\gamma)$, $X_l(\gamma) = X_r(1/\gamma)$, and $Y_l(\gamma) = Y_r(1/\gamma)$ for all $\gamma > 0$. So it suffices to check the maximum on (0, 1]. The maximum is obtained as $X_l(0) = 11/8$ when $0 < \gamma \leq 1$. The proof is complete. ∎

Now we consider "regularity and approximation property." Because p is discontinuous, it is not known whether the solution u belongs to $H^{1+\alpha}(\Omega)$, for any $\alpha > 0$. Fortunately, for CCFD, we can find a nearby smooth problem which has enough regularity, from which we shall show "regularity and approximation property" of discontinuous problem. Let p_s be a smooth function such that $p_s(1/2, y) = 2p_1p_2/(p_1 + p_2)$ for all y and $p_s(x, y) = p_1$ for $0 \leq x < 1/2 - \epsilon$, $p_s(x, y) = p_2$ for $1/2 + \epsilon < x \leq 1$ for some positive ϵ less than the size of finest grid. Clearly, this modified problem (with p_s instead of p) induces the same operator A_k as in the case of discontinuous p. Thus "regularity and approximation property" can be obtained from this modified problem, which we will show below. First, we need the following estimate.

Lemma 3.5. *We have*

$$\|(I - I_{k-1}^k)v\|^2 \leq Ch_k^2 A_{k-1}(v, v), \qquad \forall v \in V_{k-1}. \tag{3.37}$$

Proof. For simplicity, we assume that $p = 1$. By definition of prolongation and using (3.4), we have

$$\begin{aligned}(v_{i,j} - u_{I,J})^2 &= \left(v_{i,j} - \frac{9v_{i,j} + 3v_{i+1,j} + 3v_{i,j+1} + v_{i+1,j+1}}{16}\right)^2 \\ &= \frac{(4(v_{i,j} - v_{i+1,j}) + 3(v_{i,j} - v_{i,j+1}) + (v_{i+1,j} - v_{i+1,j+1}))^2}{256} \\ &\leq \frac{4(v_{i,j} - v_{i+1,j})^2 + 3(v_{i,j} - v_{i,j+1})^2 + (v_{i+1,j} - v_{i+1,j+1})^2}{32}.\end{aligned} \tag{3.38}$$

Because $\|(I - I_{k-1}^k)v\|^2 = h_k^2 \sum_{E_{i,j}^{k-1} \in \mathscr{E}^{k-1}} ((v_{i,j} - u_{I,J})^2 + (v_{i,j} - u_{I_1,J})^2 + (v_{i,j} - u_{I,J_1})^2 + (v_{i,j} - u_{I_1,J_1})^2)$, it follows from (3.38) that (3.37) holds. ∎

Lemma 3.6. *Let the operator P_{k-1} be defined by (2.9). Then (2.8) holds for $\alpha = \frac{1}{2}$.*

Proof. The proof is quite similar to that of Lemma 2.2 in [10], but we include it here for completeness. Fix $u \in V_k$ and let w be the solution of the following boundary value problem:

$$\begin{aligned}-\nabla \cdot p_s \nabla w &= A_k u \qquad \text{in } \Omega, \\ w &= 0 \qquad \text{on } \partial\Omega.\end{aligned} \tag{3.39}$$

Because u is the cell-centered approximation of w, we have [18],

$$|A_k(u - Q_k w, u - Q_k w)| \leq Ch_k^2 \|A_k u\|^2. \tag{3.40}$$

By definition of I_{k-1}^k, we have

$$\begin{aligned}A_k((I - I_{k-1}^k P_{k-1})u, u) &= A_k(u, u) - A_{k-1}(P_{k-1}u, P_{k-1}u) \\ &= A_k(u - Q_k w, u) + A_{k-1}(Q_{k-1}w - P_{k-1}u, P_{k-1}u) \\ &\quad + A_k(Q_k w, u) - A_{k-1}(Q_{k-1}w, P_{k-1}u). \quad (3.41)\end{aligned}$$

Applying the Cauchy-Schwarz inequality to the first term, by (3.40) we have

$$|A_k(u - Q_k w, u)| \le A_k(u - Q_k w, u - Q_k w)^{1/2} A_k(u, u)^{1/2} \le Ch_k \|A_k u\| A_k(u, u)^{1/2}. \quad (3.42)$$

If we define the operator $\tilde{P}_{k-1} : V_k \to V_{k-1}$ by

$$A_{k-1}(\tilde{P}_{k-1}u, v) = A_k(u, v) \qquad u \in V_k,\ v \in V_{k-1}, \quad (3.43)$$

then the second term can be estimated similarly by noting that $\tilde{P}_{k-1}u$ is the cell-centered approximation of w in V_{k-1}. By (2.9) and (3.43), we have

$$\begin{aligned}A_{k-1}(Q_{k-1}w - P_{k-1}u, P_{k-1}u) &= A_{k-1}(Q_{k-1} - \tilde{P}_{k-1}u, P_{k-1}u) + A_{k-1}(\tilde{P}_{k-1}u - P_{k-1}u, P_{k-1}u) \\ &= A_{k-1}(Q_{k-1}w - \tilde{P}_{k-1}u, P_{k-1}u) - A_k(u, (I - I_{k-1}^k)P_{k-1}u). \quad (3.44)\end{aligned}$$

By similar arguments as (3.42), we see that

$$|A_{k-1}(Q_{k-1}w - \tilde{P}_{k-1}u, P_{k-1}u)| \le Ch_k \|A_k u\| A_{k-1}(P_{k-1}u, P_{k-1}u)^{1/2}. \quad (3.45)$$

By Lemma 3.5, we have

$$|A_k(u, (I - I_{k-1}^k)P_{k-1}u)| \le \|A_k u\| \cdot \|(I - I_{k-1}^k)P_{k-1}u\| \le Ch_k A_{k-1}(P_{k-1}u, P_{k-1}u)^{1/2} \|A_k u\|. \quad (3.46)$$

Because, by (2.7),

$$\begin{aligned}A_{k-1}(P_{k-1}u, P_{k-1}u) = A_k(u, I_{k-1}^k P_{k-1}u) &\le A_k(u, u)^{1/2} A_k(I_{k-1}^k P_{k-1}u, I_{k-1}^k P_{k-1}u)^{1/2} \\ &\le C_*^{1/2} A_k(u, u)^{1/2} A_{k-1}(P_{k-1}u, P_{k-1}u)^{1/2}, \quad (3.47)\end{aligned}$$

we have

$$A_{k-1}(P_{k-1}u, P_{k-1}u) \le C_* A_k(u, u). \quad (3.48)$$

Substituting (3.45) and (3.46) into (3.44), we get from (3.48) that the second term in (3.41) satisfies

$$\begin{aligned}|A_{k-1}(Q_{k-1}w - P_{k-1}u, P_{k-1}u)| &\le Ch_k \|A_k u\| A_{k-1}(P_{k-1}u, P_{k-1}u)^{1/2} \\ &\le Ch_k \|A_k u\| A_k(u, u)^{1/2}. \quad (3.49)\end{aligned}$$

We now estimate the third and fourth term in (3.41). By Lemma 3.5, we have that

$$\|(I - I_{k-1}^k)Q_{k-1}w\| \le Ch_k A_{k-1}(Q_{k-1}w, Q_{k-1}w)^{1/2}. \tag{3.50}$$

So we have

$$\begin{aligned}|A_k(Q_k w, u) - A_{k-1}(Q_{k-1}w, P_{k-1}u)| &\le |A_k(Q_k w - Q_{k-1}w, u)| + |A_k((I - I_{k-1}^k)Q_{k-1}w, u)| \\ &\le \|Q_k w - Q_{k-1}w\|\,\|A_k u\| + \|(I - I_{k-1}^k)Q_{k-1}w\|\,\|A_k u\| \le Ch_k\|w\|_1\|A_k u\| \\ &\quad + Ch_k A_{k-1}(Q_{k-1}w, Q_{k-1}w)^{1/2}\|A_k u\| \le Ch_k\|w\|_1\|A_k u\|, \end{aligned} \tag{3.51}$$

where $\|\cdot\|_1$ is the Sobolev norm of order one. It remains to bound $\|w\|_1$, which can be done by exactly the same way as in [6]. Thus, we have

$$|A_k(Q_k w, u) - A_{k-1}(Q_{k-1}w, P_{k-1}u)| \le Ch_k\|A_k u\| A_k(u, u)^{1/2}. \tag{3.52}$$

Combining estimates (3.42), (3.49), and (3.52), together with the bound $\lambda_k \le Ch_k^{-2}$, we obtain (2.8). ∎

Now we obtain the following result from the discussion in Section 2 (Theorem 2.2).

Theorem 3.7. *Let $E_k = I - B_k A_k$ in multigrid algorithm with $s = 2$(W-cycle algorithm). Then, for any m, we have*

$$A_k(E_k u, E_k u) \le \delta_k A_k(u, u), \qquad \forall u \in V_k, \tag{3.53}$$

where $\delta_k = M/(M + \sqrt{m})$. In particular, W-cycle works with one smoothing.

C. Junction Discontinuity

We now consider the case of junction discontinuity. For simplicity, we assume p is piecewise constant as follows:

$$p = \begin{cases} p_1 & 0 \le x < \frac{1}{2},\ 0 \le y < \frac{1}{2}, \\ p_2 & \frac{1}{2} \le x \le 1,\ 0 \le y < \frac{1}{2}, \\ p_3 & 0 \le x < \frac{1}{2},\ \frac{1}{2} \le y \le 1, \\ p_4 & \frac{1}{2} \le x \le 1,\ \frac{1}{2} \le y \le 1. \end{cases}$$

The case when p is piecewise smooth can be handled similarly.

Lemma 3.8. *There is a constant C_* such that for any quadruple of positive constants p_1, p_2, p_3, and p_4,*

$$A_k(I_{k-1}^k v, I_{k-1}^k v) \le C_* A_{k-1}(v, v), \qquad \forall v \in V_{k-1}$$

holds.

Proof. The proof is quite similar to that of Proposition 3.2. We shall again count the contributions of the terms of the form $(v_{i+1,j} - v_{i,j})^2$ from those of $(u_{I,J} - u_{I_1,J})^2$, $(u_{I,J_1} - u_{I_1,J_1})^2$, etc., and compare the coefficient between the two forms $A_k(u, u)$ and $A_{k-1}(v, v)$. Let (I, J) be the index of the cell located at the upper right corner of quadrant $[0, 1/2] \times [0, 1/2]$ [Fig. 1(b)]. In this case, we have

$$u_{I,J} = \frac{9p_1 v_{i,j} + 3p_2 v_{i+1,j} + 3p_3 v_{i,j+1} + p_4 v_{i+1,j+1}}{9p_1 + 3p_2 + 3p_3 + p_4},$$

$$u_{I_1,J} = \frac{9p_1 v_{i,j} + 3p_1 v_{i-1,j} + 3p_3 v_{i,j+1} + p_3 v_{i-1,j+1}}{12p_1 + 4p_3}.$$

To simplify computations, we introduce some notations: For any natural numbers i, j, k, l, let $Q_{ijkl} = 9p_i + 3p_j + 3p_k + p_l$. By taking the difference and multiplying the denominator, we have

$$\begin{aligned}
Q_{1234}Q_{1133}(u_{I,J} - u_{I_1,J}) &= (9p_1Q_{1133} - 9p_1Q_{1234})v_{i,j} + 3p_2Q_{1133}v_{i+1,j} - 3p_1Q_{1234}v_{i-1,j} \\
&\quad + (3p_3Q_{1133} - 3p_3Q_{1234})v_{i,j+1} + p_4Q_{1133}v_{i+1,j+1} - p_3Q_{1234}v_{i-1,j+1} \\
&= (9p_1Q_{1133} - 9p_1Q_{1234})v_{i,j} + 3p_2Q_{1133}(v_{i+1,j} - v_{i,j}) + 3p_2Q_{1133}v_{i,j} \\
&\quad + 3p_1Q_{1234}(v_{i,j} - v_{i-1,j}) - 3p_1Q_{1234}v_{i,j} + (3p_3Q_{1133} - 4p_3Q_{1234})v_{i,j+1} \\
&\quad + p_4Q_{1133}(v_{i+1,j+1} - v_{i,j+1}) + p_4Q_{1133}v_{i,j+1} + p_3Q_{1234}(v_{i,j+1} - v_{i-1,j+1}) \\
&= 3p_1Q_{1234}(v_{i,j} - v_{i-1,j}) + 12(p_1p_4 - p_2p_3)(v_{i,j+1} - v_{i,j}) + 3p_2Q_{1133}(v_{i+1,j} - v_{i,j}) \\
&\quad + p_3Q_{1234}(v_{i,j+1} - v_{i-1,j+1}) + p_4Q_{1133}(v_{i+1,j+1} - v_{i,j+1}),
\end{aligned}$$

where the last equality is obtained by noting the unpaired coefficients of $v_{i,j}$ is

$$9p_1Q_{1133} + 3p_2Q_{1133} - 12p_1Q_{1234} = 12(p_2p_3 - p_1p_4),$$

and that of $v_{i,j+1}$ is

$$3p_3Q_{1133} - 4p_3Q_{1234} + p_4Q_{1133} = -12(p_2p_3 - p_1p_4).$$

Dividing by $Q_{1234}Q_{1133}$ and squaring, we see

$$\begin{aligned}
(u_{I,J} - u_{I_1,J})^2 &\leq \frac{(3p_1 + p_3)Q_{1234} + 12(p_1p_4 + p_2p_3) + (3p_2 + p_4)Q_{1133}}{Q_{1234}^2Q_{1133}^2} \\
&\quad \times [3p_1Q_{1234}(v_{i,j} - v_{i-1,j})^2 + 12(p_1p_4 + p_2p_3)(v_{i,j+1} - v_{i,j})^2 + 3p_2Q_{1133}(v_{i+1,j} - v_{i,j})^2 \\
&\quad + p_3Q_{1234}(v_{i,j+1} - v_{i-1,j+1})^2 + p_4Q_{1133}(v_{i+1,j+1} - v_{i,j+1})^2]. \qquad (3.54)
\end{aligned}$$

Now for the term $(u_{I,J_1} - u_{I_1,J_1})^2$, simple shifting along y does not give correct coefficient as in (3.27) because p is not constant along y direction. Instead, a closer look at u_{I,J_1} and u_{I_1,J_1} shows that it is obtained by reflecting with respect to $y = 1/2$, then shifting j by $-1(j + 1 \rightarrow j - 1$, $p_3 \rightarrow p_1$, $p_4 \rightarrow p_2)$. Hence

$$(u_{I,J_1} - u_{I_1,J_1})^2 \le \frac{(3p_1 + p_1)Q_{1212} + 12(p_1p_2 + p_2p_1) + (3p_2 + p_2)Q_{1111}}{Q_{1212}^2 Q_{1111}^2} \times [3p_1Q_{1212}(v_{i,j} - v_{i-1,j})^2 + 12(p_1p_2 + p_2p_1)(v_{i,j-1} - v_{i,j})^2 + 3p_2Q_{1111}(v_{i+1,j} - v_{i,j})^2 + p_1Q_{1212}(v_{i,j-1} - v_{i-1,j-1})^2 + p_2Q_{1111}(v_{i+1,j-1} - v_{i,j-1})^2]. \quad (3.55)$$

The term $(u_{I^2,J} - u_{I^1,J})^2$ is obtained by reflecting $(u_{I,J} - u_{I_1,J})^2$ with respect to $x = 1/2(i \leftrightarrow i + 1, i - 1 \leftrightarrow i + 2, p_2 \leftrightarrow p_1, p_4 \leftrightarrow p_3)$ we see

$$(u_{I^2,J} - u_{I^1,J})^2 \le \frac{(3p_2 + p_4)Q_{2143} + 12(p_2p_3 + p_1p_4) + (3p_1 + p_3)Q_{2244}}{Q_{2143}^2 Q_{2244}^2} \times [3p_2Q_{2143}(v_{i+1,j} - v_{i+2,j})^2 + 12(p_2p_3 + p_1p_4)(v_{i+1,j+1} - v_{i+1,j})^2 + 3p_1Q_{2244}(v_{i+1,j} - v_{i,j})^2 + p_4Q_{2143}(v_{i+1,j+1} - v_{i+2,j+1})^2 + p_3Q_{2244}(v_{i+1,j+1} - v_{i,j+1})^2]. \quad (3.56)$$

Similarly, $(u_{I^2,J_1} - u_{I^1,J_1})^2$ is obtained by reflecting $(u_{I,J_1} - u_{I_1,J_1})^2$ with respect to $x = 1/2(i + 1 \leftrightarrow i, i + 2 \leftrightarrow i - 1, p_1 \leftrightarrow p_2)$. Thus,

$$(u_{I^2,J_1} - u_{I^1,J_1})^2 \le \frac{(3p_2 + p_2)Q_{2121} + 12(p_2p_1 + p_1p_2) + (3p_1 + p_1)Q_{2222}}{Q_{2121}^2 Q_{2222}^2} \times [3p_2Q_{2121}(v_{i+1,j} - v_{i+2,j})^2 + 12(p_2p_1 + p_1p_2)(v_{i+1,j-1} - v_{i+1,j})^2 + 3p_1Q_{2222}(v_{i+1,j} - v_{i,j})^2 + p_2Q_{2121}(v_{i+1,j-1} - v_{i+2,j-1})^2 + p_1Q_{2222}(v_{i+1,j-1} - v_{i,j-1})^2]. \quad (3.57)$$

Collecting the coefficient of $(v_{i+1,j} - v_{i,j})^2$ from (3.54)–(3.57) after multiplication by $p_{I_{1/2},J}$, $p_{I_{1/2},J_1}$, etc., we have

$$p_1 \frac{[(3p_1 + p_3)Q_{1234} + 12(p_1p_4 + p_2p_3) + (3p_2 + p_4)Q_{1133}] \times 3p_2Q_{1133}}{Q_{1234}^2 Q_{1133}^2} \quad (3.58)$$

$$+p_1 \frac{[(3p_1 + p_1)Q_{1212} + 12(p_1p_2 + p_2p_1) + (3p_2 + p_2)Q_{1111}] \times 3p_2Q_{1111}}{Q_{1212}^2 Q_{1111}^2} \quad (3.59)$$

$$+p_2 \frac{[(3p_2 + p_4)Q_{2143} + 12(p_2p_3 + p_1p_4) + (3p_1 + p_3)Q_{2244}] \times 3p_1Q_{2244}}{Q_{2143}^2 Q_{2244}^2} \quad (3.60)$$

$$+p_2 \frac{[(3p_2 + p_2)Q_{2121} + 12(p_2p_1 + p_1p_2) + (3p_1 + p_1)Q_{2222}] \times 3p_1Q_{2222}}{Q_{2121}^2 Q_{2222}^2}. \quad (3.61)$$

Let us compare (3.58) with $2p_1p_2/(p_1 + p_2)$, the coefficient of $(v_{i+1,j} - v_{i,j})^2$ in $A_{k-1}(v, v)$. The ratio is

$$\frac{3(p_1 + p_2)[(3p_1 + p_3)Q_{1234} + 12(p_1p_4 + p_2p_3) + (3p_2 + p_4)Q_{1133}]}{2Q_{1234}^2 Q_{1133}}. \quad (3.62)$$

Because $Q_{1234} \ge 3(p_1 + p_2)$ and $Q_{1133} = 4(3p_1 + p_3)$, the first term of (3.62) is bounded by

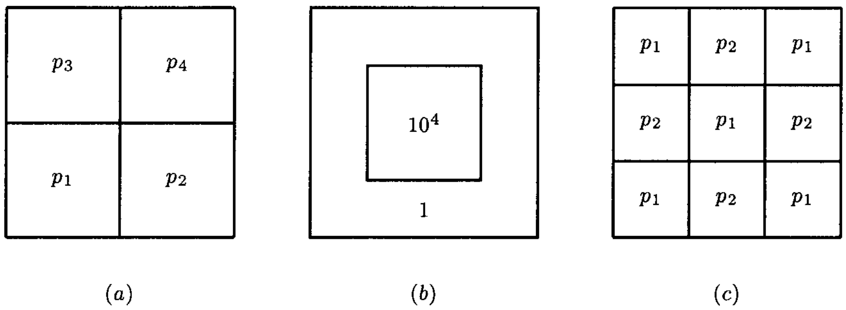

FIG. 2. Test problems.

$$\frac{3(p_1 + p_2)(3p_1 + p_3)}{2Q_{1234}Q_{1133}} \le \frac{1}{8}. \tag{3.63}$$

For the second term, we use the inequality, $Q_{1234}Q_{1133} \ge 12(p_1p_4 + p_2p_3)$ and $Q_{1234} \ge 3(p_1 + p_2)$ to see

$$\frac{3(p_1 + p_2) \cdot 12(p_1p_4 + p_2p_3)}{2Q_{1234}^2Q_{1133}} \le \frac{3(p_1 + p_2)}{2Q_{1234}} \le \frac{1}{2}. \tag{3.64}$$

The third term is

$$\frac{3(p_1 + p_2)(3p_2 + p_4)}{2Q_{1234}^2} \le \frac{1}{2}. \tag{3.65}$$

Hence the ratio from the term (3.58) is bounded by 9/8. All other terms in (3.59)–(3.61) can be bounded similarly.

Now the contributions from $(u_{I^1,J} - u_{I,J})^2$ and $(u_{I^1,J_1} - u_{I,J_1})^2$ are similar, but cannot be obtained by a simple shift or reflection. Because

$$u_{I^1,J} = \frac{9p_2v_{i+1,j} + 3p_1v_{i,j} + 3p_4v_{i+1,j+1} + p_3v_{i,j+1}}{9p_2 + 3p_1 + 3p_4 + p_3},$$

$$u_{I,J} = \frac{9p_1v_{i,j} + 3p_2v_{i+1,j} + 3p_3v_{i,j+1} + p_4v_{i+1,j+1}}{9p_1 + 3p_2 + 3p_3 + p_4},$$

TABLE I. $V(1, 1)$-cycle with modified bilinear prolongation, $p_1 = 1, p_2 = 10^4, p_3 = 10, p_4 = 10^2$ [Fig. 2(a)].

h_J	$\lambda_{\min}$	$\lambda_{\max}$	K	δ
1/32	0.589	0.999	1.696	0.169
1/64	0.562	0.999	1.779	0.191
1/128	0.557	0.999	1.795	0.210

TABLE II. $V(1, 1)$-cycle with modified bilinear prolongation [Fig. 2(b)].

h_J	$\lambda_{\min}$	$\lambda_{\max}$	K	δ
1/32	0.522	1.206	2.313	0.229
1/64	0.495	1.193	2.409	0.255
1/128	0.477	1.188	2.492	0.273

we see by direct computation

$$(u_{I^1,J} - u_{I,J})^2 \le \frac{24p_1(3p_2 + 2p_4) + 8p_3(6p_2 + p_4)}{Q_{1234}^2 Q_{2143}^2} \times [24(3p_1p_2 + p_1p_4)(v_{i+1,j} - v_{i,j})^2 + 24(p_1p_4 + p_2p_3)(v_{i+1,j+1} - v_{i+1,j})^2 + (24p_2p_3 + 8p_3p_4)(v_{i+1,j+1} - v_{i,j+1})^2]. \quad (3.66)$$

We again compare the ratio of $(v_{i+1,j} - v_{i,j})^2$ between $A_k(u, u)$ and $A_{k-1}(v, v)$. It suffices to estimate the first term of (3.66). Because $Q_{2143} \ge 3(3p_2 + 2p_4)/2$ and $Q_{2143} \ge 3(3p_2 + p_4) \ge 3(6p_2 + p_4)/2$, we have

$$\frac{24p_1(3p_2 + 2p_4) + 8p_3(6p_2 + p_4)}{Q_{1234}^2 Q_{2143}^2} \times 24 \times (3p_1p_2 + p_1p_4) \le \frac{64}{27}.$$

Shifting $J \to J - 1 (j + 1 \leftrightarrow j - 1, p_3 \to p_1, p_4 \to p_2)$, we see

$$(u_{I^1,J_1} - u_{I,J_1})^2 \le \frac{176p_1p_2}{Q_{1212}^2 Q_{2121}^2} [96p_1p_2(v_{i+1,j} - v_{i,j})^2 + 48p_1p_2(v_{i+1,j-1} - v_{i+1,j})^2 + 32p_1p_2(v_{i+1,j-1} - v_{i,j-1})^2]. \quad (3.67)$$

Hence the ratio we are interested in is

$$\frac{176p_1p_2 \cdot 96p_1p_2}{Q_{1212}^2 Q_{2121}^2} \le \frac{11}{24}.$$

There are contributions from y direction also, which can be guessed from the term $(v_{i,j+1} - v_{i,j})^2$ in (3.54) and $(v_{i+1,j+1} - v_{i+1,j})^2$ in (3.66), which can be similarly shown to be bounded.

Finally, the ratio of the term along y direction like $(v_{i,j+1} - v_{i,j})^2$ can be similarly shown to be bounded. ■

The "regularity and approximation property" can be shown similarly as in Lemma 3.6 and we deduce the following result.

TABLE III. Contraction number δ_k of Problems 1 to 3 of W-cycle.

h_J	Problem 1	Problem 2	Problem 3 ($j = 2$)	Problem 3 ($j = 3$)
1/32 (1/24)	0.101	0.160	0.121	0.122
1/64 (1/48)	0.104	0.163	0.120	0.157
1/128 (1/96)	0.106	0.165	0.121	0.199

TABLE IV. Variable $V(m(k), m(k)))$-cycle with $m(k) = 2^{J-k}$ for Problem 1.

h_J	$\lambda_{\min}$	$\lambda_{\max}$	K
1/32	0.680	0.999	1.471
1/64	0.676	0.999	1.480
1/128	0.673	0.999	1.485

Theorem 3.9. *Let p be described as above. Then for W-cycle with the prolongation described above, (3.53) holds for sufficiently large m.*

Remark. Our proof works for problems with more than one junction points. Also, the case of more general discontinuity can be handled similarly as long as the discontinuity arise as jumps along some line segments parallel to the axes.

IV. NUMERICAL RESULTS

We consider the following problem on the unit square:

$$\begin{aligned} -\nabla \cdot p\nabla\tilde{u} &= f \qquad \text{in } \Omega = (0, 1)^2, \\ \tilde{u} &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

We study the performance of our algorithms with Gauss-Seidel relaxation. We compare our modified bilinear interpolation with the bilinear interpolation, which shows similar behavior as one in [10].

Problem 1. We partition the domain into 4 pieces as in Fig. 2(a) and take $p_1 = 1$, $p_2 = 10^4$, $p_3 = 10$, and $p_4 = 10^2$. We report the eigenvalues, condition numbers, and reduction factors of V-cycle multigrid with one smoothing [$V(1, 1)$]. Here we used the power method to estimate the eigenvalues of $B_k A_k$ and obtain the average reduction factors δ_k in 15 iterations for the homogeneous problem starting from a random initial guess. We see that it converges well with modified bilinear prolongation (see Table I). We observed that behavior of multigrid with standard bilinear prolongation is unacceptably slow, even with many smoothings.

Problem 2. We take p as follows

$$p = \begin{cases} 10^4 & \text{when } \left(x - \frac{1}{4}\right)\left(x - \frac{3}{4}\right) < 0 \text{ and } \left(y - \frac{1}{4}\right)\left(y - \frac{3}{4}\right) < 0, \\ 1 & \text{otherwise.} \end{cases}$$

In this case, we also see $V(1, 1)$-cycle works well (see Table II).

TABLE V. Variable $V(m(k), m(k)))$-cycle with $m(k) = 2^{J-k}$ for Problem 2.

h_J	$\lambda_{\min}$	$\lambda_{\max}$	K
1/32	0.663	0.999	1.509
1/64	0.651	0.999	1.536
1/128	0.645	0.999	1.550

TABLE VI. Variable $V(m(k), m(k))$)-cycle with $m(k) = 2^{J-k}$ and $j = 2$ for Problem 3.

h_J	$\lambda_{\min}$	$\lambda_{\max}$	K
1/24	0.602	1.178	1.955
1/48	0.599	1.154	1.926
1/96	0.597	1.140	1.909

Problem 3. We take

$$p = \begin{cases} 10^j & \text{when } \left(x - \frac{1}{3}\right)\left(y - \frac{1}{3}\right)\left(x - \frac{2}{3}\right)\left(y - \frac{2}{3}\right) > 0, \\ 1 & \text{otherwise.} \end{cases}$$

In this case, for $j = 2$, $V(1, 1)$-cycle diverges but $W(1, 1)$-cycle converges. As j increases, we see that $W(1, 1)$ is divergent; hence, we need more smoothings; $W(2, 2)$-cycle is convergent for $j = 3$, for example. This is in accordance with the theory that we need sufficiently many smoothings when the energy norm is greater than 2. In Table III, the contraction numbers of all three problems and that of $W(2, 2)$-cycle for problem 3 with $j = 3$ are reported.

Next, we report the behavior of variable V-cycle for all examples. We see variable V-cycle is a good preconditioner in all these three cases (see Tables IV–VI).

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