Comparison of V-cycle Multigrid Method for Cell-centered Finite Difference on Triangular Meshes

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We consider a multigrid algorithm (MG) for the cell centered finite difference scheme (CCFD) on general triangular meshes using a new prolongation operator. This prolongation is designed to solve the diffusion equation with strongly discontinuous coefficient as well as with smooth one. We compare our new prolongation with the natural injection and the weighted operator in Kwak, Kwon, and Lee (Appl Math Comput 21 (1999), 552–564) and the behaviors of these three prolongation are discussed. Numerical experiments show that (i) for smooth problems, the multigrid with our new prolongation is fastest, the next is the weighted prolongation, and the third is the natural injection; and (ii) for nonsmooth problems, our new prolongation is again fastest, the next is the natural injection, and the third is the weighted prolongation. In conclusion, our new prolongation works better than the natural injection and the weighted operator for both smooth and nonsmooth problems. Numer Methods Partial Differential Eq 22: 1080–1089, 2006

Keywords: discontinuous coefficient; cell-centered finite difference methods; finite volume methods; multigrid methods

I. INTRODUCTION

The multigrid method has been widely used and proven to be effective for a large class of problems and has been the subject of extensive research [1, 3, 4, 6–8, 12]. One of the main idea of multigrid method is the use of multi-levels and the prolongation/restriction operators between these levels play key role in transferring vector components or errors. For conforming finite element methods, the finite element spaces corresponding to these levels are all nested, hence natural injections between levels can be used as a prolongation.

The multigrid method with this prolongation works well for various kinds of problems. But for nonconforming finite elements, in general, either there does not exist a natural injection, or the

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injection does not provide good convergence. The cell centered finite difference method, which is widely used because of its local mass conservation, is a nonconforming example. On rectangular grids, a few authors have studied the behavior of various multigrid methods for smooth and nonsmooth problems [5, 9, 10, 12]. Bramble et al. analyzed W-cycle multigrid convergence [3] with the natural injection as a prolongation, while Kwak showed V-cycle multigrid convergence with certain weighted prolongation [7]. On the other hand, on triangular grids, Kwak et al. in [8] proposed a weighted prolongation and numerical results show that the weighted prolongation converges faster than the natural injection for smooth problems. But the weighted prolongation does not show a good convergence for nonsmooth problems as we show in section 4. In fact, it is worse than the natural injection when the diffusion coefficient is strongly discontinuous. Additionally, the motivation of the weighted prolongation becomes vague when the triangles are not equilateral. In this article, we design a new prolongation that works for both smooth and nonsmooth problems and makes sense on more general triangulation. This prolongation is motivated by the continuity of flux and use P_1-nonconforming interpolation. Some prolongation operators based on the flux continuity have been considered and successful in accelerating convergence of multigrid algorithm [2, 9, 10]. This article is organized as follows: In section 2, we briefly introduce the cell-centered finite difference (CCFD) method on triangular grid and multigrid algorithms. In section 3, we propose a new prolongation operator and compare it with other prolongations for smooth and nonsmooth problems. In section 4, we test various problems and report the eigenvalues, condition numbers, and contraction numbers.

II. MULTIGRID ALGORITHM FOR THE CELL-CENTERED METHOD

In this section, we briefly describe CCFD and the multigrid algorithm. We consider the following model problem:

$$-\nabla \cdot p\nabla u = f, \qquad \text{in } \Omega, \tag{2.1}$$

$$u = 0, \qquad \text{on } \partial\Omega. \tag{2.2}$$

Let Ω be a unit parallelogram with unit side length. For $k = 1, 2, \ldots, J$, Ω is divided by $N_k := 2^{2k+1}$ equilateral triangles. Such triangulations are denoted by $\{\mathcal{T}_k\}$. Given a coarse triangulation $\{\mathcal{T}_k\}$, we connect midpoints of edges of triangles in $\{\mathcal{T}_k\}$ to obtain $\{\mathcal{T}_{k+1}\}$. Each triangle T in $\{\mathcal{T}_k\}$ is called a cell. The grid point of each cell is the circumcenter of the triangle. For $k = 1, 2, \ldots, J$, we let V_k denote the space of functions, which are piecewise constant on each cell.

The cell centered discretization is obtained as follows: Integrate the model equation (2.1) on each cell T_j and then use divergence theorem to get

$$-\int_{\partial T_j} p\frac{\partial u}{\partial n} = \int_{T_j} f\,dx, \tag{2.3}$$

for $j = 1, 2, \ldots, N_k$. For the sake of simplicity, we assume that p is piecewise constant on each triangles. We approximate the equation (2.3) using functions in V_k by central difference method. Let u in V_k and u_i denote its value on ith cell. For two adjacent triangles T_i and T_j, we let x_i and x_j be the circumcenters of T_i and T_j, respectively. We denote h by the length of the edge of a triangle and h' by $|x_i - x_j|$. Let $\theta = h/h'$. Then we take $p_{ij}\theta(u_j - u_i)$ as an approximation to $p\partial u/\partial n_{ij}$, where n_{ij} is the unit normal vector from the center of T_i to that of T_j and p_{ij} is the value of $p(x)$ at center of the common edge of T_i and T_j. When p is discontinuous across the common edge of T_i and T_j, p_{ij} is defined by the harmonic average of p_i and p_j. That is

$$p_{ij} = \frac{2p_i p_j}{p_i + p_j}.$$

Here, p_i and p_j are values of p at circumcenters of T_i and T_j, respectively. See [11] for the details. Note that there are only three directions of n_{ij}. When one of the edges coincides with the boundary of Ω, we assume a fictitious value by reflection. Collecting these, we have a system of algebraic equations

$$A_k \bar{u} = \bar{f}, \tag{2.4}$$

where A_k is symmetric positive definite, and $\bar{u}$ and $\bar{f}$ are vectors whose entries are u_j and the integral of f over T_j, respectively. Define a discrete L^2-inner product on V_k by

$$(v, w)_k = \sum_{i=1}^{N_k} h_k^2 v_i w_i, \qquad \forall v,\ w \in V_k,$$

where h_k denotes the length of edge of a triangle in $\{\mathcal{T}_k\}$. If we identify A_k with a quadratic form on $V_k \times V_k$ defined by

$$A_k(v, w) = (A_k v, w)_k,$$

then the problem (2.4) is equivalent to: Find $u \in V_k$ satisfying

$$A_k(\bar{u}, \phi) = (\bar{f}, \phi), \qquad \forall \phi \in V_k. \tag{2.5}$$

The error analysis for the triangular CCFD method has been presented by Vassilevsk et al. [11]. Let Q_k be the $L^2(\Omega)$ projection onto V_k. If u is the solution of (2.5), then

$$A_k(\bar{u} - Q_k u, \bar{u} - Q_k u) \le C h_k^2 \|f\|^2.$$

Now, we will describe the multigrid algorithm for the CCFD methods on triangular mesh. First, we need certain intergrid operators between V_{k-1} and V_k. Assuming we are given a certain prolongation operator I_{k-1}^k: $V_{k-1} \to V_k$, we define the restriction operator I_k^{k-1}: $V_k \to V_{k-1}$ as its adjoint with respect to $(\cdot,\cdot)$:

$$\left(I_k^{k-1} u, v\right)_{k-1} = \left(u, I_{k-1}^k v\right)_k \qquad \forall u \in V_k, v \in V_{k-1}.$$

The multigrid method also requires linear smoothing operators. Let $R_k^{(l)}$: $V_k \to V_k$ be a smoother and its adjoint R_k^t with respect to $(\cdot,\cdot)_k$. The smoother R_k, as usual, can be taken as the Jacobi, Red-Black, or Gauss-Seidel relaxation. To define, multigrid algorithm, we define

$$R_k^{(l)} = \begin{cases} R_k & \text{if } l \text{ is odd,} \\ R_k^t & \text{if } l \text{ is even.} \end{cases}$$

Now multigrid algorithm for solving (2.4) is defined as follows.

Multigrid Algorithm

Set $B_1 = A_1^{-1}$. For $1 < k \leq J$, assume that B_{k-1} has been defined and define $B_k f$ for $f \in V_k$ as follows:

1. Set $x^0 = 0$ and $q^0 = 0$.
2. Define x^l for $l = 1, \ldots, m$ by

$$x^l = x^{l-1} + R_k^{(l+m)}(f - A_k x^{l-1}).$$

3. Define $y^m = x^m + I_{k-1}^k q^s$, where q^i for $i = 1, \ldots, s$ is defined by

$$q^i = q^{i-1} + B_{k-1}\left[I_k^{k-1}(f - A_k x^m) - A_{k-1} q^{i-1}\right]$$

4. Define y^l for $l = m+1, \ldots, 2m$ by

$$y^l = y^{l-1} + R_k^{(l+m)}(f - A_k y^{l-1}).$$

5. Set $B_k f = y^{2m}$.

Here, m is a fixed positive integer that is the number of smoothings and s is a positive integer. If $s = 1$, we obtain V-cycle and if $s = 2$, we obtain W-cycle.

III. INTERPOLATION OPERATOR USING FLUX CONTINUITY AND NONCONFORMING INTERPOLATION

In this section, we briefly describe two interpolation operators in [3, 8] and introduce a new prolongation operator. For the Laplace equation, the natural injection is used in [3] and W-cycle convergence is proved by showing that the energy norm of the natural injection is $\sqrt{2}$. And V-cycle with this injection is very slow. Kwak et al. proposed a weighted prolongation operator whose energy norm is still $\sqrt{2}$ on regular triangular grids [8], where even the V-cycle with one pre-post smoothing works almost as well as a conforming example, i.e., the contraction number is independent of levels. On the other hand, the multigrid convergence with the weighted prolongation degrades as is shown in section 4 when the diffusion coefficient is nonsmooth (for example, discontinuous). To describe these prolongation, we assume, for simplicity, that triangles are equilateral and interfaces are aligned with coarse grid sides and the diffusion coefficient is piecewise constant. For subtriangles in a fixed triangle T in $\mathcal{T}_{k-1}$, there are two cases (see Fig. 1).

Case I. The triangle of $\mathcal{T}_k$ is the interior subtriangle of $\mathcal{T}_{k-1}$, i.e., those represented by black beads v_0, v_1, v_2, and v_3.

Case II. The triangle of $\mathcal{T}_k$ is sharing edges with two triangles of $\mathcal{T}_{k-1}$, i.e., those represented by white beads u_0, u_1, u_2, and u_3.

Note that u_0 and v_0 are values at the same point and the black beads are fine grid point and a coarse grid one at the same time. The natural injection is defined as follows. In Case I and II, we define

$$u_0 = \left(I_{k-1}^k v\right)_0 = v_0 = u_1 = \left(I_{k-1}^k v\right)_1 = u_2 = \left(I_{k-1}^k v\right)_2 = u_3 = \left(I_{k-1}^k v\right)_3.$$

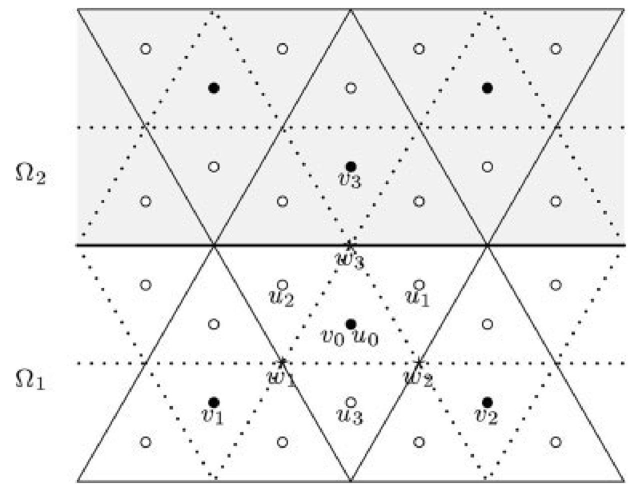

FIG. 1. $\star$: fictitious points; $\bullet$: coarse grid points; $\circ$: fine grid points.

The weighted prolongation operator in [8] is defined as follows: In Case I, we define $u_0 = (I_{k-1}^k v)_0 = v_0$. In Case II,

$$u_1 = \left(I_{k-1}^k v\right)_1 = \frac{2v_0 + v_2 + v_3}{4},$$
$$u_2 = \left(I_{k-1}^k v\right)_2 = \frac{2v_0 + v_1 + v_3}{4},$$
$$u_3 = \left(I_{k-1}^k v\right)_3 = \frac{2v_0 + v_1 + v_2}{4}.$$

Before we introduce our new prolongation, we briefly compare the two prolongation operators. In the case of smooth problems, the weighted interpolation works better than the natural injection since the former gives a better approximation in case of smooth functions. On the other hand, in case of nonsmooth problems, the solution is not so smooth across the interface, the weighted interpolation is not a good approximation. For example, we assume that the diffusion coefficient on the region Ω_1 is different from that on Ω_2 (see Fig. 1). Then u_3 defined by the weighted prolongation is a good approximation but u_1, u_2 is not. In the design of a prolongation operator to accelerate multigrid convergence, interpolations based on the flux continuity have been successful [1,2,9,10]. In these articles, they impose flux continuity across the interface and derive the relation of fine grid points and neighboring coarser grid points. However, this idea cannot be used directly to CCFD on triangular grids because of different grid structures. Fortunately, we could combine this idea with an interpolation by P_1-nonconforming finite element.

Now we define the prolongation. In Case I, we define $u_0 = (I_{k-1}^k v)_0 := v_0$. In Case II, we will first define the values w_1, w_2, w_3 by using flux continuity as follows:

$$p_0 \frac{v_0 - w_1}{\frac{h_{k-1}}{2\sqrt{3}}} = p_1 \frac{w_1 - v_1}{\frac{h_{k-1}}{2\sqrt{3}}}. \tag{3.1}$$

From (3.1), we obtain a formula for w_1,

$$w_1 = \frac{p_0 v_0 + p_1 v_1}{p_0 + p_1}.$$

Similarly, we have formulas for w_2 and w_3

$$w_2 = \frac{p_0 v_0 + p_2 v_2}{p_0 + p_2} \qquad \text{and} \qquad w_3 = \frac{p_0 v_0 + p_3 v_3}{p_0 + p_3}.$$

Next, we consider the P_1-nonconforming function having w_1, w_2, and w_3 as its values at three midpoints of edges. The values u_1, u_2, and u_3 are interpolated using this P_1-nonconforming function. Thus, substituting v_1, v_2, and v_3 for w_1, w_2, and w_3, we obtain the following formulas for u_1, u_2, and u_3:

$$\begin{aligned}
u_1 = \left(I_{k-1}^k v\right)_1 &:= \frac{2w_2 + 2w_3 - w_1}{3} \\
&= \frac{1}{3}\left[p_0 \left(\frac{2}{p_0 + p_2} + \frac{2}{p_0 + p_3} - \frac{1}{p_0 + p_1}\right) v_0 \right. \\
&\qquad \left. + p_2 \frac{2}{p_0 + p_2} v_2 + p_3 \frac{2}{p_0 + p_3} v_3 - p_1 \frac{1}{p_0 + p_1} v_1 \right], \\
u_2 = \left(I_{k-1}^k v\right)_2 &:= \frac{2w_1 + 2w_3 - w_2}{3} \\
&= \frac{1}{3}\left[p_0 \left(\frac{2}{p_0 + p_1} + \frac{2}{p_0 + p_3} - \frac{1}{p_0 + p_2}\right) v_0 \right. \\
&\qquad \left. + p_1 \frac{2}{p_0 + p_1} v_1 + p_3 \frac{2}{p_0 + p_3} v_3 - p_2 \frac{1}{p_0 + p_2} v_2 \right], \\
u_3 = \left(I_{k-1}^k v\right)_3 &:= \frac{2w_1 + 2w_2 - w_3}{3} \\
&= \frac{1}{3}\left[p_0 \left(\frac{2}{p_0 + p_1} + \frac{2}{p_0 + p_2} - \frac{1}{p_0 + p_3}\right) v_0 \right. \\
&\qquad \left. + p_1 \frac{2}{p_0 + p_1} v_1 + p_2 \frac{2}{p_0 + p_2} v_3 - p_3 \frac{1}{p_0 + p_3} v_3 \right].
\end{aligned}$$

When the diffusion coefficient is constant, these values are as follows:

$$\begin{aligned}
u_0 &= v_0, & u_1 &= \frac{3v_0 + 2v_2 + 2v_3 - v_1}{6}, \\
u_2 &= \frac{3v_0 + 2v_1 + 2v_3 - v_2}{6}, & u_3 &= \frac{3v_0 + 2v_1 + 2v_2 - v_3}{6}.
\end{aligned}$$

Since the values w_1, w_2, and w_3 are obtained by imposing flux continuity, we believe that they are good approximations at the edge points even for nonsmooth problems. Note that our new prolongation operator is different from the weighted one, even when the diffusion p is constant.

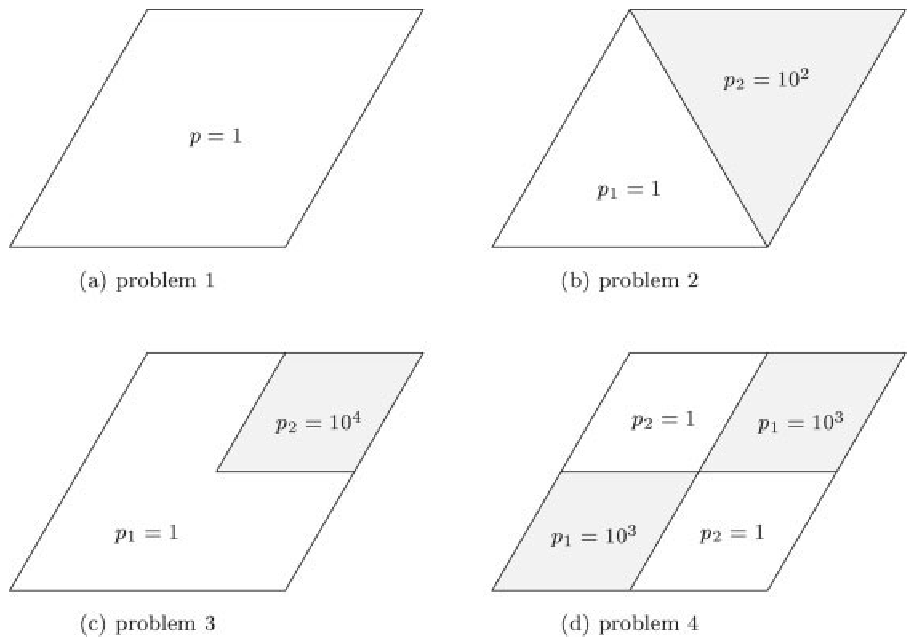

FIG. 2. Problems.

Remark. *From the definition of our new prolongation, it can be easily adapted to general triangular grids.*

IV. NUMERICAL RESULTS

We consider the following problem on the unit parallelogram:

$$
\begin{aligned}
-\nabla \cdot p\nabla u &= f, \qquad \text{in } \Omega,\\
u &= 0, \qquad \text{on } \partial\Omega.
\end{aligned}
$$

with various diffusion coefficients p (see Fig. 2). We test multigrid algorithms with three prolongations described above: the natural injection, the weighted prolongation in [8] with the weight $w = 4$ and our new prolongation. In all three cases, we use the V-cycle multigrid algorithm with one Red-Black Gauss-Seidel pre-post smoothing. We report eigenvalues and condition numbers of $B_k A_k$ and reduction rate δ_k. First, we test three algorithms for the problem with smooth coefficient ($p = 1$). Our new prolongation performs better than the other two (see Tables I–III). The algorithm with new prolongation is slightly better than one with the weighted prolongation.

Second, we test the problem with jump coefficient [see (b) in Fig. 2 and Tables IV–VI]. In this test, the weighted prolongation does not provide any convergence and has large condition numbers, which grow as the number of levels (see Tables IV–VI). Our new prolongation converges independent of the number of levels and does much faster than the natural one.

Next, we test two more problem with larger magnitude of jumps and different interfaces [see (c) and (d) in Fig. 2]. The numerical result of the weighted prolongation is unacceptable. Now, we report and compare the natural injection and new prolongation. Tables VII–X show that new

TABLE I. V-cycle result with the natural injection for problem 1, $m = 1$.

h_J	$\lambda_{\min}$	$\lambda_{\max}$	K	δ
1/32	0.503	1.757	3.491	0.725
1/64	0.496	1.896	3.824	0.853
1/128	0.492	2.014	4.093	0.962
1/256	0.487	2.118	4.349	>1

TABLE II. V-cycle result with the weighted prolongation for problem 1, $m = 1$.

h_J	$\lambda_{\min}$	$\lambda_{\max}$	K	δ
1/32	0.506	1.161	2.295	0.379
1/64	0.498	1.202	2.414	0.385
1/128	0.492	1.242	2.524	0.392
1/256	0.486	1.279	2.629	0.398

TABLE III. V-cycle result with new prolongation for problem 1, $m = 1$.

h_J	$\lambda_{\min}$	$\lambda_{\max}$	K	δ
1/32	0.558	1.135	2.035	0.357
1/64	0.556	1.135	2.043	0.365
1/128	0.555	1.135	2.043	0.371
1/256	0.555	1.135	2.043	0.375

TABLE IV. V-cycle result with the natural injection for problem 2, $m = 1$.

h_J	$\lambda_{\min}$	$\lambda_{\max}$	K	δ
1/32	0.506	1.763	3.487	0.739
1/64	0.497	1.903	3.824	0.879
1/128	0.492	2.021	4.103	0.990
1/256	0.489	2.112	4.319	>1

TABLE V. V-cycle result with the weighted prolongation for problem 2, $m = 1$.

h_J	$\lambda_{\min}$	$\lambda_{\max}$	K	δ
1/32	0.293	4.715	16.120	>1
1/64	0.258	5.734	22.243	>1
1/128	0.234	6.655	28.479	>1
1/256	0.216	7.478	34.558	>1

TABLE VI. V-cycle result with new prolongation for problem 2, $m = 1$.

h_J	$\lambda_{\min}$	$\lambda_{\max}$	K	δ
1/32	0.560	1.134	2.026	0.374
1/64	0.557	1.134	2.038	0.380
1/128	0.556	1.134	2.039	0.384
1/256	0.555	1.134	2.042	0.390

TABLE VII. V-cycle result with natural injection for problem 3, $m = 1$.

h_J	$\lambda_{\min}$	$\lambda_{\max}$	K	δ
1/32	0.504	1.766	3.503	0.648
1/64	0.497	1.909	3.842	0.755
1/128	0.492	2.035	4.135	0.866
1/256	0.488	2.153	4.412	0.956

TABLE VIII. V-cycle result with new prolongation for problem 3, $m = 1$.

h_J	$\lambda_{\min}$	$\lambda_{\max}$	K	δ
1/32	0.559	1.216	2.175	0.360
1/64	0.556	1.267	2.277	0.367
1/128	0.555	1.314	2.364	0.372
1/256	0.556	1.338	2.407	0.376

TABLE IX. V-cycle result with natural injection for problem 4, $m = 1$.

h_J	$\lambda_{\min}$	$\lambda_{\max}$	K	δ
1/32	0.476	1.670	3.511	0.644
1/64	0.458	1.821	3.979	0.795
1/128	0.446	1.950	4.367	0.916
1/256	0.439	2.062	4.696	>1

TABLE X. V-cycle result with new prolongation for problem 4, $m = 1$.

h_J	$\lambda_{\min}$	$\lambda_{\max}$	K	δ
1/32	0.561	1.122	2.000	0.368
1/64	0.558	1.127	2.020	0.377
1/128	0.556	1.128	2.028	0.382
1/256	0.556	1.128	2.029	0.385

prolongation converges faster and has smaller condition number than the natural injection. In particular, Tables VIII and X show that the condition numbers and contraction numbers of our new one do not change much as the number of levels grows. On the other hand, Tables VII and IX show that those of the natural one become large as the number of levels grows. As a conclusion, our new prolongation is better than the other two for both smooth and non-smooth examples. Also, the contraction number is independent of the number of levels, the jumps of coefficient and the shape of interfaces.

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