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Multigrid algorithm for cell centered finite difference on triangular meshes [1]

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Abstract

We consider a multigrid algorithm for the cell centered difference scheme on triangular meshes using a new prolongation operator. The energy norm of this prolongation is shown to be less than $\sqrt{2}$. Thus the $\mathscr{W}$-cycle is guaranteed to converge. Numerical experiments show that our operator is better than the trivial injection. © 1999 Elsevier Science Inc. All rights reserved.

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1. Introduction

Multigrid convergence theory for elliptic problems is mainly based on two fundamental properties of finite element spaces. One is "regularity and approximation". The other is the boundedness of the prolongation operator [1–4]. For conforming finite elements, an injection operator is used as the prolongation and its energy norm is one. But for nonconforming elements, the situation is more complicated. Either there does not exist a natural injection or the norm is greater than one [5,6].

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The cell centered finite difference is a nonconforming example. This method can be characterized as a finite volume method having piecewise constant basis functions. There are some analyses on the multigrid methods for the cell centered finite difference on rectangular meshes [5,7,8], where the trivial injection was chosen as the prolongation.

Recently, Kwak [9] proposed a new prolongation operator whose energy norm is one in the case of rectangular meshes. We investigate a similar prolongation operator on triangular meshes. It turns out that the energy norm of this new prolongation is the same as that of the trivial injection. However, the convergence speed is much faster.

2. Multigrid algorithm for the cell centered method

In this section, we consider a multigrid algorithm of the cell centered difference for a model problem:

$$-\Delta u = f \text{ in } \Omega, \tag{1}$$

$$u = 0 \text{ on } \Gamma. \tag{2}$$

Let Ω be taken as a parallelogram with unit side length, $\Gamma = \partial\Omega$. For $k = 1, 2, \ldots, J$, Ω is divided by $N_k := 2^{2k+1}$ regular triangles. Such triangulations are denoted by $\{\mathscr{T}_k\}$. Given a coarse triangulation $\{\mathscr{T}_k\}$, we connect midpoints of edges of triangles in $\{\mathscr{T}_k\}$ to obtain $\{\mathscr{T}_{k+1}\}$. Each triangle T in $\{\mathscr{T}_k\}$ is called a cell. The grid point of each cell is the circumcenter of the triangle. For $k = 1, 2, \ldots, J$, we let V_k denote the space of functions which are piecewise constant on each cell.

The cell centered discretization is obtained as follows: Integrate Eq. (1) against test functions in V_k and then use divergence theorem to get

$$-\int_{\partial T_j} \frac{\partial u}{\partial n}\,\mathrm{d}s = \int_{T_j} f\,\mathrm{d}x, \tag{3}$$

for $j = 1, 2, \ldots, N_k$.

We approximate Eq. (3) using functions in V_k by central difference method. Let u in V_k and u_i denote its value on i-th cell. For two adjacent triangles T_i and T_j, we let x_i and x_j be the circumcenters of T_i and T_j, respectively. We denote h by the length of the edge of a triangle and h' by $|x_i - x_j|$. Let $\theta = h/h'$. Then we take $\theta(u_j - u_i)$ as an approximation to $\partial u/\partial n_{ij}$ where n_{ij} is the unit normal vector from the center of T_i to that of T_j. Note that there are only three directions of n_{ij}. When one of the edge coincides with the boundary of Ω, we assume a fictitious value by reflection [10].

Collecting these, we have a system of algebraic equations

$$A_k u = \bar{f}, \tag{4}$$

where A_k is symmetric positive definite, and u and $\bar{f}$ are vectors whose entries are u_j and the integral of f over T_j, respectively. Define a discrete L^2-inner product on V_k by

$$(v, w)_k = \sum_{i=1}^{N_k} h_k^2 v_i w_i, \quad \forall v, w \in V_k,$$

where h_k denotes the length of edge of a triangle in $\{\mathscr{T}_k\}$. If we identify A_k with a quadratic form on $V_k \times V_k$ defined by

$$A_k(v, w) = (A_k v, w)_k,$$

then the problem (4) is equivalent to: Find $u \in V_k$ satisfying

$$A_k(u, \phi) = (f, \phi), \quad \forall \phi \in V_k. \tag{5}$$

The error estimate for the triangular cell centered finite difference is well known in Ref. [10]. Let Q_k be the $L^2(\Omega)$ projection onto V_k. If u is the solution of Eq. (5), then

$$A_k(u - Q_k\bar{u}, u - Q_k\bar{u}) \leqslant Ch_k^2 \|f\|^2,$$

where $\bar{u}$ is the solution of Eq. (1).

In order to describe the multigrid algorithm, we need certain intergrid transfer operators between V_{k-1} and V_k. Let a certain coarse-to-fine operator $I_k : V_{k-1} \to V_k$ be given. The fine-to-coarse operator $P_{k-1}^0 : V_k \to V_{k-1}$ is defined to be the transpose of I_k, i.e.,

$$(P_{k-1}^0 u, v)_{k-1} = (u, I_k v)_k, \quad \forall u \in V_k, \ v \in V_{k-1}.$$

The multigrid algorithm which we shall consider also requires linear smoothing operators $R_k : V_k \to V_k$ for $1 < k \leqslant J$. Let R_k^t denote the adjoint of R_k with respect to the $(\cdot,\cdot)_k$ inner product and define

$$R_k^{(l)} = \begin{cases} R_k & \text{if } l \text{ is odd,} \\ R_k^t & \text{if } l \text{ is even.} \end{cases}$$

The multigrid operator $B_k : V_k \to V_k$ is now defined by induction.

Multigrid Algorithm

Set $B_1 = A_1^{-1}$. For $1 < k \leqslant J$, assume that B_{k-1} has been defined and define $B_k g$ for $g \in V_k$ as follows:

1. Set $x^0 = 0$ and $q^0 = 0$.
2. Define x^l for $l = 1, \ldots, m$ by

$$x^l = x^{l-1} + R_k^{(l+m)}(g - A_k x^{l-1}).$$

3. Define $y^m = x^m + I_k q^p$, where q^i for $i = 1, \ldots, p$ is defined by

$$q^i = q^{i-1} + B_{k-1}\left[P^0_{k-1}(g - A_k x^m)\right].$$

4. Define y^l for $l = m+1, \ldots, 2m$ by

$$y^l = y^{l-1} + R_k^{(l+m)}(g - A_k y^{l-1}).$$

5. Set $B_k g = y^{2m}$.

In this algorithm, m is a fixed positive integer which denotes the number of smoothings and p is a positive integer. The cases $p = 1$ and $p = 2$ correspond respectively to the symmetric $\mathscr{V}$ and $\mathscr{W}$-cycles of the multigrid algorithm.

3. A weighted prolongation operator

In this section, we discuss prolongation operators. Since the trial function space V_k's are nested, a usual choice for the prolongation is the trivial injection operator I_k^t. To see whether the multigrid algorithm converges or not, we need to estimate the energy norm of the prolongation operator. To do so, we investigate the structure of $A_{k-1}(\cdot,\cdot)$. It is easy to see from [3,5]

$$A_{k-1}(v, v) = \theta \sum_{i \neq j} (v_i - v_j)^2, \tag{6}$$

where the sum is taken for all pairs of adjacent triangles i and j. Let $u = I_k^t v$. Then

$$A_k(u, u) = \theta \sum_{I \neq J} (u_I - u_J)^2, \tag{7}$$

where the sum is taken for all pair of indices I, J as Eq. (6). It can be easily shown that

$$A_k(I_k^t v, I_k^t v) = 2\theta \sum_{i \neq j} (v_i - v_j)^2.$$

In other words, the energy norm of the trivial injection operator is $\sqrt{2}$. From this, we can proceed as in Ref. [5] to conclude the $\mathscr{W}$-cycle with one smoothing is convergent. However, no conclusion can be deduced for the $\mathscr{V}$-cycle. Indeed, the numerical experiment in Section 4 shows that the $\mathscr{V}$-cycle does not converge.

Now we introduce a weighted operator I_k^w. Let $v \in V_{k-1}$ and $u = I_k^w v$. Referring to Fig. 1, we shall use $i, i+2$ to denote the nodes and the triangles in $\{\mathscr{T}_{k-1}\}$ while we use $0, 1, 2, \ldots$ to denote the nodes of triangles in $\{\mathscr{T}_k\}$. Fix a triangle i in $\{\mathscr{T}_{k-1}\}$. For a subtriangle j in i, there are two cases (see Fig. 2):

Case I. The subtriangle j is the interior subtriangle of i.

Case II. The subtriangle j is sharing edges with two triangles in $\{\mathscr{T}_{k-1}\}$, i.e., $i-1$, $i-2$. Define $I_k^w v$ as follows:

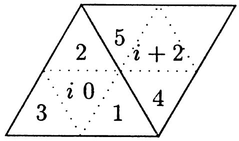

Fig. 1. Triangle i and $i+2$ with subdivisions.

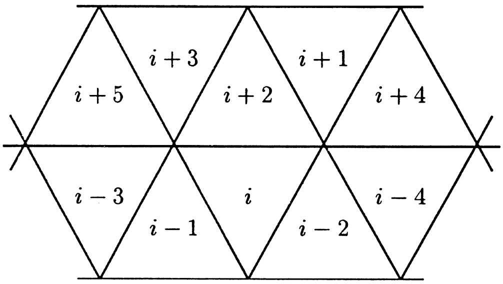

Fig. 2. Numbering of elements.

$$(I_k^w v)_j = \begin{cases} v_i & \text{if } j \text{ is Case I,} \\ ((w-2)v_i + v_{i-1} + v_{i-2})/w & \text{if } j \text{ is Case II.} \end{cases}$$

We shall estimate $A_k(u,u)$ when $u = I_k^w v$. The sum (7) can be divided into two cases. The first case is that two subtriangles belong to the same triangle i in $\{\mathscr{T}_{k-1}\}$. In this case, we denote it by S_i^1. The other case is that the terms come from adjacent triangles i, j of T_{k-1}. It is denoted by $S_{i,j}^2$. For example, we have as in Fig. 1,

$$S_i^1 = \theta \sum_{j=1}^{3} (u_0 - u_j)^2, \qquad S_{i,i+2}^2 = \theta\Big[(u_1 - u_4)^2 + (u_2 - u_5)^2\Big].$$

Referring to Fig. 2, we can represent the above two sums as follows:

$$S_i^1 = \frac{\theta}{w^2}\Big[(-2v_i + v_{i+2} + v_{i-2})^2 + (-2v_i + v_{i+2} + v_{i-1})^2 + (-2v_i + v_{i-2} + v_{i-1})^2\Big], \tag{8}$$

$$S_{i,i+2}^2 = \frac{\theta}{w^2}[(w-2)(v_i - v_{i+2}) + v_{i-2} - v_i + v_{i+2} - v_{i+1}]^2 + \frac{\theta}{w^2}[(w-2)(v_i - v_{i+2}) + v_{i-1} - v_i + v_{i+2} - v_{i+3}]^2. \tag{9}$$

We shall count the terms to $(v_i - v_{i+2})^2$. In Eq. (8), the contribution from $(w^2/\theta)S_i^1$ for the interior subtriangle of i is

$$2\Big[(v_i - v_{i+2})^2 + (v_i - v_{i+2})(v_i - v_{i-2}) + (v_i - v_{i+2})(v_i - v_{i-1})\Big]. \tag{10}$$

The contribution from $(w^2/\theta)S^1_{i+2}$ for the interior subtriangle of $i+2$ is

$$2\Big[(v_i - v_{i+2})^2 + (v_{i+2} - v_i)(v_{i+2} - v_{i+1}) + (v_{i+2} - v_i)(v_{i+2} - v_{i+3})\Big]. \tag{11}$$

In Eq. (9), the contribution from $(w^2/\theta)S^2_{i,i+2}$ is

$$\begin{aligned}2\Big[&(w-2)^2(v_i - v_{i+2})^2 + (w-2)(v_i - v_{i+2})((v_{i-2} - v_i) + (v_{i+2} - v_{i+1}))\\ &+ (w-2)(v_i - v_{i+2})((v_{i-1} - v_i) + (v_{i+2} - v_{i+3}))\Big].\end{aligned} \tag{12}$$

The contribution from $(w^2/\theta)S^2_{i,i-1}$ is

$$(v_i - v_{i+2})^2 + 2(v_i - v_{i+2})[(w-2)(v_{i-1} - v_i) + (v_{i-3} - v_{i-1})]. \tag{13}$$

The contribution from $(w^2/\theta)S^2_{i,i-2}$ is

$$(v_i - v_{i+2})^2 + 2(v_i - v_{i+2})[(w-2)(v_{i-2} - v_i) + (v_{i-4} - v_{i-2})]. \tag{14}$$

The contribution from $(w^2/\theta)S^2_{i+2,i+1}$ is

$$(v_i - v_{i+2})^2 + 2(v_i - v_{i+2})[(w-2)(v_{i+2} - v_{i+1}) + (v_{i+1} - v_{i+4})]. \tag{15}$$

The contribution from $(w^2/\theta)S^2_{i+2,i+3}$ is

$$(v_i - v_{i+2})^2 + 2(v_i - v_{i+2})[(w-2)(v_{i+2} - v_{i+3}) + (v_{i+3} - v_{i+5})]. \tag{16}$$

Adding the last two terms of Eqs. (10)–(12), some terms cancel out and we get

$$\begin{aligned}&2(w-3)(v_i - v_{i+2})(v_{i-2} - v_i) + 2(w-3)(v_i - v_{i+2})(v_{i+2} - v_{i+1})\\ &\quad + 2(w-3)(v_i - v_{i+2})(v_{i-1} - v_i) + 2(w-3)(v_i - v_{i+2})(v_{i+2} - v_{i+3}).\end{aligned}$$

By arithmetic–geometric inequality, the above sum is less than

$$\begin{aligned}&(w-3)(v_i - v_{i+2})^2 + (w-3)(v_{i-2} - v_i)^2 + (w-3)(v_i - v_{i+2})^2\\ &\quad + (w-3)((v_{i+2} - v_{i+1})^2 + (w-3)(v_i - v_{i+2})^2 + (w-3)(v_{i-1} - v_i)^2\\ &\quad + (w-3)(v_i - v_{i+2})^2 + (w-3)(v_{i+2} - v_{i+3})^2.\end{aligned} \tag{17}$$

Similarly, the sum of the second terms of Eqs. (13)–(16) is less than

$$\begin{aligned}&8(v_i - v_{i+2})^2 + 4(w-2)(v_i - v_{i+2})^2 + (v_{i-3} - v_{i-1})^2\\ &\quad + (w-2)(v_{i-1} - v_i)^2 + (v_{i-4} - v_{i-2})^2 + (w-2)(v_{i-2} - v_i)^2\\ &\quad + (v_{i+1} - v_{i+4})^2 + (w-2)(v_{i+2} - v_{i+1})^2 + (v_{i+3} - v_{i+5})^2\\ &\quad + (w-2)(v_{i+2} - v_{i+3})^2.\end{aligned} \tag{18}$$

In summary, the sum of the coefficients of $(v_i - v_{i+2})^2$ from Eqs. (13)–(18)

$$2 + 2 + 2(w-2)^2 + 4(w-3) + 8 + 4(w-2) = 2w^2.$$

Thus we have proved the following lemma.

Lemma 3.1. *For all $w \geqslant 3$, we have*

$$A_k(I_k^w v, I_k^w v) \leqslant 2A_{k-1}(v,v), \quad v \in V_{k-1}. \tag{19}$$

The next ingredient to prove multigrid convergence theory is so-called "regularity and approximation" condition.

Lemma 3.2. *Let the operator P_{k-1} be defined by*

$$A_{k-1}(P_{k-1}u, v) = A_k(u, I_k^w v), \quad \forall u \in V_k, \ v \in V_{k-1}.$$

Then there is a constant C independent of v such that for all $k = 1, \ldots, J$

$$A_k((I - I_k^w P_{k-1})v, v) \leqslant C\left(\frac{\|A_k v\|^2}{\lambda_k}\right)^{1/2} A_k(v,v)^{1/2}, \quad \forall v \in V_k. \tag{20}$$

Proof. The proof is similar to the rectangular case (cf. Refs. [5] or [9]). We omit the proof. □

We now apply the general framework of [3] to have the following result.

Theorem 3.1. *Let $p = 2$ and $m = 1$ in the multigrid algorithm. If we let $E_k = I - B_k A_k$, then we have*

$$A_k(E_k u, E_k u) \leqslant \delta_k^2 A_k(u,u), \quad \forall u \in V_k,$$

where $\delta_k < 1$. In other words, the $\mathscr{W}$-cycle converges with just one smoothing.

4. Numerical result

We consider the following problem on the unit parallelogram:

$$-\Delta u = f \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega.$$

We solve this problem by two multigrid algorithms: First we use the trivial injection operator. Next, we use the weighted prolongation operator with $w = 4$. In both algorithms, we use one Gauss–Seidel pre-post relaxation.

We report eigenvalues, condition numbers of $B_k A_k$ and reduction rate δ_k. Tables 1 and 2 show the results of $\mathscr{V}$-cycles with one smoothing. The algorithm with weighted prolongation operator converges while that of trivial injection does not. Tables 3–6 show that both algorithms converge with more smoothings or $\mathscr{W}$-cycle. In all of the cases, the new algorithm is better.

Table 1
V-cycle result with the trivial injection, $m = 1$

h_J	$\lambda_{\min}$	$\lambda_{\max}$	K	δ
1/32	0.443	2.013	4.541	>1
1/64	0.419	2.266	5.408	>1
1/128	0.408	2.523	6.184	>1

Table 2
V-cycle result with the weighted prolongation, $m = 1$

h_J	$\lambda_{\min}$	$\lambda_{\max}$	K	δ
1/32	0.491	1.089	2.218	0.217
1/64	0.477	1.114	2.335	0.222
1/128	0.466	1.140	3.466	0.227

Table 3
V-cycle result with the trivial injection, $m = 2$

h_J	$\lambda_{\min}$	$\lambda_{\max}$	K	δ
1/32	0.791	1.256	1.59	0.060
1/64	0.779	1.298	1.67	0.080
1/128	0.760	1.333	1.75	0.098

Table 4
V-cycle result with the weighted prolongation, $m = 2$

h_J	$\lambda_{\min}$	$\lambda_{\max}$	K	δ
1/32	0.763	1.020	1.34	0.050
1/64	0.756	1.029	1.36	0.053
1/128	0.749	1.037	1.38	0.055

Table 5
W-cycle result with the trivial injection, $m = 1$

h_J	$\lambda_{\min}$	$\lambda_{\max}$	K	δ
1/32	0.492	1.372	2.79	0.214
1/64	0.483	1.375	2.85	0.222
1/128	0.479	1.375	2.87	0.228

Table 6
W-cycle result with the weighted prolongation, $m = 1$

h_J	$\lambda_{\min}$	$\lambda_{\max}$	K	δ
1/32	0.547	1.097	2.00	0.180
1/64	0.546	1.100	2.01	0.180
1/128	0.545	1.105	2.02	0.180

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