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Multigrid analysis for higher order finite difference scheme

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Abstract — We introduce and analyze a multigrid algorithm for higher order finite difference schemes for elliptic problems on a nonuniform rectangular mesh. These schemes are presented by 9-point stencils. We prove the V-cycle convergence adopting the theory developed for finite element methods to these schemes. To be more precise, we show that the energy norm of the prolongation operator is less than one and hence obtain the conclusion using the approximation and regularity property as in [2]. In the numerical experiment section, we report contraction numbers, eigenvalues and condition numbers of the multigrid algorithm. The numerical test shows that for higher order schemes the multigrid algorithm converges much faster than for low order schemes. We also test the case of a nonuniform grid with a line smoother which also shows good convergence behavior.

Keywords: multigrid methods, higher order finite difference method

1. INTRODUCTION

The multigrid method has been widely used and proven to be effective for a large class of problems and has been the subject of extensive research [2,4,5,6,8,11,14]. V-cycle multigrid convergence has been analyzed for 2-nd order elliptic equations using finite element, finite difference, cell-centered finite difference methods and its behavior is well known [2,6,8,11]. On the other hand, compact schemes are shown to be effective for a class of problems including convection-diffusion equations when combined with multigrid algorithms [4,5]. These schemes are of higher order which are often used to solve the Laplace equation on a domain which can be partitioned into rectangular subdomains. These schemes can be adjusted to handle a nonuniform grid which has different mesh sizes along two coordinate axes [1]. Numerical experiments in [1] show relatively higher order convergence even for less smooth problems. The purpose of this paper is to introduce a multigrid algorithm for these higher order schemes and prove V-cycle convergence.

While there are plenty of finite element multigrid theories, few results are available for finite difference methods (see the references cited above). One of the reason is that in most finite element methods, all the spaces involved in the algorithm are nested. Hence, the bilinear forms on all levels are inherited and many well known

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finite element theories apply. For the cell-centered finite difference (CCFD) or the low order finite difference method (FDM) on a rectangular domain, V-cycle convergence is proved [2,8] using the fact that the stencil of FDM on a rectangular domain is similar to, or the same as that of the finite element method (FEM) on a certain triangular grid. However, there is no rigorous V-cycle multigrid convergence proof for higher order finite difference schemes although they have been used for many practical problems and show good convergence behavior [1,7,9,10,12,15].

In this paper, we prove the convergence by estimating the energy norm of the prolongation operator, thus adopting the theory developed by Bramble *et al.* [2]. Our approach can be adapted to show multigrid convergence for other finite difference schemes. The rest of this paper is organized as follows. In Section 2, we briefly introduce multigrid algorithms together with energy norm estimation. In Section 3, we report the eigenvalues, condition numbers and contraction numbers of the multigrid algorithm for a model problem.

2. MULTIGRID METHOD FOR HIGHER ORDER FINITE DIFFERENCE SCHEME

In this section, we briefly consider some families of higher order finite difference schemes in [1] and introduce a multigrid algorithm for these schemes. Here, as a model problem, we consider the following reaction-diffusion equation with Dirichlet boundary condition:

$$\begin{cases} -\Delta u + qu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

where $q \geqslant 0$ and Ω is any region in $\mathbb{R}^2$ covered by rectangles. For simplicity, we assume Ω is the unit square. Let h_x and $h_y = \vartheta h_x$ be the mesh sizes of x-axis and y-axis, respectively. We consider a class of fourth-order discretizations introduced in [1], denoted by $P(h,1,\vartheta,q)$, $Q(h,1,\vartheta,q)$, $P(h,1/2,1,q)$, and $Q(h,1/2,1,q)$. In this analysis, we choose the scheme $P(h,1,\vartheta,q)$:

$$\begin{aligned} \frac{1}{6h_xh_y}\big[&20u_{i,j} - au_{i-1,j} - au_{i+1,j} - bu_{i,j-1} - bu_{i,j+1} \\ &- (u_{i-1,j-1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i+1,j+1})\big] \\ &+2rq[8u_{i,j} + u_{i,j-1} + u_{i-1,j} + u_{i+1,j} + u_{i,j+1}] = F_{i,j} \end{aligned}$$

where $a = (10\vartheta^2 - 2)/(1+\vartheta^2)$, $b = (10 - 2\vartheta^2)/(1+\vartheta^2)$, $r = \vartheta/(12(1+\vartheta^2))$, and $F_{i,j} = 2r(8f_{i,j} + f_{i,j-1} + f_{i-1,j} + f_{i+1,j} + f_{i,j+1})$ (see Fig. 1). Other schemes can be analyzed similarly.

This nine-point discretization has a truncation error of $O(h^8)$ over a square mesh (i.e., a convergence order of $O(h^6)$). The resulting linear system can be written as

$$A_k u = f \tag{2.1}$$

$$\frac{1}{6h_xh_y}\begin{array}{|c|c|c|c|c|}\hline -1 & & -b & & -1 \\ \hline & & & & \\ \hline -a & & 20 & & -a \\ \hline & & & & \\ \hline -1 & & -b & & -1 \\ \hline \end{array}\; u \;+\; 2r \begin{array}{|c|c|c|c|c|}\hline & & 1 & & \\ \hline & & & & \\ \hline 1 & & 8 & & 1 \\ \hline & & & & \\ \hline & & 1 & & \\ \hline \end{array}\; qu$$

$$= \; 2r \begin{array}{|c|c|c|c|c|}\hline & & 1 & & \\ \hline & & & & \\ \hline 1 & & 8 & & 1 \\ \hline & & & & \\ \hline & & 1 & & \\ \hline \end{array}\; f$$

Figure 1. Stencil of $P(h,1,\vartheta,f)$, where $a=(10\vartheta^2-2)/(1+\vartheta^2)$, $b=(10-2\vartheta^2)/(1+\vartheta^2)$, $r=\vartheta/(12(1+\vartheta^2))$.

where A_k is a sparse, symmetric, positive definite matrix and u is the vector whose entries are $u_{i,j}$, and f is the vector whose entries are $F_{i,j}$. To define the multigrid algorithm, we introduce a sequence of nested grid sets Ω_k, for $k=1,2,\ldots,J$. Let h_x^k and h_y^k be the mesh sizes of the x-axis and the y-axis at level k. Define Ω_k to be the space of points $(x_i,y_j)=(ih_x^k,jh_y^k)$ for $i=0,1,\ldots,n(k)$, $j=0,1,\ldots,m(k)$ and V_k to be the vector space of functions defined on Ω_k. To describe the multigrid algorithm, we need certain intergrid transfer operators between two grids. Assuming that we are given a certain prolongation operator $I_{k-1}^k:\ V_{k-1}\longrightarrow V_k$, we define the restriction operator $I_k^{k-1}:\ V_k\longrightarrow V_{k-1}$ as its adjoint with respect to $(\cdot,\cdot)_k$:

$$(I_k^{k-1}u,v)_{k-1}=(u,I_{k-1}^k v)_k \qquad \forall u\in V_k,\ \ \forall v\in V_{k-1}$$

where $(\cdot,\cdot)_k$ is the discrete inner product defined by $(u,v)_k=h_x^kh_y^k\sum u_{i,j}v_{i,j}$. Now, the multigrid algorithm for solving (2.1) with a certain smoother R_k^i is defined as follows.

Multigrid Algorithm *V(m,m)*. Set $B_1=A_1^{-1}$. Assume that B_{k-1} has been defined and define B_kg for $g\in V_k$ as follows:

Step 1 (Pre-relaxation). Set $v^0=0$ and define v^i for $i=1,2,\ldots,m$ by

$$v^i=v^{i-1}+R_k^i(g-A_kv^{i-1})$$

Step 2. Define $w^m=v^m+I_{k-1}^kB_{k-1}[I_k^{k-1}(g-A_kv^m)]$.

Step 3 (Post-relaxation). Define w^i for $i = m+1, \ldots, 2m$ by

$$w^i = w^{i-1} + R_k^{(i+m)}(z - A_k w^{i-1}).$$

Step 4. Set $B_k z = w^{2m}$.

Let the bilinear form $A_k(\cdot,\cdot)$ be defined as follows:

$$A_k(u,v) = (A_k u, v)_k = D_k(u,v) + q_k(u,v).$$

Here, $D_k(u,v)$ and $q_k(u,v)$ corresponds to the second-order and zeroth-order part of $A_k(u,v)$ respectively. Specifically, $D_k(u,v)$ and $q_k(u,v)$ has the following form:

$$D_k(u,v) := \frac{1}{6}\sum_{i,j}\big[\,20u_{i,j} - au_{i-1,j} - au_{i+1,j} - bu_{i,j-1} - bu_{i,j+1})$$
$$- (u_{i-1,j-1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i+1,j+1})\big]\, v_{i,j}$$

and

$$q_k(u,v) := h_x^k h_y^k \sum_{i,j} q_{i,j} u_{i,j} v_{i,j}.$$

To define a prolongation operator, we proceed as follows: Fix a level $k-1$ and a cell $E_{i,j}$ at level $k-1$. Let $u_{i,j}, u_{i+1,j}, u_{i,j+1}, u_{i+1,j+1}$ be the values of u at vertices of $E_{i,j}$. Subdividing $E_{i,j}$ by half, we obtain four subcells $e_{i,j}, e_{i+1/2,j}, e_{i+1/2,j+1/2}, e_{i,j+1/2}$ which we label counterclockwise. The values at vertices of subcells will be denoted by $u_{i,j}$, $u_{i+1/2,j}$, $u_{i+1/2,j+1/2}$, $u_{i,j+1/2}$, *etc.*, as in Figure 2. Now, we define the prolongation operator I_{k-1}^k to be the bilinear interpolation through four points $u_{i,j}, u_{i+1,j}, u_{i,j+1}$ and $u_{i+1,j+1}$. First, $u_{i,j}$, $u_{i+1,j}$, $u_{i,j+1}$, and $u_{i+1,j+1}$ of level k has the same value as in level $k-1$. The mid point values $u_{i+1/2,j}$, $u_{i+1,j+1/2}$, $u_{i+1/2,j+1}$, and $u_{i,j+1/2}$ are given as follows:

$$u_{i+1/2,j} = \frac{u_{i,j} + u_{i+1,j}}{2}, \qquad u_{i+1,j+1/2} = \frac{u_{i+1,j} + u_{i+1,j+1}}{2}$$
$$u_{i+1/2,j+1} = \frac{u_{i,j+1} + u_{i+1,j+1}}{2}, \qquad u_{i,j+1/2} = \frac{u_{i,j} + u_{i,j+1}}{2}.$$

The value $u_{i+1/2,j+1/2}$ at the center point is the average of $u_{i,j}$, $u_{i+1,j}$, $u_{i,j+1}$, and $u_{i+1,j+1}$:

$$u_{i+1/2,j+1/2} = \frac{u_{i,j} + u_{i+1,j} + u_{i,j+1} + u_{i+1,j+1}}{4}.$$

Note that this bilinear prolongation does not depend on the ratio $\vartheta = h_y/h_x$.

Now, we have the following energy norm estimation.

Theorem 2.1. *When $q \geqslant 0$ is constant, we have*

$$A_k(I_{k-1}^k u, I_{k-1}^k u) \leqslant A_{k-1}(u,u) \quad \forall u \in V_{k-1}.$$

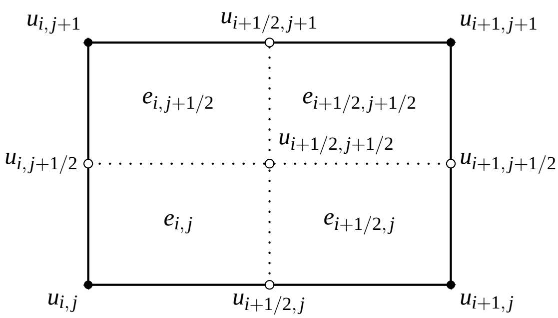

Figure 2. Cell $E^k_{i,j}$ and its subcells.

Proof. Since the quadratic form $A_k(I^k_{k-1}u, I^k_{k-1}u)$ can be decomposed into two parts, the conclusion will follow if the following inequalities

$$D_k(I^k_{k-1}u, I^k_{k-1}u) \leqslant D_{k-1}(u,u) \tag{2.2}$$

and

$$q_k(I^k_{k-1}u, I^k_{k-1}u)_k \leqslant q_{k-1}(u,u) \tag{2.3}$$

hold for all $u \in V_{k-1}$. These inequalities are shown in the following two lemmas. □

Lemma 2.1. *We have*

$$D_k(I^k_{k-1}u, I^k_{k-1}u) \leqslant D_{k-1}(u,u) \quad \forall u \in V_{k-1}.$$

Proof. Using the symmetry of the form D_{k-1} in i, j, we have

$$\begin{aligned} D_{k-1}(u,u) = \frac{1}{6}\sum_{i,j}\big[&20u_{i,j} - (au_{i-1,j} + bu_{i,j-1} + au_{i+1,j} + bu_{i,j+1}) \\ &- (u_{i-1,j-1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i+1,j+1})\big]u_{i,j} \\ = \frac{1}{6}\sum_{i,j}\big[&a(u_{i,j} - u_{i-1,j})^2 + b(u_{i,j} - u_{i,j-1})^2 \\ &+ a(u_{i,j} - u_{i+1,j})^2 + b(u_{i,j} - u_{i,j+1})^2\big] \\ + \frac{1}{6}\sum_{i,j}\big[&(u_{i,j} - u_{i-1,j-1})^2 + (u_{i,j} - u_{i+1,j-1})^2 \\ &+ (u_{i,j} - u_{i-1,j+1})^2 + (u_{i,j} - u_{i+1,j+1})^2\big]. \end{aligned}$$

Writing this cell-wise, we have

$$D_{k-1}(u,u) = \frac{1}{6}\sum_{E_{i,j}} S_{i,j} \tag{2.4}$$

where

$$\begin{aligned} S_{i,j} =& \frac{a}{2}(u_{i+1,j+1} - u_{i,j+1})^2 + \frac{a}{2}(u_{i+1,j} - u_{i,j})^2 \\ &+ \frac{b}{2}(u_{i+1,j+1} - u_{i+1,j})^2 + \frac{b}{2}(u_{i,j+1} - u_{i,j})^2 \\ &+ (u_{i+1,j+1} - u_{i,j})^2 + (u_{i+1,j} - u_{i,j+1})^2. \end{aligned} \tag{2.5}$$

Note that $S_{i,j}$ reflects the sum of two diagonal differences and the half of four axis-parallel differences in $E_{i,j}$. Equation (2.4) means that $D_{k-1}(u,u)$ can be written as the sum of local cell differences $S_{i,j}$. Set $v = I_{k-1}^{k} u$. Then, similar to derive (2.4), we obtain

$$D_k(v,v) = \frac{1}{6}\sum_{E_{i,j}} (s_{i,j} + s_{i+1/2,j} + s_{i,j+1/2} + s_{i+1/2,j+1/2})$$

where $s_{i,j}$, $s_{i+1/2,j}$, $s_{i,j+1/2}$, and $s_{i+1/2,j+1/2}$ are defined as in case of $S_{i,j}$. By definition of the prolongation operator, we see

$$\begin{aligned} s_{i,j} =& \frac{a}{8}(u_{i+1,j} - u_{i,j})^2 + \frac{a}{32}(u_{i+1,j+1} + u_{i+1,j} - u_{i,j} - u_{i,j+1})^2 \\ &+ \frac{b}{8}(u_{i,j+1} - u_{i,j})^2 + \frac{b}{32}(u_{i+1,j+1} + u_{i,j+1} - u_{i+1,j} - u_{i,j})^2 \\ &+ \frac{1}{4}(u_{i+1,j+1} - u_{i,j})^2 + \frac{1}{16}(3u_{i+1,j} - u_{i+1,j+1} - u_{i,j+1} - u_{i,j})^2. \end{aligned} \tag{2.6}$$

Similarly, $s_{i+1/2,j}, s_{i+1/2,j+1/2}$ and $s_{i,j+1/2}$ are given as follows:

$$\begin{aligned} s_{i+1/2,j} =& \frac{a}{8}(u_{i+1,j} - u_{i,j})^2 + \frac{a}{32}(u_{i+1,j+1} + u_{i+1,j} - u_{i,j} - u_{i,j+1})^2 \\ &+ \frac{b}{8}(u_{i+1,j+1} - u_{i+1,j})^2 + \frac{b}{32}(u_{i+1,j+1} + u_{i,j+1} - u_{i+1,j} - u_{i,j})^2 \\ &+ \frac{1}{4}(u_{i+1,j+1} - u_{i,j})^2 + \frac{1}{16}(3u_{i+1,j} - u_{i+1,j+1} - u_{i,j+1} - u_{i,j})^2 \end{aligned} \tag{2.7}$$

$$\begin{aligned} s_{i+1/2,j+1/2} =& \frac{a}{8}(u_{i+1,j+1} - u_{i,j+1})^2 + \frac{a}{32}(u_{i+1,j+1} + u_{i+1,j} - u_{i,j+1} - u_{i,j})^2 \\ &+ \frac{b}{8}(u_{i+1,j+1} - u_{i+1,j})^2 + \frac{b}{32}(u_{i+1,j+1} + u_{i,j+1} - u_{i+1,j} - u_{i,j})^2 \\ &+ \frac{1}{4}(u_{i,j+1} - u_{i+1,j})^2 + \frac{1}{16}(3u_{i+1,j+1} - u_{i,j+1} - u_{i+1,j} - u_{i,j})^2 \end{aligned} \tag{2.8}$$

$$\begin{aligned} s_{i,j+1/2} = {} & \frac{a}{8}(u_{i+1,j+1} - u_{i,j+1})^2 + \frac{a}{32}(u_{i+1,j+1} + u_{i+1,j} - u_{i,j+1} - u_{i,j})^2 \\ & + \frac{b}{8}(u_{i,j+1} - u_{i,j})^2 + \frac{b}{32}(u_{i+1,j+1} + u_{i,j+1} - u_{i+1,j} - u_{i,j})^2 \\ & + \frac{1}{4}(u_{i+1,j+1} - u_{i,j})^2 + \frac{1}{16}(3u_{i,j+1} - u_{i+1,j+1} - u_{i+1,j} - u_{i,j})^2. \end{aligned} \tag{2.9}$$

We compare $S_{i,j}$ with the sum of $s_{i,j} + s_{i+1/2,j} + s_{i,j+1/2} + s_{i+1/2,j+1/2}$. In this way, (2.2) will be proved, if we show the inequality $S_{i,j} - s_{i,j} - s_{i+1/2,j} - s_{i,j+1/2} - s_{i+1/2,j+1/2} \geqslant 0$. Summing up (2.6) through (2.9), we have

$$\begin{aligned} & s_{i,j} + s_{i+1/2,j} + s_{i+1/2,j+1/2} + s_{i,j+1/2} \\ & \quad = \frac{a}{4}(u_{i+1,j+1} - u_{i,j+1})^2 + \frac{a}{4}(u_{i+1,j} - u_{i,j})^2 \\ & \qquad + \frac{a}{8}(u_{i+1,j+1} + u_{i+1,j} - u_{i,j+1} - u_{i,j})^2 \\ & \qquad + \frac{b}{4}(u_{i+1,j+1} - u_{i+1,j})^2 + \frac{b}{4}(u_{i,j+1} - u_{i,j})^2 \\ & \qquad + \frac{b}{8}(u_{i+1,j+1} + u_{i,j+1} - u_{i+1,j} - u_{i,j})^2 \\ & \qquad + \frac{1}{2}(u_{i,j+1} - u_{i+1,j})^2 + \frac{1}{2}(u_{i+1,j+1} - u_{i,j})^2 \\ & \qquad + \frac{1}{16}(3u_{i,j} - u_{i+1,j+1} - u_{i+1,j} - u_{i,j+1})^2 \\ & \qquad + \frac{1}{16}(3u_{i+1,j} - u_{i+1,j+1} - u_{i,j+1} - u_{i,j})^2 \\ & \qquad + \frac{1}{16}(3u_{i+1,j+1} - u_{i,j+1} - u_{i+1,j} - u_{i,j})^2 \\ & \qquad + \frac{1}{16}(3u_{i,j+1} - u_{i+1,j+1} - u_{i+1,j} - u_{i,j})^2. \end{aligned} \tag{2.10}$$

Note that the last four terms can be simplified as follows:

$$\begin{aligned} & \frac{1}{4}(u_{i+1,j+1} - u_{i,j+1})^2 + \frac{1}{4}(u_{i,j+1} - u_{i,j})^2 + \frac{1}{4}(u_{i+1,j} - u_{i,j})^2 \\ & \quad + \frac{1}{4}(u_{i+1,j+1} - u_{i+1,j})^2 + \frac{1}{4}(u_{i+1,j+1} - u_{i,j})^2 + \frac{1}{4}(u_{i+1,j} - u_{i,j+1})^2. \end{aligned} \tag{2.11}$$

Subtracting the first three terms in (2.10) from the first two terms in (2.5), we see the difference is as follows:

$$\begin{aligned}
&\frac{a}{2}(u_{i+1,j+1}-u_{i,j+1})^2+\frac{a}{2}(u_{i+1,j}-u_{i,j})^2-\Big[\frac{a}{4}(u_{i+1,j+1}-u_{i,j+1})^2\\
&\qquad+\frac{a}{4}(u_{i+1,j}-u_{i,j})^2+\frac{a}{8}(u_{i+1,j+1}+u_{i+1,j}-u_{i,j+1}-u_{i,j})^2\Big]\\
&=\frac{a}{4}(u_{i+1,j+1}-u_{i,j+1})^2+\frac{a}{4}(u_{i+1,j}-u_{i,j})^2-\frac{a}{8}(u_{i+1,j+1}-u_{i,j+1}+u_{i+1,j}-u_{i,j})^2\\
&=\frac{a}{8}(u_{i+1,j+1}-u_{i,j+1}-u_{i+1,j}+u_{i,j})^2.
\end{aligned} \tag{2.12}$$

Similarly, the terms regarding b become

$$\begin{aligned}
&\frac{b}{2}(u_{i+1,j+1}-u_{i+1,j})^2+\frac{b}{2}(u_{i,j+1}-u_{i,j})^2-\Bigg[\frac{b}{4}(u_{i+1,j+1}-u_{i+1,j})^2\\
&\qquad+\frac{b}{4}(u_{i,j+1}-u_{i,j})^2+\frac{b}{8}(u_{i+1,j+1}+u_{i,j+1}-u_{i+1,j}-u_{i,j})^2\Bigg]\\
&=\frac{b}{8}(u_{i+1,j+1}-u_{i,j+1}-u_{i+1,j}+u_{i,j})^2.
\end{aligned} \tag{2.13}$$

Noting that $a+b=8$, the sum of (2.12) and (2.13) is equal to

$$(u_{i+1,j+1}-u_{i,j+1}-u_{i+1,j}+u_{i,j})^2. \tag{2.14}$$

By elementary algebra, with (2.11) and (2.14), we see

$$\begin{aligned}
S_{i,j}&-s_{i,j}-s_{i+1/2,j}-s_{i,j+1/2}-s_{i+1/2,j+1/2}\\
&=(u_{i+1,j+1}-u_{i,j+1}-u_{i+1,j}+u_{i,j})^2\\
&\quad+\frac{1}{4}(u_{i+1,j+1}-u_{i,j})^2+\frac{1}{4}(u_{i+1,j}-u_{i,j+1})^2\\
&\quad-\frac{1}{4}(u_{i+1,j+1}-u_{i,j+1})^2-\frac{1}{4}(u_{i,j+1}-u_{i,j})^2\\
&\quad-\frac{1}{4}(u_{i+1,j}-u_{i,j})^2-\frac{1}{4}(u_{i+1,j+1}-u_{i+1,j})^2\\
&=\frac{3}{4}(u_{i,j}-u_{i+1,j}+u_{i+1,j+1}-u_{i,j+1})^2\geqslant 0.
\end{aligned}$$

This completes the proof. □

Lemma 2.2. *For a nonnegative constant q, we have*

$$q_k(I_{k-1}^k u, I_{k-1}^k u)\leqslant q_{k-1}(u,u)\quad \forall u\in V_{k-1}.$$

Proof. For simplicity, we let $q = 1$. Using symmetry, we have

$$\begin{aligned} q_{k-1}(u,u) &= h_x^{k-1} h_y^{k-1} \sum_{i,j} \left[8u_{i,j} + u_{i-1,j} + u_{i,j-1} + u_{i+1,j} + u_{i,j+1}) \right] u_{i,j} \\ &= h_x^{k-1} h_y^{k-1} \sum_{i,j} \left[4u_{i,j}^2 + (u_{i,j} + u_{i-1,j})^2/2 + (u_{i,j} + u_{i,j-1})^2/2 \right. \\ &\qquad \left. + (u_{i,j} + u_{i+1,j})^2/2 + (u_{i,j} + u_{i,j+1})^2/2 \right] \\ &= h_x^{k-1} h_y^{k-1} \sum_{E_{i,j}} \left[u_{i,j}^2 + u_{i+1,j}^2 + u_{i,j+1}^2 + u_{i+1,j+1}^2 \right. \\ &\qquad + (u_{i,j} + u_{i-1,j})^2/2 + (u_{i,j} + u_{i,j-1})^2/2 \\ &\qquad \left. + (u_{i,j} + u_{i+1,j})^2/2 + (u_{i,j} + u_{i,j+1})^2/2 \right]. \end{aligned}$$

Writing this cell-wise as in the proof Lemma 2.1, we have

$$q_{k-1}(u,u) = h_x^{k-1} h_y^{k-1} \sum_{E_{i,j}} T_{i,j}$$

where

$$\begin{aligned} T_{i,j} = {} & u_{i,j}^2 + u_{i+1,j}^2 + u_{i,j+1}^2 + u_{i+1,j+1}^2 \\ & + (u_{i,j} + u_{i-1,j})^2/2 + (u_{i,j} + u_{i,j-1})^2/2 \\ & + (u_{i,j} + u_{i+1,j})^2/2 + (u_{i,j} + u_{i,j+1})^2/2. \end{aligned}$$

Set $v = I_{k-1}^k u$ then, similarly, we have

$$q_k(v,v) = h_x^k h_y^k \sum_{E_{i,j}} (t_{i,j} + t_{i+1/2,j} + t_{i,j+1/2} + t_{i+1/2,j+1/2})$$

where $t_{i,j}$, $t_{i+1/2,j}$, $t_{i,j+1/2}$, $t_{i+1/2,j+1/2}$ are defined as in the case of $T_{i,j}$. To complete the proof, noting that $h_x^{k-1} h_y^{k-1} = 4 h_x^k h_y^k$, we only need to show the following inequality

$$t_{i,j} + t_{i+1/2,j} + t_{i,j+1/2} + t_{i+1/2,j+1/2} \leqslant 4T_{i,j}$$

which is a result of some algebraic calculation

$$\begin{aligned} & 4T_{i,j} - (t_{i,j} + t_{i+1/2,j} + t_{i,j+1/2} + t_{i+1/2,j+1/2}) \\ &\quad = (u_{i,j} - u_{i+1,j+1})^2 + (u_{i+1,j} - u_{i,j+1})^2 \\ &\qquad + \frac{1}{2}(u_{i,j} - u_{i+1,j})^2 + \frac{1}{2}(u_{i+1,j} - u_{i+1,j+1})^2 \\ &\qquad + \frac{1}{2}(u_{i+1,j+1} - u_{i,j+1})^2 + \frac{1}{2}(u_{i,j+1} - u_{i,j})^2 \geqslant 0. \end{aligned}$$

This completes the proof. □

To prove the multigrid convergence theory, we need the following property, so-called, “approximation and regularity”: There exist a number $0 < \alpha \leqslant 1$ and a constant C_α such that for all $k = 1, \ldots, J$,

$$A_k((I - I_{k-1}^k P_{k-1})u, u) \leqslant C_\alpha \left(\frac{\|A_k u\|_k^2}{\lambda_k} \right)^\alpha A_k(u,u)^{1-\alpha} \quad \forall u \in V_k\,. \tag{2.15}$$

Here, λ_k is the largest eigenvalue of A_k and P_{k-1} is the elliptic projection defined by

$$A_{k-1}(P_{k-1}u, v) = A_k(u, I_{k-1}^k v) \qquad \forall u \in V_k,\ v \in V_{k-1}. \tag{2.16}$$

The following lemma can be proved in a similar manner as presented in [8].

Lemma 2.3. *Let the operator P_{k-1} be defined by* (2.16). *Then* (2.15) *holds for* $\alpha = 1/2$.

Now we state the main result. Because of Theorem 2.1 and Lemma 2.3, we can employ the framework presented in [2] to obtain the following convergence theorem.

Theorem 2.2. *Let $E_k = I - B_k A_k$ in algorithm V(m,m). Then we have*

$$A_k(E_k u, u) \leqslant \delta_k A_k(u,u) \quad \forall u \in V_k$$

where $\delta_k = Ck/(Ck + \sqrt{m})$.

3. NUMERICAL EXPERIMENTS

We consider the following Dirichlet problem on the unit square:

$$\begin{aligned} -\triangle u + qu &= f \qquad \text{in } \Omega = (0,1)^2 \\ u &= \varphi \qquad \text{on } \partial\Omega. \end{aligned}$$

First, we report the maximum, minimum eigenvalues, condition numbers and contractions numbers of $V(1,1)$-cycle algorithms for the model problem with $q = 0$. Here, we use Gauss-Seidel smoothing and 9-point interpolation. We see that the number of $V(1,1)$-cycle iterations is about 7 or 8 to obtain machine accuracy. The contraction number δ was estimated as $\delta = (A_k(E_k^N u, u)/A_k(u,u))^{1/N}$ when $N \approx 15$. When the multigrid algorithm is used as a preconditioner, it is known that the contraction number is asymptotically

$$\frac{\sqrt{K} - 1}{\sqrt{K} + 1} \approx 0.05.$$

Table 1.
Higher order scheme for the model problem with $q = 0$.

h^J	$\lambda_{\min}$	$\lambda_{\max}$	K	δ
$1/16$	0.826	0.999	1.211	0.119
$1/32$	0.823	0.999	1.215	0.123
$1/64$	0.823	0.999	1.215	0.125
$1/128$	0.822	0.999	1.216	0.126

Table 2.
5-point scheme for the model problem with $q = 0$.

h^J	$\lambda_{\min}$	$\lambda_{\max}$	K	δ
$1/16$	0.776	0.999	1.289	0.167
$1/32$	0.765	0.999	1.307	0.173
$1/64$	0.758	0.999	1.316	0.175
$1/128$	0.756	0.999	1.321	0.176

For comparison, we also list the result of the 5-point stencil. Table 1 and 2 show that the multigrid algorithm for the higher order scheme converges faster than that for the 5-point scheme. Both algorithms have contraction numbers independent of the mesh size h and the number of levels J.

Next, we test the model problem with $q = 10$. In this case, the V-cycle still converges as fast as in the case of the pure Laplace problem (see Tables 3 and 4). For the last test, we simulate the algorithms when $h_x \neq h_y$ $(a/b \neq 1)$. The analysis in [13,14] shows that Gauss-Seidel smoothing works well in the case of $a/b \approx 1$,

Table 3.
Higher order scheme for the model problem with $q = 10$.

h^J	$\lambda_{\min}$	$\lambda_{\max}$	K	δ
$1/16$	0.830	0.999	1.204	0.112
$1/32$	0.824	0.999	1.213	0.118
$1/64$	0.823	0.999	1.216	0.122
$1/128$	0.822	0.999	1.216	0.125

Table 4.
5-point scheme for the model problem with $q = 10$.

h^J	$\lambda_{\min}$	$\lambda_{\max}$	K	δ
$1/16$	0.783	0.999	1.276	0.161
$1/32$	0.770	0.999	1.298	0.169
$1/64$	0.761	0.999	1.314	0.173
$1/128$	0.758	0.999	1.319	0.175

Table 5.
Higher order scheme with line smoother.

$\vartheta = h_y^J/h_x^J$	$\lambda_{\min}$	$\lambda_{\max}$	K	δ
$1/2$	0.802	0.999	1.247	0.151
$1/4$	0.802	0.999	1.247	0.147
$1/8$	0.801	0.999	1.247	0.143

in other words, $\vartheta \approx 1$. But when ϑ goes to ∞ or 0, it does not reduce the high frequency error and the multigrid convergence is not as good as in the case of $\vartheta \approx 1$. In this case, it is known [3] that the x-axis or the y-axis line smoother is necessary. We report the result with varying $\vartheta = 1/2, 1/4, 1/8$ in Table 5.

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