A NORM ESTIMATE OF ADI METHOD FOR NONSYMMETRIC PROBLEM

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Abstract. In this paper, we give a norm estimate of alternating direction implicit method for convection diffusion type of nonsymmetric elliptic problem on a rectangular domain. We estimate certain norm of the iteration matrix in terms of coefficients of convective terms and mesh size. The norm is aymtotically shown to be of the form $(1 - Ch)/(1 + Ch)$, where C is the same constant as the symmetric case.

We also show that the optimal size of parameter is the same as symmetric case. As a consequence, we conclude that the convergence behavior is as good as that of symmetric case and does not deteriorate as the size of convective terms grow. Numerical experiment shows that our anaysis is sharp.

1. Introduction

Since the introduction of alternating direction implicit method by Peaceman-Rachford[10] for solving elliptic boundary value problem for the model of oil reservoirs, there has been extensive research on theoretical aspect([6],[2],[3],[5]) as well as applications to physical problems([1],[4],[7],[8], [9], and [11]). But the convergence was known for a while only for the case when the splitting matrices commute. It is only recently that the convergence of ADI method in the case of noncommuting splitting was shown[4]. But they used commuting part as a preconditioner to resolve the convection dominated problem through eigenvalue analysis of the iteration mtrix. Moreover, they assumed either $bh/2 < 1$ or $ch/2 < 1$ holds and could not analyze the case when both of the terms are greater than 1. In this paper, we estimate the norm of iteration matrix resulting from ADI formulation for nonsymmetric problem using eigenvectors of Laplace operators and show that the norm is of the form $(1 - Ch)/(1 + Ch)$ even if both $bh/2, ch/2$ are greater than 1. We also show that the asymptotic convergence factor is as good as the symmetric case with the same optimal choice of the iteration parameter, which is not shown in any literature. As a consequence, the convergence behavior is not affected by the size of convective terms.

The rest of our paper is organized as follows: In section 2, we give a basic ADI formulation for solving stationary convection-diffusion type of problems. We use a weighted norm to estimate the norms of the iteration matrices corresponding to the horizontal sweep and vertical sweep, thus obtaining the estimate of the complete iteration matrix of ADI method. In section 3, we give a numerical example. It is

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shown that the number of iterations grows linearly with the number of unknowns, which indicates the sharpness of our estimate .

2. Problem formulation

Consider a second order elliptic partial differential equation on a rectangular domain Ω, which we take the unit square for simplicity:

$$\begin{aligned} -\Delta u + bu_x + cu_y =& f \text{ in } \Omega \\ u =& g \text{ on } \partial\Omega, \end{aligned} \tag{2.1}$$

where b, c are constants. Applying central difference scheme gives rise to a system of algebraic equation of the form, after multiplication by h^2:

$$Ax = f, \quad A = H_P + V_Q, \quad H_P = H + P, \quad V_Q = V + Q,$$

where H, V are horizontal and vertical operator corresponding to $-\Delta$ and P, Q are horizontal and vertical operator corresponding to bu_x and cu_y respectively. Explicitly, we have

$$H = \begin{bmatrix} J & 0 & 0 & 0 & 0 \\ 0 & J & 0 & 0 & 0 \\ & & \cdots & & \\ & & \cdots & & \\ 0 & 0 & 0 & 0 & J \end{bmatrix}, \text{ where } J = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ & & \cdots & & \\ & & \cdots & & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

and

$$V = \begin{bmatrix} 2I & -I & 0 & 0 & 0 \\ -I & 2I & -I & 0 & 0 \\ & & \cdots & & \\ & & \cdots & & -I \\ 0 & 0 & 0 & -I & 2I \end{bmatrix}.$$

P, Q are block matrices of the form

$$P = \begin{bmatrix} B & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 \\ 0 & 0 & B & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & 0 & 0 & B \end{bmatrix}, \text{ where } B = \frac{h}{2}\begin{bmatrix} 0 & b & 0 & 0 & 0 \\ -b & 0 & b & 0 & 0 \\ 0 & -b & 0 & b & 0 \\ & & \cdots & & b \\ 0 & 0 & 0 & -b & 0 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} 0 & C & 0 & 0 & 0 \\ -C & 0 & C & 0 & 0 \\ 0 & -C & 0 & C & 0 \\ & & \cdots & & C \\ 0 & 0 & 0 & -C & 0 \end{bmatrix}, \text{where } C = \frac{h}{2}\text{diag}[c, c, \cdots, c].$$

Now we consider an ADI or Peaceman-Rachford iteration in the following form:

$$\begin{aligned} x^{k+1} =& (I + sV_Q)^{-1}(I + sH_P)^{-1}(I - sH_P)(I - sV_Q)x^k \\ &+ (I + sV_Q)^{-1}(I + sH_P)^{-1}2sAx. \end{aligned} \tag{2.2}$$

$$x^{k+1} = G_s x^k + \tilde{f}, \tag{2.3}$$

where s is certain positive number to minimize the norm of the matrix G_s. For Laplace operator in rectangular domain, s is chosen as the reciprocal of geometric mean of two extreme eigenvalues. Let

$$\begin{aligned}\tilde{G}_s =&(I+sV_Q)G_s(I+sV_Q)^{-1}\\ =&(I+sH_P)^{-1}(I-sH_P)(I-sV_Q)(I+sV_Q)^{-1}.\end{aligned} \tag{2.4}$$

Let $S=(I+sV_Q)$. Then $\tilde{G}_s = SG_sS^{-1}$ and if we introduce a new norm $\|x\|_S$ defined by

$$\|x\|_S = (Sx,Sx)^{1/2},$$

we have

$$\begin{aligned}\|\tilde{G}_s\| &= \sup_{\|x\|=1} \|SG_sS^{-1}x\|\\ &= \sup_{\|Sy\|=1} \|SG_sy\|\\ &= \sup_{\|y\|_S=1} \|G_sy\|_S\\ &= \|G\|_S\end{aligned}$$

Thus to estimate $\|G_s\|_S$, it suffices to estimate $\|(I+sH_P)^{-1}(I-sH_P)\|$ and $\|(I-sV_Q)(I+sV_Q)^{-1}\|$.

Since $H_P = \mathrm{diag}[J_B,\cdots,J_B]$, where $J_B = J+B$, we see that $\|H_P\| = \|J_B\|$. Thus to estimate $\|(I+sH_P)^{-1}(I-sH_P)\|$, it suffices to estimate

$$\|(I+sJ_B)^{-1}(I-sJ_B)\|. \tag{2.5}$$

Let $h=\frac{1}{n+1}$ and let $x_j = [\sin(j\pi h),\cdots,\sin(nj\pi h)]^T$ be the eigenvector of J corresponding to the eigenvalue $\lambda_j = 2(1-\cos(j\pi h))$.

Lemma 2.1. *We have*

$$JBx_j - BJx_j = bh\sin(j\pi h)[1,0,\cdots,0,(-1)^j] \tag{2.6}$$

and

$$B^2x_j = -b^2h^2\sin^2(j\pi h)\cdot x_j + \frac{b^2h^2}{2}\sin(j\pi h)\cdot[1,0,\cdots,0,(-1)^{j-1}]. \tag{2.7}$$

Proof. Since

$$\begin{aligned}Bx_j =&\frac{bh}{2}[\sin(2j\pi h),\sin(3j\pi h)-\sin(j\pi h),\cdots,\sin((n-1)j\pi h]\\ =&bh\sin(j\pi h)[\cos(j\pi h),\cos(2j\pi h),\cdots,\cos(nj\pi h)],\end{aligned} \tag{2.8}$$

we have

$$
\begin{aligned}
JBx_j =& bh\sin(j\pi h)[2\cos(j\pi h)-\cos(2j\pi h), 2\cos(2j\pi h)-\cos(j\pi h)\\
&-\cos(3j\pi h),\cdots,2\cos(nj\pi h)-\cos((n-1)j\pi h)]\\
=&2bh\sin(j\pi h)(1-\cos(j\pi h))[0,\cos(2j\pi h),\cdots,\cos((n-1)j\pi h),0]\\
&+bh\sin(j\pi h)[2\cos(j\pi h)-\cos(2j\pi h),\cdots,0,\\
&2\cos(nj\pi h)-\cos((n-1)j\pi h)]
\end{aligned}
\tag{2.9}
$$

and

$$
\begin{aligned}
BJx_j =& 2(1-\cos(j\pi h))Bx_j\\
=& 2bh\sin(j\pi h)(1-\cos(j\pi h))[\cos(j\pi h),\cos(2j\pi h),\cdots,\cos(nj\pi h)].
\end{aligned}
\tag{2.10}
$$

Thus, we have

$$
\begin{aligned}
JBx_j-BJx_j =& bh\sin(j\pi h)[2\cos(j\pi h)-\cos(2j\pi h),0,\cdots,0,\\
&2\cos(nj\pi h)-\cos((n-1)j\pi h)]\\
&-2b\sin(j\pi h)(1-\cos(j\pi h))[\cos(j\pi h),0,\cdots,0,\cos(nj\pi h)]\\
=& bh\sin(j\pi h)[1,0,\cdots,0,(-1)^j].
\end{aligned}
$$

For (2.7), we see from (2.8)

$$
\begin{aligned}
B^2x_j =& bh\sin(j\pi h)B[\cos(j\pi h),\cos(2j\pi h),\cdots,\cos(nj\pi h)]\\
=&\frac{b^2h^2}{2}\sin(j\pi h)[\cos(2j\pi h),\cos(3j\pi h)-\cos(j\pi h),\cdots,\\
&\cos(nj\pi h)-\cos((n-2)j\pi h),-\cos((n-1)j\pi h)]\\
=&-b^2h^2\sin^2(j\pi h)[0,\sin(2j\pi h),\cdots,\sin((n-1)j\pi h),0]\\
&+\frac{h^2}{2}\sin(j\pi h)[\cos(2j\pi h),0,\cdots,0,-\cos((n-1)j\pi h)]\\
=&-b^2h^2\sin^2(j\pi h)\cdot x_j+h^2\sin^2(j\pi h)[\sin(j\pi j),0,\cdots,0,\sin(nj\pi h)]\\
&+\frac{b^2h^2}{2}\sin(j\pi h)[\cos(2j\pi h),0,\cdots,0,-\cos((n-1)j\pi h)]\\
=&-b^2h^2\sin^2(j\pi h)\cdot x_j+\frac{b^2h^2}{2}\sin(j\pi h)\cdot[1,0,\cdots,0,(-1)^{j-1}],
\end{aligned}
$$

which completes the proof.

Lemma 2.2. *We have*

$$
\begin{aligned}
x_i^TJ_B^TJ_Bx_j =& \delta_{ij}(\lambda_j^2+b^2h^2\sin^2(j\pi h))x_i^Tx_j+bh\sin(j\pi h)(1+(-1)^{i+j-1})\sin(i\pi h)\\
&-\frac{b^2h^2}{2}\sin(j\pi h)(1+(-1)^{i+j})\sin(i\pi h).
\end{aligned}
$$

Proof. Note that
$$J_B^T J_B = J^2 + JB - BJ - B^2.$$
From lemma 2.2, we have
$$\begin{aligned} J_B^T J_B x_j =& \lambda_j^2 x_j + bh\sin(j\pi h)[1,0,\cdots,0,(-1)^j] + b^2h^2\sin^2(j\pi h)\cdot x_j \\ &- \frac{b^2h^2}{2}\sin(j\pi h)[1,0,\cdots,0,(-1)^{j-1}] \\ =&(\lambda_j^2 + b^2h^2\sin^2(j\pi h))x_j + bh\sin(j\pi h)[1,0,\cdots,0,(-1)^j] \\ &- \frac{b^2h^2}{2}\sin(j\pi h)\cdot[1,0,\cdots,0,(-1)^{j-1}]. \end{aligned}$$
Thus the result follows by noting that $\sin(ni\pi h) = (-1)^{i-1}\sin(i\pi h)$.

We need an estimate on the size of eigenvectors x_j.

Lemma 2.3. *We have*
$$\|x_j\|^2 = \sum_{k=1}^{n}\sin^2(jk\pi h) = \frac{n+1}{2}$$
and
$$\sum_{k=odd}^{n}\sin^2(jk\pi h) \sim \frac{n+1}{4}.$$

Corollary.
$$\sum_i c_i\sin(i\pi h) \le (\sum c_i^2)^{1/2}(\sum \sin^2(i\pi h))^{1/2} = (\sum c_i^2)^{1/2}\sqrt{\frac{n+1}{2}}.$$

Proof of Lemma 2.3. Using a trigonometry identity, we see that
$$\begin{aligned}\sum_{k=1}^{n}\sin^2(jk\pi h) &= \sum_{k=1}^{n}\frac{1-\cos(2jk\pi h)}{2} \\ &= \frac{n}{2} - \frac{1}{2}\sum_{k=1}^{n}\cos(2jk\pi h) \\ &= \frac{n}{2} - \frac{1}{2}\mathrm{Re}\sum_{k=1}^{n}e^{ik\theta} = \frac{n}{2} - \frac{1}{2}\mathrm{Re}\frac{z(1-z^n)}{1-z},\end{aligned}$$
where $\theta = 2j\pi h$. Now we compute $\mathrm{Re}\frac{z(1-z^n)}{1-z}$.
$$\begin{aligned}\mathrm{Re}\frac{z(1-z^n)}{1-z} &= \mathrm{Re}\frac{z(1-\bar z)(1-z^n)}{(1-z)(1-\bar z)} = -\mathrm{Re}\frac{(1-z-z^n+z^{n+1})}{(1-z)(1-\bar z)} \\ &= \frac{-(1-\cos\theta)+\cos n\theta - \cos(n+1)\theta}{2(1-\cos\theta)} \\ &= -\frac{1}{2} - \frac{2\sin(\frac{2n+1}{2}\theta)\sin(-\frac{\theta}{2})}{2(1-\cos\theta)}.\end{aligned}$$

Noting that $\sin(\frac{2n+1}{2}\theta) = \sin(-\frac{\theta}{2})$, the first estimate follows. The second estimate is now obvious.

Remark. $\nu_j^2 = \lambda_j^2 + b^2h^2\sin^2(j\pi h)$ can be regarded as approximate eigenvalues of $J_B^TJ_B$ since

$$\frac{\|x_j^TJ_B^TJ_Bx_j\|}{\|x_j\|^2} = (\lambda_j^2 + b^2h^2\sin^2(j\pi h)) - \frac{b^2h^2\sin^2(j\pi h)}{\|x_j\|^2}.$$

Theorem 2.4. *Assume $\sigma(J) \subset [d_0^2h^2, d_1^2]$ and $s = \frac{1}{d_0d_1h}$. Then,*

$$\|(I+sJ_B)^{-1}(I-sJ_B)x\| \le r_H\|x\|, \tag{2.11}$$

where $r_H^2 = r_H^2(n) = \frac{\tilde{b}+(1-s\lambda_n)^2}{\tilde{b}+(1+s\lambda_n)^2}$ and $\tilde{b} = \frac{|b|(2+|b|h)}{d_0^2d_1^2h}$. Here the choice of $1/s$ as the geometric mean of λ_1 and λ_n, as in the symmetric case, is the best possible.

Proof. Let $v_j = x_j/\|x_j\|$ be the normalized eigenvectors of J and put $y = (I+sJ_B)^{-1}x = \sum c_jv_j$. Without loss of generality, we may assume y is normalized so that $\sum c_i^2 = 1$.

Noting that $(I-sJ_B)(I+sJ_B) = (I+sJ_B)(I-sJ_B)$, (2.11) is equivalent to

$$\|(I-sJ_B)y\|^2 \le r_H^2\|(I+sJ_B)y\|^2. \tag{2.12}$$

Expanding we have

$$y^T(I - sJ_B^T - sJ_B + s^2J_B^TJ_B)y \le r_H^2y^T(I + sJ_B^T + sJ_B + s^2J_B^TJ_B)y$$

or

$$\begin{aligned}(1-r_H^2)[y^Ty + s^2y^TJ_B^TJ_By] &\le s(1+r_H^2)y^T(J_B^T+J_B)y \\ &= 2s(1+r_H^2)\sum_i c_i^2\lambda_i^2.\end{aligned} \tag{2.13}$$

But by Lemma 2.1, 2.2, 2.3 and Cauchy-Schwarz inequality,

$$\begin{aligned}y^TJ_B^TJ_By &= y^T(J^2 + JB - BJ - B^2)y \\ &= \sum c_i^2\nu_i^2 + bh\sum_{i,j}(1+(-1)^{i+j-1})c_ic_j\sin(i\pi h)\sin(j\pi h)/\|x_i\|\cdot\|x_j\| \\ &\quad + O(h^2) \\ &\le \sum c_i^2\nu_i^2 + 2|b|h \le \sum c_i^2\lambda_i^2 + b^2h^2 + 2|b|h,\end{aligned}$$

where $\nu_i^2 = \lambda_i^2 + b^2h^2\sin^2(i\pi h)$. Thus, it suffices to show

$$(1-r_H^2)\left(1 + s^2(\sum c_i^2\nu_i^2 + b^2h^2 + 2|b|h)\right) \le 2s(1+r_H^2)\sum c_i^2\lambda_i^2$$

or

$$\sum c_i^2(1-s\lambda_i)^2+\tilde{b}\le r_H^2\left[\sum c_i^2(1+s\lambda_i)^2+\tilde{b}\right]. \tag{2.14}$$

When there are no convective terms, it is well known that([2], [3],[5]), the optimal choice of $1/s$ is the geometric mean of two extreme eigenvalues to minimize r_H^2. But when $b\neq 0$, we do not have an optimal choice a prioi. Instead, let $1/s$ be any number in $\sigma(J)$ unspecified for the time being, and let n_0 be the index for which $\max_i\left(\frac{1-s\lambda_i}{1+s\lambda_i}\right)^2$ is attained and let $r_H^2=r_H^2(n_0)$. Let $\bar{r}^2=\left(\frac{1-s\lambda_{n_0}}{1+s\lambda_{n_0}}\right)^2=\max_i\left(\frac{1-s\lambda_i}{1+s\lambda_i}\right)^2$. Then $\bar{r}^2<r_H^2(n_0)$ since $\tilde{b}>0$. Thus the left hand side of (2.14) is less than

$$\begin{aligned}
&\bar{r}^2\sum c_i^2(1+s\lambda_i)^2+\tilde{b}\\
&=r_H^2\left(\sum c_i^2(1+s\lambda_i)^2+\tilde{b}\right)+(\bar{r}^2-r_H^2)\left(\sum c_i^2(1+s\lambda_i)^2\right)+(1-r_H^2)\tilde{b}.\\
&\le r_H^2\left(\sum c_i^2(1+s\lambda_i)^2+\tilde{b}\right)+(\bar{r}^2-r_H^2)(1+s\lambda_n)^2+(1-r_H^2)\tilde{b}.
\end{aligned}$$

We claim that the sum of the last terms is less than or equal to zero.

$$\begin{aligned}
\bar{r}^2-r_H^2&=\left(\frac{1-s\lambda_{n_0}}{1+s\lambda_{n_0}}\right)^2-\frac{\tilde{b}+(1-s\lambda_{n_0})^2}{\tilde{b}+(1+s\lambda_{n_0})^2}\\
&=\frac{-4s\lambda_{n_0}\tilde{b}}{(1+s\lambda_{n_0})^2[\tilde{b}+(1+s\lambda_{n_0})^2]}.
\end{aligned}$$

But

$$(1-r_H^2)\tilde{b}=\frac{4s\lambda_{n_0}\tilde{b}}{\tilde{b}+(1+s\lambda_{n_0})^2}.$$

Thus, sum of the last two terms is less than or equal to zero because $\lambda_n\ge\lambda_{n_0}$. This proves (2.14).

Now we discuss the optimal choice of s. Let $s_0=\frac{1}{d_0d_1h}=1/\sqrt{\lambda_1\lambda_n}$ be the optimal choice for the symmmetric case. First assume $s<s_0$. Then the maximum of $(\frac{1-s\lambda_i}{1+s\lambda_i})^2$ is attained at $\lambda_1(n_0=1)$. In this case we claim that $r_H^2(n)$ is better(smaller) than $r_H^2(1)$. Indeed, we compute

$$\frac{\tilde{b}+(1-s\lambda_1)^2}{\tilde{b}+(1+s\lambda_1)^2}-\frac{\tilde{b}+(1-s_0\lambda_n)^2}{\tilde{b}+(1+s_0\lambda_n)^2}. \tag{2.15}$$

Its numerator is

$$4(s_0\lambda_n-s\lambda_1)(\tilde{b}+1-ss_0\lambda_n\lambda_1). \tag{2.16}$$

The first factor $(s_0\lambda_n-s\lambda_1)$ is obviously positive and $\tilde{b}+1-ss_0\lambda_n\lambda_1$ is positive because $s_0\lambda_n=\sqrt{\lambda_n/\lambda_1}$, $s\lambda_1<s_0\lambda_1=\sqrt{\lambda_1/\lambda_n}$. Thus s_0 is a better choice resulting in the convergrence factor $r_H^2=r_H^2(n)$. Now assume $s>s_0$. In this case the maximum is attained at $\lambda_n(n_0=n)$. Now the first factor of (2.16)(λ_n in place of λ_1), $(s_0-s)\lambda_n$, is negative while the second factor $\tilde{b}+1-ss_0\lambda_n\lambda_n$ is negative also asymptotically because $\tilde{b}+1=O(h^{-1})$ and $ss_0\lambda_n\lambda_n>s_0^2\lambda_n^2>O(h^{-2})$. In either case, we conclude that s_0 is the best choice and $r_H^2=r_H^2(n)$.

Theorem 2.5. *Assume* $\sigma(J) \subset [d_0^2 h^2, d_1^2]$ *and* $s = \frac{1}{d_0 d_1 h}$. *Then,*

$$\|(I - sV_Q)(I + sV_Q)^{-1}x\| \leq r_V \|x\| \tag{2.17}$$

where r_V^2 *is similar to that of Theorem 2.4, with* c *replacing* b.

Proof. Since V_Q has the same block structure as H_P by reordering the nodes, the proof is the same as Theorem 2.4.

Now we have the following:

Theorem 2.6. *The convergence factor of the ADI iteration in the nonsymmetric case is asymptotically the same as the symmetric case, i.e,*

$$\begin{aligned}\rho(G_s) = \rho(\tilde{G}_s) &\leq \|\tilde{G}_s\| \\ &\leq \|(I + sH_P)^{-1}(I - sH_P)\| \cdot \|(I - sV_Q)(I + SV_Q)^{-1}\| \\ &\leq r_H r_V = \frac{(1 - Ch)}{(1 + Ch)},\end{aligned}$$

where the constant C *is the same as that of symmetric case. Furthermore, the convergence is unaffected by the presence of nonsymmetric terms.*

Proof. Write $\tilde{b} = \frac{2\tilde{b}}{h}$. Then one can easily see that

$$r_H^2 = \frac{1 + \frac{2(\tilde{b} - D_1)h}{D_1^2} + \frac{h^2}{D_1^2}}{1 + \frac{2(\tilde{b} + D_1)h}{D_1^2} + \frac{h^2}{D_1^2}}, \tag{2.18}$$

where $D_1 = d_1/d_0$. If $\tilde{b} = 0$, this reduces to the symmetric case

$$\frac{1 - \frac{2h}{D_1} + \frac{h^2}{D_1^2}}{1 + \frac{2h}{D_1} + \frac{h^2}{D_1^2}}. \tag{2.19}$$

We subtract (2.19) from (2.18) neglecting second order terms, to get

$$\frac{4\tilde{b}h^2}{D_1^3(1 + O(h))}.$$

This is second order in h which means, asymptotically, the convergence factor is as good as the symmetric case. The esimate for r_V^2 follows exactly the same way. As a consequence the second assertion follows.

3. Numerical Examples

Example 3.1. *We solve*

$$\begin{aligned}-\Delta u + \vec{b} \cdot \nabla u &= f \ \text{ in } \Omega \\ u &= g \ \text{ on } \partial\Omega\end{aligned}$$

with $\vec{b} = (b, c)$, *and* Ω *is the unit square. We stopped when the residual error is less than* 10^{-6}. *The numerical experiment shows that the number of iteration grows linearly with* $\sqrt{N} = 1/h$, *which is in agreement with the predicted theory. The entries with* $*$ *show that the algorithm converges when both* $|b|h/2$ *abd* $|c|h/2$ *are greater than 1. When* b *or* c *becomes larger, the solution is not available for some* h. *This is due to the indefiniteness of matrix* A. *If convective terms are large, the discretization only makes sense when* h *is sufficiently small. ,*

Number of Iterations				
$h \backslash b = c$	5.0	10.0	30.0	50.0
1/5	7	9	NA	NA
1/10	13	11	*21	NA
1/15	19	17	16	NA
1/20	25	22	17	*27
1/25	31	27	22	22
1/30	37	31	30	20
1/50	60	50	45	39

Table 3.1

Both experiments show that the number of iterations grows linearly with $\sqrt{N} = \frac{1}{h}$, which is in agreement with the predicted theory.

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