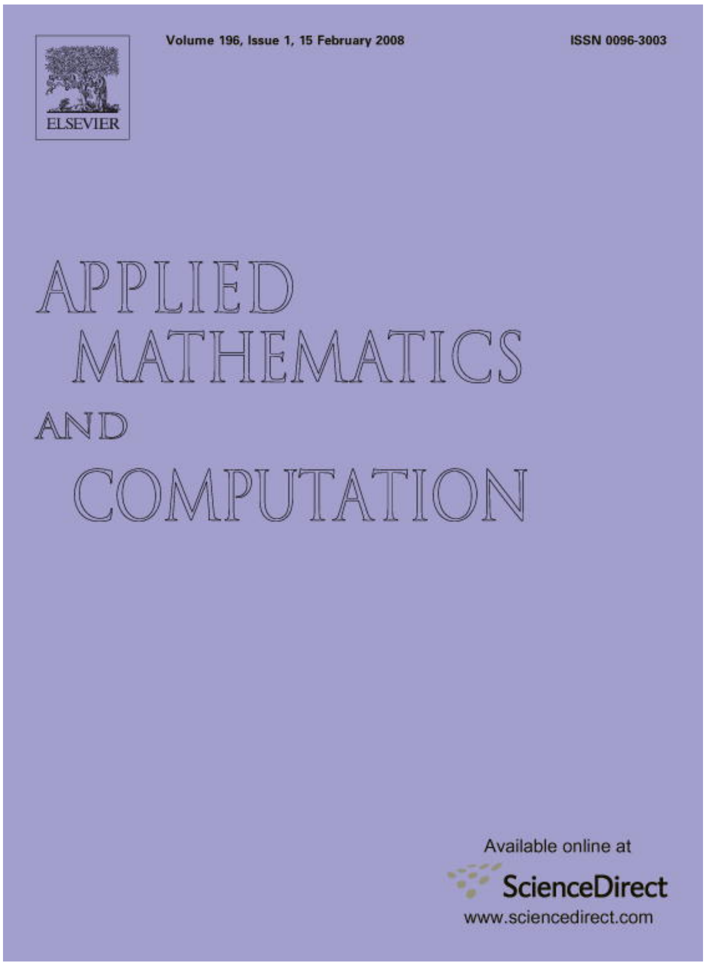
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A nonconforming covolume method for elliptic problems

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Abstract

We consider a control volume (covolume) method for second-order elliptic PDEs with the rotated-Q_1 nonconforming finite element on rectangular grids. The coefficient κ may a variable, diagonal tensor, or discontinuous. We prove first-order convergence in H^1 norm and second order convergence in L^2 norm when the partition is square. Our numerical experiments show that our covolume scheme has about 30% less error than FEM even when κ is discontinuous tensor.

Keywords: Covolume method; Rotated bilinear finite element method; Duality; Optimal order

1. Introduction

In this paper, we consider a covolume scheme with Q_1 nonconforming finite element method (FEM) for the second-order elliptic equations with tensor coefficients. We prove first-order convergence in H^1 norm and second-order convergence in L^2 norm. This scheme was first introduced for Stokes problems in [3,6]. We apply it to elliptic problems with variable, discontinuous, and tensor coefficient. For other type of finite volume methods, we refer to [1,2,4–7,11–13,15] and for finite element method, we refer to [8,9,14].

The analysis of these covolume methods can be well described if we introduce a transfer operator γ from usual FEM space to the space of piecewise constant. As a result, the scheme can be viewed as a Galerkin scheme rather than a Petrov–Galerkin scheme.

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ with the boundary $\partial\Omega$. We consider the following second-order elliptic boundary value problem:

$$\begin{aligned} -\mathrm{div}(\kappa\nabla u) &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\kappa = \kappa(x) := \mathrm{diag}(\kappa_1(x), \kappa_2(x))$ is a diagonal and uniformly positive definite matrix. For Q_1 nonconforming finite element method, we only consider the case where Ω is a union of axi-parallel domain. Let $h > 0$ be a

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parameter and let $\mathcal{T}_h = \{K\}$ be a regular family of partition into rectangles of Ω in the sense that there exist constants $c > 0$, $C > 0$ such that

$$ch^2 \leqslant |K| \leqslant Ch^2 \quad \text{for all} \quad K \in \mathcal{T}_h.$$

Now we describe a Q_1 nonconforming finite element method. Let $S_K = \{v_h = a + bx_1 + cx_2 + d(x_1^2 - x_2^2) \text{ on } K\}$ and let

$$N_h = \left\{ v_h|_K \in S_K, \int_{K\cap\partial K} v_h \, d\sigma = \int_{K'\cap\partial K'} v_h \, d\sigma \quad \text{if } K, K' \text{are adjacent, and } \int_{\partial K\cap\partial\Omega} v_h \, d\sigma = 0 \right\}. \tag{2}$$

The usual nonconforming variational formulation of (1) is defined through element-wise form, i.e, we let

$$a_h(v_h, w_h) = \sum_K (\kappa \nabla v_h, \nabla w_h)_K, \quad \text{for } v_h, w_h \in N_h.$$

Let $\|\cdot\|_{0,D}$ (dropping D when $D = \Omega$) denote the usual $L^2(D)$ norm on a domain D; $D = K$ or Ω, ∂K, $\partial\Omega$, etc., and $|\cdot|_{1,h} = |a_h(\cdot,\cdot)|^{1/2}$ be the discrete energy norm induced by $a_h(\cdot,\cdot)$. Then it is well-known [14,4,12,8] that the nonconforming finite element solution defined by

$$a_h(\tilde{u}_h, v_h) = (f, v_h), \quad \text{for all } v_h \in N_h,$$

satisfies

$$\|u - \tilde{u}_h\|_0 + h|u - \tilde{u}_h|_1 \leqslant Ch^2 \|f\|_0. \tag{3}$$

Now we consider a covolume formulation of the problem (1). For that purpose, we need to subdivide the given partition by connecting the vertices of each element with its center C, resulting in four subtriangles. Now the region consisting of two adjacent triangles sharing a common edge is denoted by K^*, called a covolume. The midpoints of edges are denoted by $m_i, i = 1, \ldots, 4$. To define the covolume method, we need another space (Fig. 1)

$$W_h = \{v_h|_{K^*} \quad \text{is constant on each } K^*, \text{ and 0 on the boundary covolumes}\} \tag{4}$$

and an operator connecting N_h to W_h. Let K_j^* be the covolume with associated edge e_j, and m_j denote its mid point. Let $\bar{v}_h = \frac{1}{|e_j|}\int_{e_j} v_h d\sigma$ be the average of v_h on the edge e_j. Then we introduce a transfer operator $\gamma : N_h \to W_h$ by

$$\gamma v_h|_{K^*} = \sum_{K_j^*} \bar{v}_h(m_j)\chi_j(x),$$

where χ_j is the characteristic function of K_j^*.

Find $u_h^* \in N_h$ such that

$$a_h^*(u_h^*, v_h) = (f, \gamma v_h) \quad \text{for all } v_h \in N_h, \tag{5}$$

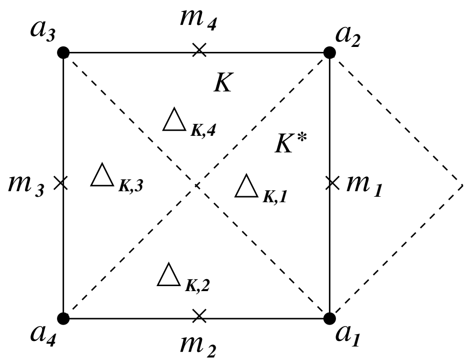

Fig. 1. A typical element K and its covolume partition K^*.

where

$$a_h^*(v_h, z_h) = -\sum_{K^*} \int_{\partial K^*} \kappa \nabla v_h \cdot \mathbf{n} \gamma z_h \, d\sigma, \quad \text{for } v_h, z_h \in N_h. \tag{6}$$

Then the covolume scheme can be regard as a Galerkin method.

Now we show an optimal order error estimate.

2. Constant coefficient

In this section, we shall prove error estimates in case of constant tensor coefficient, possibly discontinuous.

Lemma 2.1. *When κ is constant, we have*

$$a_h^*(z_h, v_h) = a_h(z_h, v_h) \tag{7}$$

for all $z_h, v_h \in N_h$.

Proof. First arranging the terms with respect to K, we see

$$\begin{aligned} a_h^*(z_h, v_h) &= -\sum_{K^*} \int_{\partial K^*} \kappa \nabla z_h \cdot \mathbf{n} \bar{v}_h(m_i) \, d\sigma \\ &= -\sum_K \sum_{j=1}^4 \bar{v}_h(m_j) \int_{a_{j+1} c a_j} \kappa \nabla z_h \cdot \mathbf{n} \, d\sigma \\ &\quad \text{since } \kappa \text{ is constant, we have by divergence theorem} \\ &= -\sum_K \sum_{j=1}^4 \bar{v}_h(m_j) \left[\int_{\partial \Delta_{K,j}} \kappa \nabla z_h \cdot \mathbf{n} \, d\sigma - \int_{a_j a_{j+1}} \kappa \nabla z_h \cdot \mathbf{n} \, d\sigma \right] \\ &= \sum_K \sum_{j=1}^4 \bar{v}_h(m_j) \int_{a_j a_{j+1}} \kappa \nabla z_h \cdot \mathbf{n} \, d\sigma \\ &\quad \text{rearranging terms on each edges of } K \\ &= -\sum_K \sum_{j=1}^2 \left[\bar{v}_h(m_{j+2}) \int_{a_{j+2} a_{j+3}} \kappa_j \frac{\partial z_h}{\partial x_j} \, d\sigma - \bar{v}_h(m_j) \int_{a_j a_{j+1}} \kappa_j \frac{\partial z_h}{\partial x_j} \, d\sigma \right]. \end{aligned}$$

Here we adopted the convention that $a_{j+4} = a_j$, $j = 1, \ldots, 4$. Now fix a K and let us look at terms in each sum. Since $\frac{\partial z_h}{\partial x_1}$ is constant in x_2 and $\frac{\partial z_h}{\partial x_2}$ is constant in x_1, we have using the notation $|e_j|$ = the length of the edge $a_j a_{j+1}$ or $a_{j+2} a_{j+3}$,

$$\begin{aligned} &-\sum_{j=1}^2 \left[\bar{v}_h(m_{j+2}) \int_{a_{j+2} a_{j+3}} \kappa_j \frac{\partial z_h}{\partial x_j} \, d\sigma - \bar{v}_h(m_j) \int_{a_j a_{j+1}} \kappa_j \frac{\partial z_h}{\partial x_j} \, d\sigma \right] \\ &= -\sum_{j=1}^2 |e_j| \left(\kappa_j \frac{\partial z_h}{\partial x_j} \bar{v}_h(m_{j+2}) - \kappa_j \frac{\partial z_h}{\partial x_j} \bar{v}_h(m_j) \right) \\ &= -\sum_{j=1}^2 \left(\int_{a_{j+2} a_{j+3}} \kappa_j \frac{\partial z_h}{\partial x_j} v_h \, d\sigma - \int_{a_j a_{j+1}} \kappa_j \frac{\partial z_h}{\partial x_j} v_h \, d\sigma \right) = \int_{\partial K} \kappa \nabla z_h \cdot \mathbf{n} v_h \, d\sigma = \int_K \kappa \nabla z_h \cdot \nabla v_h \, d\mathbf{x}. \end{aligned}$$

The last equality is obtained by divergence theorem and the fact that v_h is harmonic and κ is constant on K. Now summing over all K, we obtain $a_h(z_h, v_h)$ which completes the proof. □

Now we show H^1 error estimate. Since the usual Q_1 nonconforming finite element solution $\tilde{u}_h$ satisfies

$$a_h(\tilde{u}_h, v_h) = \sum_K (\kappa \nabla u_h, \nabla v_h)_K = (f, v_h), \quad v_h \in N_h \tag{8}$$

and by above lemma the covolume solution satisfies

$$a_h(u_h^*, v_h) = (f, \gamma v_h), \quad v_h \in N_h. \tag{9}$$

Subtracting,

$$a_h(\tilde{u}_h - u_h^*, v_h) = (f, (I-\gamma)v_h), \quad v_h \in N_h. \tag{10}$$

Set $v_h = \tilde{u}_h - u_h^*$. Then by coerciveness,

$$|\tilde{u}_h - u_h^*|_{1,h}^2 \leqslant C\|f\|_0\|(I-\gamma)(\tilde{u}_h - u_h^*)\|_0 \leqslant Ch\|f\|_0|\tilde{u}_h - u_h^*|_{1,h}. \tag{11}$$

Hence

$$|\tilde{u}_h - u_h^*|_{1,h} \leqslant Ch\|f\|_0. \tag{12}$$

Thus, we have the following result for an H^1 error analysis.

Theorem 2.2. *Let $u \in H^2(\Omega)$ be the solution of problem (1), and let u_h^* be the solution of the covolume (5). Then*

$$|u - u_h^*|_{1,h} \leqslant Ch\|f\|_0. \tag{13}$$

Proof. Let $\tilde{u}_h$ be the Q_1 nonconforming solution. Then by (3) and (12)

$$\begin{aligned}|u - u_h^*|_{1,h} &\leqslant |u - \tilde{u}_h|_{1,h} + |\tilde{u}_h - u_h^*|_{1,h} \\ &\leqslant Ch(\|u\|_2 + \|f\|_0) \leqslant Ch\|f\|_0. \quad \square\end{aligned} \tag{14}$$

2.1. Duality

For L^2 error estimate we use the duality argument. Let $e_h = u - u_h^*$ and consider

$$\begin{aligned}&-\operatorname{div}(\kappa\nabla U) = e_h \quad \text{in } \Omega, \\ &U = 0 \quad \text{on } \partial\Omega.\end{aligned} \tag{15}$$

Then the regularity of the problem (15) implies that $\|U\|_2 \leqslant C\|e_h\|_0$. Multiply (15) by e_h and integrate, we obtain

$$\|e_h\|_0^2 = (-\operatorname{div}(\kappa\nabla U), e_h) = a_h(U, e_h) - \sum_{K\in\mathcal{T}_h}\left\langle \kappa\frac{\partial U}{\partial \nu}, e_h\right\rangle_{\partial K\setminus\partial\Omega}. \tag{16}$$

We present the proof when the domain is partitioned into squares.

Lemma 2.3. *For any $v \in H^2(\Omega)$, $\psi_h \in N_h$, and $\psi \in C^0(\Omega) \cap H_0^1(\Omega)$, we have*

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K} v(\psi_h - \bar{\psi}_h)\,\mathrm{d}\sigma \leqslant Ch|v|_{1,h}|\psi_h - \psi|_{1,h}. \tag{17}$$

The proof of this lemma can be found in [10].

Lemma 2.4. *Suppose the solution u of (1) belongs to $H^3(\Omega)$ and u_h^* is the solution of covolume scheme. Then for any $\psi_h \in N_h$, and $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$, we have*

$$a_h(u - u_h^*, \psi_h) \leqslant Ch\|f\|_0|\psi - \psi_h|_{1,h} + Ch^2\|u\|_3|\psi_h|_{1,h} + Ch^2\|f\|_0|\psi|_2. \tag{18}$$

The proof of this lemma essentially follows from Lemma 3.8 of [12] as long as we use

$$\int_K (\psi_h - \gamma\psi_h)dx = 0, \quad \text{for} \quad \psi_h \in N_h,$$

which holds when K is a square.

Now, for the error bound of $a_h(U, e_h)$ in (16) we need the following lemma:

Lemma 2.5. *Let u be the solution of the problem (1). Assume that $u \in H^3(\Omega)$. Then for any $\psi \in H^2(\Omega)$, $\psi_h \in N_h$, we have*

$$a_h(\psi, u - u_h^*) \leqslant C|\psi - \psi_h|_{1,h}|u - u_h^*|_{1,h} + Ch\|f\|_0|\psi_h - \psi|_{1,h} + Ch^2\|u\|_3|\psi_h|_{1,h} + Ch^2\|f\|_0|\psi|_2. \quad (19)$$

Proof. Since $a_h(\cdot,\cdot)$ is symmetric, we see that

$$a_h(\psi, u - u_h^*) = a_h(\psi - \psi_h, u - u_h^*) + a_h(u - u_h^*, \psi_h). \quad (20)$$

Now, the estimate of the second term is obtained from Lemma 2.4. Since the first term of (20) can be estimated trivially, we complete proof. □

Now we can prove the following theorem:

Theorem 2.6. *Let u be the solution of problem (1). Let u_h^* be the solution of the (5). Further, assume that u is in $H^3(\Omega)$. Then*

$$\|u - u_h^*\|_0 \leqslant Ch^2(\|u\|_3 + \|f\|_0). \quad (21)$$

Proof. Let U_h be the usual nonconforming FEM solution of (15), and let $\psi = U$ in Lemma 2.5. Then the first term in right hand side of (16) satisfies

$$\begin{aligned}|a_h(U, e_h)| &\leqslant C(|U - U_h|_{1,h}|e_h|_{1,h} + Ch\|f\|_0|U_h - U|_{1,h} + Ch^2\|u\|_3|U_h|_{1,h} + Ch^2\|f\|_0|U|_2) \\ &\leqslant Ch^2\|U\|_2(|f\|_0 + \|u\|_3) \leqslant Ch^2\|e_h\|_0(\|f\|_0 + \|u\|_3).\end{aligned} \quad (22)$$

The second term in (16) can be estimated by Lemma 2.3 as

$$\left|\sum_{K\in\mathcal{T}_h}\left\langle \kappa\frac{\partial U}{\partial v}, e_h\right\rangle_{\partial K\setminus\partial\Omega}\right| \leqslant Ch\|U\|_2|e_h|_{1,h} \leqslant Ch\|e_h\|_0|e_h|_{1,h} \leqslant Ch^2\|e_h\|_0\|f\|_0. \quad (23)$$

From (22), (23), we obtain (21). □

3. Nonconstant coefficient case

In this section, we define a covolume scheme using the projected coefficient and prove the optimal error estimates when κ is of class $C^1(\overline{K})$ for all element K. We do this by comparing the finite element solution with our covolume solution with projected coefficient. Let $\tilde{u}_h$ satisfy

$$\sum_{K\in\mathcal{T}_h}(\kappa\nabla\tilde{u}_h, \nabla v_h)_K = (f, v_h)_\Omega, \quad v_h \in N_h \quad (24)$$

while u_h^* satisfy

$$\sum_{K\in\mathcal{T}_h}(\bar{\kappa}\nabla u_h^*, \nabla v_h)_K = (f, \gamma v_h)_\Omega, \quad v_h \in N_h, \quad (25)$$

where $\bar{\kappa} = \frac{1}{|K|}\int_K \kappa\,dx$. Then we have

Lemma 3.1. *Let $u \in H^2(\Omega)$ be the solution of problem (1), and let $\tilde{u}_h$, and u_h^* be the solution of (24) and (25), respectively. Then we have*

$$\|u_h^* - \tilde{u}_h\|_{1,h} \leqslant Ch\|f\|_0. \quad (26)$$

Proof. Subtracting (24) and (25), then we have

$$\sum_{K\in\mathcal{T}_h}(\bar{\kappa}\nabla(\tilde{u}_h - u_h^*), \nabla v_h)_K = (f, (I-\gamma)v_h)_\Omega - \sum_K((\kappa - \bar{\kappa})\nabla\tilde{u}_h, \nabla v_h)_K.$$

Table 1
Errors and orders of convergence for the rectangular meshes with $\kappa = 1 + 10x + y$ on $\Omega = [0,1]^2$

	Nonconforming covolume				Nonconforming FEM			
	$\|u-u_h\|_{L^2(\Omega)}$		$\|u-u_h\|_{1,h}$		$\|u-u_h\|_{L^2(\Omega)}$		$\|u-u_h\|_{1,h}$	
n	Error	Order	Error	Order	Error	Order	Error	Order
4	1.7827e−03		2.6063e−01		2.3529e−03		2.5284e−01	
8	4.5171e−04	1.99	1.3136e−01	0.99	6.0136e−04	1.96	1.3036e−01	0.95
16	1.1335e−04	1.99	6.5800e−02	0.99	1.5112e−04	1.99	6.5675e−02	0.98
32	2.8365e−05	1.99	3.2915e−02	0.99	3.7828e−05	1.99	3.2899e−02	0.99
64	7.0930e−06	1.99	1.6459e−02	0.99	9.4601e−06	1.99	1.6457e−02	0.99

Table 2
Errors and orders of convergence for the rectangular meshes with $\kappa = \mathrm{diag}(10,10)$ for $0 < x < 0.5$, $\kappa = diag(1,1)$ for $0.5 < x < 1$ on $\Omega = [0,1]^2$

	Nonconforming covolume				Nonconforming FEM			
	$\|u-u_h\|_{L^2(\Omega)}$		$\|u-u_h\|_{1,h}$		$\|u-u_h\|_{L^2(\Omega)}$		$\|u-u_h\|_{1,h}$	
n	Error	Order	Error	Order	Error	Order	Error	Order
4	1.3953e−02		3.2646e−01		1.9137e−02		3.1250e−01	
8	3.6163e−03	1.94	1.6855e−01	0.95	5.0480e−03	1.92	1.6674e−01	0.90
16	9.1251e−04	1.98	8.4862e−02	0.99	1.2788e−03	1.98	8.4635e−02	0.97
32	2.2866e−04	1.99	4.2502e−02	0.99	3.2077e−04	1.99	4.2473e−02	0.99
64	5.7198e−05	1.99	2.1260e−02	0.99	8.0259e−05	1.99	2.1256e−02	0.99

Set $v_h = e_h := \tilde{u}_h - u_h^*$. Then

$$|e_h|_{1,h}^2 \leqslant Ch(\|f\|_0 + |\kappa|_{1,\infty}|\tilde{u}_h|_{1,h})|e_h|_{1,h}, \leqslant Ch(\|f\|_0 + \|u\|_2)|e_h|_{1,h} \leqslant Ch\|f\|_0|e_h|_{1,h}.$$

This completes proof. □

4. Numerical results

In this section, we present some numerical simulations which confirm our theory. We test various cases:

Example 1. Let $\overline{\Omega} = [0,1]^2$ be partitioned into $n_2(n = 2^k, k = 2,\ldots,6)$, square grids of size $h \times h$, and let $\kappa = 1 + 10x + y$. We compare with FEM solutions. The results are shown in Table 1. The L^2 error with covolume scheme smaller than FEM about 30%.

Example 2. In this example, we test discontinuous coefficient case on the unit square with n^2 square grids. Let

$$\kappa = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} \quad \text{for } 0 < x < 0.5, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for } 0.5 < x < 1,$$

where the exact solution is given as

$$u = \begin{cases} (-20x^2 + 30x - 10)(y - y^2) & \text{for } 0 < x < 0.5, \\ (2x^2 - x)(y - y^2) & \text{for } 0.5 < x < 1. \end{cases} \tag{27}$$

The results are shown in Table 2. The L^2 error with covolume scheme smaller than FEM about 30%.

References

[1] T. Arbogast, C.N. Dawson, P. Keenan, M.F. Wheeler, I. Yotov, Enhanced cell-centered finite differences for elliptic equations on general geometry, SIAM J. Sci. Comput. 19 (1998) 404–425.

[2] Z. Cai, S. McCormick, On the accuracy of the finite volume element method for diffusion equations on composite grids, SIAM J. Numer. Anal. 27 (1990) 636–656.
[3] S.H. Chou, Analysis and convergence of a covolume methods for the generalized Stokes problem, Math. Comp. 66 (1997) 85–104.
[4] S.H. Chou, D.Y. Kwak, Mixed Covolume Methods on Rectangular Grids for Elliptic Problems, SIAM J. Numer. Anal., 37(3), 758–771.
[5] S.H. Chou, D.Y. Kwak, Analysis and convergence of a MAC-like scheme for the generalized Stokes problem, Numer. Methods Part. Diff. Eq. 13 (1997) 1–16.
[6] S.H. Chou, D.Y. Kwak, A covolume method based on rotated bilinears for the generalized Stokes problem, SIAM J. Numer. Anal. 35 (1998) 494–507.
[7] S.H. Chou, D.Y. Kwak, P.S. Vassilevski, Mixed upwinding covolume methods on rectangular grids for convection-diffusion problems, SIAM J. Sci. Comput. 21 (1999) 145–165.
[8] So-Hsiang Chou, Do Y. Kwak, Kwang Y. Kim, Mixed finite volume methods on nonstaggered quadrilateral grids for elliptic problems, Math. Comput. 72 (242) (2002) 525–539.
[9] P.G. Cialet, The finite Element Method for Elliptic Equations, North-Holland, Amsterdam, 1978.
[10] M. Crouzeix, P.A. Raviart, Conforming and nonconforming finite element methods for solving the stationary Stokes equations, RAIRO Anal. Numer. 7 (1973) 33–76.
[11] R.E. Ewing, R.D. Lazarov, P.S. Vassilevski, Local refinement techniques for elliptic problems on cell-centered grids, I: Error analysis, Math. Comp. 56 (1991) 437–461.
[12] K.S. Kang, D.Y. Kwak, Error Estimate in L^2 of a Covolume Method for the Generalized Stokes Problem, Numer. Method P.D.E 22 (2005) 165–179.
[13] R.A. Nicolaides, Direct discretization of planar div–curl problems, SIAM J. Numer. Anal., 29 (1992) 32–56.
[14] R. Rannacher, S. Turek, Simple nonconforming quadrilateral Stokes element, Numer. Methods Part. Diff. Eq. 8 (1992) 97–111.
[15] T.F. Russell, Rigorous Block-centered Discretizations on Irregular Grids: Improved Simulation of Complex Reservoir Systems, Reservoir Simulation Research Corporation, Denver, Co, 1995.