# MULTIGRID ALGORITHMS FOR NONCONFORMING AND MIXED METHODS FOR NONSYMMETRIC AND INDEFINITE PROBLEMS*

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**Abstract.** In this paper we consider multigrid algorithms for nonconforming and mixed finite element methods for nonsymmetric and/or indefinite elliptic problems. We show that a simple $\mathcal{V}$-cycle multigrid iteration using conforming coarse-grid corrections converges at a uniform rate provided that the coarsest level in the multilevel iteration is sufficiently fine (but independent of the number of multigrid levels). Various types of smoothers for the nonsymmetric and indefinite problems are discussed. Extensive numerical results are presented.

**Key words.** mixed method, nonconforming method, finite elements, multigrid algorithm, convergence, nonsymmetric and/or indefinite problems

**AMS subject classifications.** 65N30, 65N22, 65F10

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**1. Introduction.** In this paper we study multigrid algorithms for nonsymmetric and/or indefinite elliptic problems. We consider the solution of the discrete systems which arises from the application of nonconforming and mixed finite element methods. We assume that the nonsymmetric/indefinite terms are a "compact perturbation"; the convection-dominated problems are not studied here.

We study the multigrid algorithms for solving the nonsymmetric and/or indefinite problems by the standard $P_1$-nonconforming finite elements. The $P_1$-nonconforming multigrid algorithms for symmetric problems have been studied in the past few years. There have been two types of multigrid algorithms for solving the symmetric problems. The first one exploits the nonconforming finite elements in both smoothing iterations and coarse-grid corrections in the multilevel iteration. For this type of multigrid algorithm, only the $\mathcal{W}$-cycle algorithms were proven to be convergent under the assumption that the number of smoothing iterations on all levels is big enough [8, 9, 7, 4, 11]. The convergence of the standard $\mathcal{V}$-cycle algorithm still remains open.

The second type of multigrid algorithm uses the nonconforming finite elements in the smoothing iterations on the finest level but the $P_1$-conforming finite elements in the coarse-grid corrections in the multilevel iteration. For this approach, uniform iterative convergence estimates for the $\mathcal{V}$-cycle multigrid algorithm with one smoothing step have been obtained for the symmetric problem [23, 18, 11].

In this paper we first consider the problem of existence and uniqueness of the solution of the discrete systems arising from application of the nonconforming method to the nonsymmetric and/or indefinite problem. We prove that the discrete systems have a unique solution and produce optimal-order error estimates provided that the size of the coarsest mesh is sufficiently small. We then provide a convergence analysis for multigrid algorithms. There has been intensive research on the multigrid

algorithms for the nonsymmetric and indefinite problem using the conforming finite elements. Paper [5] has a good survey in its introduction and is most closely related to the subject of this paper. For the nonconforming multigrid algorithm of the nonsymmetric and indefinite problem under consideration, we only analyze the second type of multigrid algorithm mentioned above. We show that the result for the symmetric problems can carry over to the nonsymmetric and indefinite case. Namely, we prove uniform iterative convergence estimates for the $\mathcal{V}$-cycle multigrid algorithm for the nonsymmetric and indefinite problem under rather weak assumptions (e.g., the domain need not be convex).

A variety of smoothers is considered here. One type of smoother is defined in terms of the corresponding symmetric problem, and the other type is entirely based on the original nonsymmetric and indefinite problem. These two types of smoothers include point and line Jacobi and Gauss–Seidel iterations.

Not only is the analysis of multigrid algorithms for nonconforming finite element methods of interest for its own sake (see, e.g., [14, 16, 17, 19] and the bibliographies therein), but it has great application to mixed finite element methods. It has been shown [2, 3, 10, 11, 13, 14] that the linear system arising from the mixed methods of the symmetric problem can be algebraically condensed to a symmetric, positive definite system for Lagrange multipliers. This linear system is identical to the system arising from the nonconforming finite element methods. Hence the analysis of multigrid algorithms for the nonconforming methods can carry over directly to the mixed methods. We here extend this approach to the nonsymmetric and indefinite case. To our knowledge, the multigrid algorithms for the nonsymmetric and indefinite problem by the nonconforming and mixed methods are analyzed here for the first time.

The rest of the paper is organized as follows. In the next section we state the continuous problem and its corresponding discrete system. Then, in section 3 we describe the multigrid method for the nonconforming method and do the analysis. Next, in section 4 we consider mixed finite element methods. Finally, in section 5, extensive numerical results for nonsymmetric problems are given to illustrate the present theories. Both types of multigrid methods mentioned above are tested for the first time. The later analysis is carried out for the two-dimensional, triangular case; it works for the three-dimensional case without substantial changes, as noticed in [11, 13, 14].

**2. Preliminaries.** In this section we consider as our model problem the following equation:

$$(2.1) \qquad \begin{aligned} -\nabla \cdot (\mathcal{A}\nabla u) + \mathcal{B} \cdot \nabla u + cu &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3 is a simply connected bounded polygonal domain with the boundary $\partial\Omega$, $f \in L^2(\Omega)$, and the coefficient $\mathcal{A} \in (L^\infty(\Omega))^{n \times n}$ satisfies the uniformly positive definite condition

$$(2.2) \qquad \xi^t \mathcal{A}(x)\xi \geq a_0 \xi^t \xi, \qquad x \in \Omega, \, \xi \in \mathbb{R}^n.$$

Further, for the analysis of our multigrid algorithm we assume that the elements of $\mathcal{A}$ are in the Sobolev space $W_{r,q}(\Omega)$ for $r > 2/q$ (see [1] for the definition of $W_{r,q}(\Omega)$), $\mathcal{B}$ is continuously differentiable on $\overline{\Omega}$ and piecewise $C^2$ with the sum of the second-order derivatives over pieces being bounded, and $|c|$ is bounded. Finally, we assume that (2.1) has a unique solution.

Problem (2.1) is recast in weak form as follows. The bilinear form $A(\cdot, \cdot)$ is given by

$$A(v, w) = (\mathcal{A}\nabla v, \nabla w) + (\mathcal{B} \cdot \nabla v, w) + (cv, w), \qquad v, w \in H^1(\Omega),$$

where $(\cdot, \cdot)$ denotes the $L^2(\Omega)$ or $(L^2(\Omega))^n$ inner product, as appropriate. The solution $u \in H_0^1(\Omega)$ of (2.1) then satisfies

$$(2.3) \qquad\qquad A(u, v) = (f, v) \qquad \forall \, v \in H_0^1(\Omega).$$

Associated with $A(\cdot, \cdot)$, we also introduce the symmetric positive definite form $\widehat{A}(\cdot, \cdot)$ by

$$\widehat{A}(v, w) = (\mathcal{A}\nabla v, \nabla w) + (v, w), \qquad v, w \in H^1(\Omega).$$

The difference form is indicated by

$$(2.4) \qquad\qquad D(v, w) = A(v, w) - \widehat{A}(v, w).$$

For $0 < h < 1$, let $\mathcal{E}_h$ be a triangulation of $\Omega$ into triangles of size $h$, and define the $P_1$-nonconforming finite element space

$$V_h = \{ v \in L^2(\Omega) : \; v|_E \text{ is linear for all } E \in \mathcal{E}_h, v \text{ is continuous}$$
$$\text{at the barycenters of interior edges and}$$
$$\text{vanishes at the barycenters of edges on } \partial\Omega \}.$$

Associated with $V_h$, we define a mesh-dependent form $A_h(\cdot, \cdot)$ by

$$A_h(v, w) = \sum_{E \in \mathcal{E}_h} \{ (\mathcal{A}\nabla v, \nabla w)_E + (\mathcal{B} \cdot \nabla v, w)_E \} + (cv, w), \qquad v, w \in V_h \oplus H_0^1(\Omega),$$

where $(\cdot, \cdot)_E$ is the $L^2(E)$ inner product. The corresponding symmetric form is denoted by $\widehat{A}_h(\cdot, \cdot)$. The nonconforming finite element solution $u_h \in V_h$ of (2.1) is given by

$$(2.5) \qquad\qquad A_h(u_h, v) = (f, v) \qquad \forall \, v \in V_h.$$

The norm induced by $(\widehat{A}_h(v, v))^{1/2}$ for $v \in V_h \oplus H_0^1(\Omega)$ is equivalent to the norm $(\sum_{E \in \mathcal{E}_h} ||\nabla v||_{L^2(E)}^2 + ||v||^2)^{1/2}$. Thus, we define

$$||v||_h = \widehat{A}_h(v, v)^{1/2} \qquad \forall v \in V_h \oplus H_0^1(\Omega).$$

Let us note the inequality

$$(2.6) \qquad\qquad |A_h(v, w)| \leq C ||v||_h ||w||_h \qquad \forall v, w \in V_h \oplus H_0^1(\Omega).$$

It is not hard to show the Garding inequality

$$(2.7) \qquad\qquad C_1 ||v||_h^2 - C_2 ||v||^2 \leq |A_h(v, v)| \qquad \forall v \in V_h \oplus H_0^1(\Omega),$$

where (and below) $C$, with or without a subscript, denotes a generic constant independent on $h$.

With the usual argument [21], we can prove the next theorem.

THEOREM 2.1. *Problem* (2.5) *has a unique solution for h sufficiently small.*
Define the projection operator $\mathcal{P}_h : H_0^1(\Omega) \to V_h$ by

$$A_h(\mathcal{P}_h u, v) = A_h(u, v) \qquad \forall v \in V_h.$$

It follows in the usual way that, if the solution of (2.1) satisfies regularity estimates of the form

(2.8) $$\|u\|_{1+\alpha} \le C\|f\|_{-1+\alpha},$$

then

(2.9) $$\|u - \mathcal{P}_h u\| \le Ch^\alpha \|u - \mathcal{P}_h u\|_h$$

and

(2.10) $$\|\mathcal{P}_h u\|_h \le C\|u\|_h.$$

In the case where regularity estimates of the form of (2.8) are not known to hold, it can be shown as in the conforming case [22] that, given $\epsilon > 0$, there exists an $h_0(\epsilon) > 0$ such that for $0 < h \le h_0$,

(2.11) $$\|u - \mathcal{P}_h u\| \le \epsilon \|u - \mathcal{P}_h u\|_h,$$

and (2.10) is satisfied. The above $\epsilon$ will appear in our later convergence result.

**3. The multigrid algorithm.** To develop a multigrid algorithm for (2.5), we need to assume a structure to our family of partitions. Let $h_1$ and $\mathcal{E}_{h_1} = \mathcal{E}_1$ be given. For each integer $1 < k \le K$, let $h_k = 2^{1-k}h_1$ and $\mathcal{E}_{h_k} = \mathcal{E}_k$ be constructed by connecting the midpoints of the edges of the triangle in $\mathcal{E}_{k-1}$, and let $\mathcal{E}_h = \mathcal{E}_K$ be the finest grid. In this and the following sections, we replace subscript $h_k$ simply by subscript $k$.

Let the mesh size of $\mathcal{E}_1$ be $d_1$; then, by similarity, the mesh size of $\mathcal{E}_k$ is $2^{1-k}d_1$. From Theorem 2.1, for (2.5) to be well behaved, the approximation grid must be sufficiently fine. As in the conforming case [5], we shall require that the coarsest grid in the multilevel algorithm be sufficiently fine. Toward that end, let the coarse grid size be determined by an integer $L$. Then the space $V_k$ has a mesh size of $h_k = 2^{1-L-k}d_1 = 2^{1-k}h_1$, $k = 1, \ldots, K$.

As noted in [5] and demonstrated in our experiments in section 5, in practice, the coarse grid can be taken considerably coarser than the solution grid. The reason for this is that we can only expect that the discrete errors depend monotonically on the grid sizes; consequently, if the fine grid approximation is reasonably accurate, we expect that there exists a sequence of coarser grids whose approximations are well defined.

Following [8, 7], the coarse-to-fine intergrid transfer operator $I_k : V_{k-1} \to V_k$ for $k = 2, \ldots, K$ is defined as follows. For $v \in V_{k-1}$, let $q$ be a midpoint of an edge of a triangle in $\mathcal{E}_k$; then we define $I_k v$ by

$$(I_k v)(q) = \begin{cases} 0 & \text{if } q \in \partial\Omega, \\ v(q) & \text{if } q \notin \partial E \text{ for any } E \in \mathcal{E}_{k-1}, \\ \frac{1}{2}\{v|_{E_1}(q) + v|_{E_2}(q)\} & \text{if } q \in \partial E_1 \cap \partial E_2 \text{ for some } E_1, E_2 \in \mathcal{E}_{k-1}. \end{cases}$$

Let $A_k : V_k \to V_k$ be the discretization operator on level $k$ ($k = 1, \ldots, K$) given by

$$(A_k v, w) = A_k(v, w) \qquad \forall\, w \in V_k.$$

Let $U_k$ indicate the $P_1$-conforming finite element space associated with $\mathcal{E}_k$. For $k = 2, \ldots, K$, we define the projection operators $P_{k-1} : V_k \oplus H^1(\Omega) \to U_{k-1}$ and $P^0_{k-1} : L^2(\Omega) \to U_{k-1}$ by

$$A_{k-1}(P_{k-1}v, w) = A_k(v, w) \qquad \forall w \in U_{k-1},$$

and

$$(P^0_{k-1}v, w) = (v, w) \qquad \forall w \in U_{k-1}.$$

Also, for each $k = 1, \ldots, K$, we introduce the conforming discretization operator $M_k : U_k \to U_k$ by

$$(M_k v, w) = A_k(v, w) \qquad \forall w \in U_k.$$

We first describe a simplest $\mathcal{V}$-cycle multigrid algorithm for iteratively computing the solution of the conforming method. Find $z_k \in U_k$ that satisfies

$$(3.1) \qquad\qquad A_k(z_k, v) = (f, v) \qquad \forall v \in U_k.$$

Algorithm 3.2 below is actually used for the nonconforming method (2.5), which requires Algorithm 3.1. In both algorithms, we smooth only as we proceed to coarser grids. Alternatively, we could consider a multigrid algorithm with just postsmoothing or both pre- and postsmoothing. They can be analyzed analogously, and are not considered here.

The following algorithm iteratively defines a multigrid operator $N_k : U_k \to U_k$. The operator $R_k : U_k \to U_k$ is a linear smoothing operator for the conforming case. A variety of examples for $R_k$ has been given in [5]; we do not repeat these examples in this paper.

MULTIGRID ALGORITHM 3.1. Set $N_1 = M_1^{-1}$. For $1 < k \leq K$, assume that $N_{k-1}$ has been defined and define $N_k g$ for $g \in U_k$ by the following:

1. Set $x_k = R_k g$.
2. Define $N_k g = x_k + q$, where $q \in U_{k-1}$ is given by

$$q = N_{k-1} P^0_{k-1} \left( g - M_k x_k \right).$$

We now define the $\mathcal{V}$-cycle algorithm for the nonconforming method (2.5), which determines a multigrid operator $B_K : V_K \to V_K$. The operator $Q_K : V_K \to V_K$ below is a linear smoothing operator. Examples of this operator will be given in section 3.1.

MULTIGRID ALGORITHM 3.2. If $K = 1$, set $B_1 = A_1^{-1}$. If $K > 1$, define $B_K g$ for $g \in V_K$ by the following:

1. Set $x_K = Q_K g$.
2. Define $B_K g = x_K + q$, where $q \in U_{K-1}$ is given by

$$q = N_{K-1} P^0_{K-1} \left( g - A_K x_K \right).$$

We remark that the coarse-grid correction in Algorithm 3.2 is defined on the conforming finite element spaces. That is, it is of the second type of multigrid algorithm, mentioned in the Introduction. It will be analyzed in section 3.2.

**3.1. Smoothers.** The smoothers presented in this subsection are the variants of those for the conforming finite element method (see, e.g., [5]). We first describe three smoothers which are based on the symmetric problem, and then three smoothers which correspond to the original nonsymmetric and indefinite problem.

The simplest smoother is given in the next example.

*Example* 1. We define

$$Q_K = \lambda_K^{-1} I,$$

where $\lambda_K$ is the largest eigenvalue of $\widehat{A}_K$.

The following two smoothers are defined in terms of subspace decompositions. To this end, let

$$V_K = \sum_{j=1}^{l(K)} V_{j,K},$$

where $V_{j,K}$ is the one-dimensional subspace spanned by a nodal basis function or the one spanned by the nodal basis functions along a line, and $l(K)$ is the number of such spaces. The smoothers in Examples 2 and 3 below are additive and multiplicative, respectively.

*Example* 2. We define

$$Q_K = \gamma \sum_{j=1}^{l(K)} \widehat{A}_{j,K}^{-1} Q_{j,K},$$

where $\widehat{A}_{j,K} : V_{j,K} \to V_{j,K}$ is the symmetric discretization operator on $V_{j,K}$ defined by

$$(\widehat{A}_{j,K} v, \varphi) = \widehat{A}_K(v, \varphi) \qquad \forall \varphi \in V_{j,K},$$

$Q_{j,K} : V_K \to V_{j,K}$ is the projection operator on $V_{j,K}$ with respect to the $L^2$ inner product $(\cdot, \cdot)$, and the constant $\gamma$ is a scaling factor which is chosen to ensure that the smoothing property is satisfied [5].

*Example* 3. Given $g \in V_K$, we define the following:
1. Set $x_0 = 0$.
2. Determine $x_i$, for $i = 1, \ldots, l(K)$, by

$$x_i = x_{i-1} + \widehat{A}_{j,K}^{-1} Q_{j,K}(g - \widehat{A}_K x_{i-1}).$$

3. Set $Q_K g = x_{l(K)}$.

The following example corresponds to the first example, and the later two examples are closely related to Examples 2 and 3.

*Example* 4. We define

$$Q_K = \lambda_K^{-2} A_K^t,$$

where $\lambda_K$ is as in Example 1 and $A_K^t$ is the adjoint operator of $A_K$ with respect to the $L^2$ inner product $(\cdot, \cdot)$.

*Example* 5. We define

$$Q_K = \gamma \sum_{j=1}^{l(K)} A_{j,K}^{-1} Q_{j,K},$$

where $A_{j,K} : V_{j,K} \to V_{j,K}$ is the discretization operator on $V_{j,K}$ given by

$$(A_{j,K}v, \varphi) = A_K(v, \varphi) \qquad \forall \varphi \in V_{j,K},$$

and $Q_{j,K} : V_K \to V_{j,K}$ and $\gamma$ are as in Example 2.

*Example* 6. Given $g \in V_K$, we define the following:

1. Set $x_0 = 0$.
2. Determine $x_i$, for $i = 1, \ldots, l(K)$, by

$$x_i = x_{i-1} + A_{j,K}^{-1} Q_{j,K}(g - A_K x_{i-1}).$$

3. Set $Q_K g = x_{l(K)}$.

**3.2. Analysis of the multigrid algorithm.** We now provide a convergence analysis for Algorithm 3.2 with the smoothers given in Examples 1–6 in the framework of [5]. All of their analysis is based on perturbation from the uniform convergence estimate for the multigrid algorithm applied to the symmetric problem. Essential use in [5] is made of a product representation of the error operator and two properties of the difference form $D(\cdot, \cdot)$ (see (2.4) in [5]). In this section we shall show that our error operator has the same structure (see Lemma 3.2 below), and the form $D(\cdot, \cdot)$ satisfies the same properties (see Lemma 3.3 below). Thus the convergence analysis given in [5] carries over to Algorithm 3.2 since the uniform iterative convergence estimate for Algorithm 3.2 applied to the symmetric problem has been shown in [23, 18, 11].

LEMMA 3.1. *It holds that*

$$B_K = Q_K + N_{K-1}P_{K-1}^0(I - A_K Q_K)$$

*and*

$$N_k = R_k + N_{k-1}P_{k-1}^0(I - M_k R_k), \qquad k = 2, \ldots, K.$$

This lemma can be easily seen from Algorithms 3.1 and 3.2.

LEMMA 3.2. *Let* $P_K = I$, $T_1 = P_1$, $T_k = R_k M_k P_k$, $k = 2, \ldots, K - 1$, *and* $T_K = Q_K A_K P_K$. *Then*

$$(3.2) \qquad I - B_K A_K = (I - T_1)(I - T_2) \cdots (I - T_K).$$

*Proof.* From the definitions of $P_{k-1}$ and $P_{k-1}^0$, we see that

$$P_{K-1}^0 A_K = M_{K-1}P_{K-1},$$

$$P_{k-1}^0 M_k = M_{k-1}P_{k-1}, \qquad k = 2, \ldots, K - 1,$$

$$P_{k-1}P_k = P_{k-1}, \qquad k = 2, \ldots, K.$$

Then it follows from Lemma 3.1 that

$$I - B_K A_K = (I - N_{K-1}M_{K-1}P_{K-1})(I - Q_K A_K)$$

and

$$I - N_{K-1}M_{K-1}P_{K-1}$$

$$= I - P_{K-1} + (I - N_{K-1}M_{K-1})P_{K-1}$$

$$= I - P_{K-1} + (I - N_{K-2}M_{K-2}P_{K-2})(I - R_{K-1}M_{K-1})P_{K-1}$$

$$= (I - N_{K-2}M_{K-2}P_{K-2})(I - P_{K-1} + (I - R_{K-1}M_{K-1})P_{K-1})$$

$$= (I - N_{K-2}M_{K-2}P_{K-2})(I - T_{K-1}).$$

Therefore, a straightforward mathematical induction argument shows the desired result (3.2) since $P_K = I$. $\quad\square$

The product representation of the error operator in Lemma 3.2 is a fundamental ingredient in the convergence analysis. The other important ingredients are the following properties of the difference operator $D(\cdot, \cdot)$. They are trivial in the conforming case; however, as shown below, the second property is not so straightforward in the nonconforming case.

LEMMA 3.3. *Under the above assumption on the coefficient $\mathcal{B}$, there is a constant $C$ independent on $k$ such that*

$$(3.3) \qquad |D(v, w)| \leq C||v||_k||w|| \qquad \forall v, w \in V_k$$

*and*

$$(3.4) \qquad |D(v, w)| \leq C||w||_k||v|| \qquad \forall v, w \in V_k.$$

*Proof.* (3.3) directly follows from the definition of $D(v, w)$:

$$D(v, w) = \sum_{E \in \mathcal{E}_k} (\mathcal{B} \cdot \nabla v, w)_E + ((c-1)v, w).$$

To prove (3.4), we apply integration by parts on each finite element to see that

$$(3.5) \qquad D(v, w) = \sum_{E \in \mathcal{E}_k} \{ (\mathcal{B} \cdot \nu_E v, w)_{\partial E} - (\nabla \cdot \mathcal{B} w + \mathcal{B} \cdot \nabla w, v)_E \} + ((c-1)v, w).$$

Evidently, it suffices to estimate the terms over edges.

Let $E_1, E_2 \in \mathcal{E}_k$ share an edge $e$ with midpoint $m^k$, and let $e$ have the parametric representation $x = x(t)$, $y = y(t)$ with $t$ as parameter. They are linear functions of $t$. Then, by the midpoint rule and the continuity at midpoints on the elements of $V_k$, we find that

$$\int_e (\mathcal{B} \cdot \nu_{E_1} vw)|_{E_1} ds + \int_e (\mathcal{B} \cdot \nu_{E_2} vw)|_{E_2} ds$$

$$= |e|\{(\mathcal{B} \cdot \nu_{E_1} vw)|_{E_1}(m^k) + (\mathcal{B} \cdot \nu_{E_2} vw)|_{E_2}(m^k)\}$$

$$+ \frac{|e|^3}{24} \left\{ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \right\}^{-1} \left\{ \frac{d^2}{dt^2}(\mathcal{B} \cdot \nu_{E_1} vw)|_{E_1}(\xi_1^k) + \frac{d^2}{dt^2}(\mathcal{B} \cdot \nu_{E_2} vw)|_{E_2}(\xi_2^k) \right\}$$

$$= \frac{|e|^3}{24} \left\{ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \right\}^{-1} \left\{ \frac{d^2}{dt^2}(\mathcal{B} \cdot \nu_{E_1} vw)|_{E_1}(\xi_1^k) + \frac{d^2}{dt^2}(\mathcal{B} \cdot \nu_{E_2} vw)|_{E_2}(\xi_2^k) \right\},$$

for some points $\xi_1^k, \xi_2^k \in e$. Note that, since $v$ and $w$ are piecewise linear, for $i = 1, 2$,

$$\frac{d^2}{dt^2}(\mathcal{B} \cdot \nu_{E_i} vw) = \frac{d^2}{dt^2}(\mathcal{B} \cdot \nu_{E_i})vw + 2\frac{d}{dt}(\mathcal{B} \cdot \nu_{E_i})\frac{d}{dt}(vw) + 2(\mathcal{B} \cdot \nu_{E_i})\frac{dv}{dt}\frac{dw}{dt}.$$

Also, by the chain rule, we have with any function $g = g(x(t), y(t))$

$$\frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt}$$

and

$$\frac{d^2 g}{dt^2} = \frac{\partial^2 g}{\partial x^2}\left(\frac{dx}{dt}\right)^2 + 2\frac{\partial^2 g}{\partial x \partial y}\frac{dx}{dt}\frac{dy}{dt} + \frac{\partial^2 g}{\partial y^2}\left(\frac{dy}{dt}\right)^2,$$

since $e$ is a line segment. Consequently, we see that

$$\left| \int_e (\mathcal{B} \cdot \nu_{E_1} vw)|_{E_1} ds + \int_e (\mathcal{B} \cdot \nu_{E_2} vw)|_{E_2} ds \right|$$

$$\leq \frac{C|e|^3}{24} \sum_{i=1}^2 \left( |v| + |\tfrac{\partial v}{\partial x}| + |\tfrac{\partial v}{\partial y}| \right) \left( |w| + |\tfrac{\partial w}{\partial x}| + |\tfrac{\partial w}{\partial y}| \right) (\xi_i^k).$$

This, together with the Cauchy–Schwarz inequality, an inverse inequality, and the fact that $v$ and $w$ are piecewise linear, implies that

$$\left| \sum_{E \in \mathcal{E}_k} (\mathcal{B} \cdot \nu_E v, w)_{\partial E} \right|$$

$$\leq Ch_k^3 \sum_{e \in \partial \mathcal{E}_k} \left( |v| + |\tfrac{\partial v}{\partial x}| + |\tfrac{\partial v}{\partial y}| \right) \left( |w| + |\tfrac{\partial w}{\partial x}| + |\tfrac{\partial w}{\partial y}| \right) (\xi_e^k)$$

$$\leq Ch_k (||v|| + ||v||_k)(||w|| + ||w||_k)$$

$$\leq C||w||_k ||v||,$$

which, by (3.5), yields the desired result (3.4). Thus the proof is complete.    □

With Lemmas 3.2 and 3.3 and the arguments presented in Theorems 5.2–5.6 of [5], we have the following theorem.

THEOREM 3.4. *Let $Q_K$ be one of the smoothers defined in Examples 1–6. Then, given $\epsilon > 0$, there exists an $h_0 > 0$ such that for $h_1 \leq h_0$,*

$$\widehat{A}_K(Ev, Ev) \leq \delta^2 \widehat{A}_K(v, v) \qquad \forall v \in V_K,$$

*where $E = I - B_K A_K$, $\delta = \widehat{\delta} + C(h_1 + \epsilon)$, and $\widehat{\delta}$ is less than one and independent on $K$.*

We remark that $\widehat{\delta}$ comes from the uniform convergence estimate of Algorithm 3.2 applied to the symmetric problem [11, 18].

**4. The multigrid algorithm for mixed methods.** In this section, we now consider a mixed finite element method for numerically solving (2.1). The Raviart–Thomas space [20] over triangles is given by

$$\Lambda_h = \left\{ v \in (L^2(\Omega))^2 : v|_E = \left( a_E^1 + a_E^2 x,\ a_E^3 + a_E^2 y \right),\ a_E^i \in \mathbb{R},\ E \in \mathcal{E}_h \right\},$$
$$W_h = \left\{ w \in L^2(\Omega) : w|_E \text{ is constant for all } E \in \mathcal{E}_h \right\},$$
$$L_h = \left\{ \mu \in L^2(\partial \mathcal{E}_h) : \mu|_e \text{ is constant},\ e \in \partial \mathcal{E}_h;\ \mu|_e = 0,\ e \subset \partial \Omega \right\},$$

where $\partial \mathcal{E}_h$ denotes the set of all interior edges. Then the hybrid form of the mixed finite element solution to (2.1) is $(\sigma_h, u_h, \lambda_h) \in \Lambda_h \times W_h \times L_h$, satisfying

$$(4.1) \quad \begin{aligned} \sum_{E \in \mathcal{E}_h} (\nabla \cdot \sigma_h, w)_E - (\mathcal{Y}_h \cdot \sigma_h, w) + (c_h u_h, w) &= (f, w) &\forall\, w \in W_h, \\ (\mathcal{X}_h \sigma_h, v) - \sum_{E \in \mathcal{E}_h} [(u_h, \nabla \cdot v)_E - (\lambda_h, v \cdot \nu_E)_{\partial E}] &= 0 &\forall\, v \in \Lambda_h, \\ \sum_{E \in \mathcal{E}_h} (\sigma_h \cdot \nu_E, \mu)_{\partial E} &= 0 &\forall\, \mu \in L_h, \end{aligned}$$

where $\nu_E$ denotes the unit outer normal to $E$, $\mathcal{X}_h = \mathcal{Q}_h \mathcal{A}^{-1}$ (componentwise), $\mathcal{Y}_h = \mathcal{X}_h(\mathcal{Q}_h \mathcal{B})$, $c_h = \mathcal{Q}_h c$, and $\mathcal{Q}_h$ denotes the $L^2(\Omega)$ projection operator onto $W_h$.

The solution $\sigma_h$ is introduced to approximate the vector field

$$\sigma = -\mathcal{A}\nabla u,$$

which is the variable of primary interest in many applications. Since $\sigma$ lies in the space

$$H(\mathrm{div}; \Omega) = \left\{ v \in (L^2(\Omega))^2 : \nabla \cdot v \in L^2(\Omega) \right\},$$

and we do not require that $\Lambda_h$ be a subspace of $H(\text{div}; \Omega)$, the last equation in (4.1) is used to enforce that the normal components of $\sigma_h$ are continuous across the interior edges in $\partial \mathcal{E}_h$, so in fact $\sigma_h \in H(\text{div}; \Omega)$. Also, the projection of the coefficients $\mathcal{A}^{-1}$, $\mathcal{B}$, and $c$ into the space $W_h$ is introduced in (4.1). The projection of coefficients gives us considerable computational savings, without any loss of accuracy [12]. Furthermore, it can be used to establish an equivalence between the triangular nonconforming method and the mixed method (4.1). This equivalence has been obtained in [2, 3, 11, 13, 14] for the symmetric case. We now extend it to the present nonsymmetric problem.

There is no continuity requirement on the spaces $\Lambda_h$ and $W_h$, so $\sigma_h$ and $u_h$ can be locally (element by element) eliminated from (4.1). In fact, applying the ideas in [11], (4.1) can be algebraically condensed to the symmetric, positive definite system for the Lagrange multiplier $\lambda_h$:

$$(4.2) \qquad\qquad M\lambda = F,$$

where the contributions of the triangle $E$ to the stiffness matrix $M$ and the right-hand side $F$ are

$$m_{ij}^E = \overline{\nu}_E^i \beta^E \overline{\nu}_E^j + \tfrac{1}{3}(\mathcal{Q}_h \mathcal{B})|_E \cdot \overline{\nu}_E^i + \tfrac{1}{3}(c,1)_E \delta_{ij},$$

$$F_i^E = -\frac{(J_E^f, \overline{\nu}_E^i)_E}{|E|} + (J_E^f, \nu_E^i)_{e_E^i},$$

where $\nu_E^i$ denotes the outer unit normal to the edge $e_E^i$ ($E$ has three edges), $\overline{\nu}_E^i = |e_E^i|\nu_E^i$, $|e_E^i|$ is the length of $e_E^i$, $\beta^E = (((\mathcal{X}_h)_{ij}, 1)_E)^{-1}$, $J_E^f = (f,1)_E(x,y)/(2|E|)$, $\delta_{ij}$ is the Kronecker symbol, and $|E|$ denotes the area of $E$. After the computation of $\lambda_h$, (4.1) can be used to recover $\sigma_h$ and $u_h$ on each element. Furthermore, with the same argument as in [11], we have the next result.

THEOREM 4.1. *System (4.2) corresponds to the linear system arising from the nonconforming problem. Find $\psi_h \in V_h$ such that*

$$\tilde{A}_h(\psi_h, \varphi) = (f_h, \varphi) \quad \forall \varphi \in V_h,$$

*where*

$$\tilde{A}_h(\psi_h, \varphi) = \sum_{E \in \mathcal{E}_h} \left\{ (\mathcal{X}_h^{-1} \nabla \psi_h, \nabla \varphi)_E + (\mathcal{Q}_h \mathcal{B} \cdot \nabla \psi_h, \varphi)_E \right\} + (c_h \psi_h, \varphi).$$

It thus follows that Algorithm 3.2 can be exploited to solve the system arising from the mixed method (4.1), and the convergence result in Theorem 3.4 is valid. It is known that the linear system arising from the mixed finite element method is a saddle point problem, which can be expensive to solve. One of the useful numerical methods for solving this saddle point problem is the inexact Uzawa algorithm (see, e.g., the reference in [15]). However, it turns out [2, 10, 11] that the nonmixed formulation approach under consideration is more efficient.

**5. Numerical examples.** In this section we report the results of numerical examples to illustrate our theory. We consider the nonsymmetric and indefinite problem

$$(5.1) \qquad \begin{aligned} -\nabla \cdot (\mathcal{A}\nabla u) + \mathcal{B} \cdot \nabla u + cu &= f \quad \text{in } \Omega = (0,1)^2, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

In (5.1), we take $\mathcal{A}$ to be the identity matrix, the right-hand side $f$ is generated randomly, and three different choices for the constants $\mathcal{B}$ and $c$ are made in our

TABLE 1
*Convergence results with one Jacobi presmoothing and with conforming corrections.*

| $c$ | $(h_K, h_1)$ | $\overline{\delta}_v$ | $\|E\|^*$ |
|---|---|---|---|
| -5 | $(1/16, 1/4)$ | 0.639436 | 0.74 |
| 10 | $(1/16, 1/4)$ | 0.634469 | 0.74 |
| 15 | $(1/16, 1/8)$ | NC | - |
| -5 | $(1/32, 1/4)$ | 0.737174 | 0.77 |
| 10 | $(1/32, 1/4)$ | 0.735298 | 0.77 |
| 15 | $(1/32, 1/8)$ | 0.732282 | 0.76 |
| -5 | $(1/64, 1/4)$ | 0.658209 | 0.75 |
| 10 | $(1/64, 1/4)$ | 0.657875 | 0.75 |
| 15 | $(1/64, 1/8)$ | 0.657205 | 0.75 |

TABLE 2
*Convergence results with one Gauss–Seidel presmoothing and with conforming corrections.*

| $c$ | $(h_K, h_1)$ | $\overline{\delta}_v$ | $\|E\|^*$ |
|---|---|---|---|
| -5 | $(1/16, 1/4)$ | 0.464180 | 0.56 |
| 10 | $(1/16, 1/4)$ | 0.428082 | 0.53 |
| 15 | $(1/16, 1/8)$ | NC | - |
| -5 | $(1/32, 1/4)$ | 0.551210 | 0.57 |
| 10 | $(1/32, 1/4)$ | 0.533360 | 0.55 |
| 15 | $(1/32, 1/8)$ | 0.523353 | 0.54 |
| -5 | $(1/64, 1/4)$ | 0.490357 | 0.56 |
| 10 | $(1/64, 1/4)$ | 0.484002 | 0.55 |
| 15 | $(1/64, 1/8)$ | 0.481043 | 0.55 |

TABLE 3
*Convergence results with one Jacobi presmoothing and with nonconforming corrections.*

| $c$ | $(h_K, h_1)$ | $\overline{\delta}_v$ | $\|E\|^*$ |
|---|---|---|---|
| -5 | $(1/16, 1/4)$ | 0.486392 | 0.56 |
| 10 | $(1/16, 1/4)$ | 0.613320 | 0.88 |
| 15 | $(1/16, 1/8)$ | 0.476986 | 0.56 |
| -5 | $(1/32, 1/4)$ | 0.651377 | 0.77 |
| 10 | $(1/32, 1/4)$ | 0.902975 | 1.25 |
| 15 | $(1/32, 1/8)$ | 0.721914 | 1.00 |
| -5 | $(1/64, 1/4)$ | 0.726687 | 0.93 |
| 10 | $(1/64, 1/4)$ | 0.978622 | 1.37 |
| 15 | $(1/64, 1/8)$ | 0.967870 | 1.24 |

experiments:

$$\mathcal{B} = (c, c) \text{ with } c = -5, \ 10, \ 15.$$

We first report the results obtained by using Algorithm 3.2 with one (point) Jacobi and Gauss–Seidel presmoothing. They are shown in Tables 1 and 2, respectively, where $(h_K, h_1)$ denotes the mesh sizes of the finest and coarsest grids, respectively. We report the average error reduction factor in 50 iterations $\overline{\delta}_v$. Also, for comparison we report the following quantity with a randomly chosen initial guess:

$$\|E\|^* = \max_{1 \leq i \leq 50} \widehat{A}_K(v_{i+1}, v_{i+1}) / \widehat{A}_K(v_i, v_i),$$

TABLE 4
*Convergence results with one Gauss–Seidel presmoothing and with nonconforming corrections.*

| $c$ | $(h_K, h_1)$ | $\overline{\delta}_v$ | $||E||^*$ |
|---|---|---|---|
| -5 | (1/16, 1/4) | 0.653408 | 0.76 |
| 10 | (1/16, 1/4) | 0.247183 | 0.30 |
| 15 | (1/16, 1/8) | 0.236150 | 0.27 |
| -5 | (1/32, 1/4) | 0.750099 | 0.88 |
| 10 | (1/32, 1/4) | 0.299764 | 0.34 |
| 15 | (1/32, 1/8) | 0.289018 | 0.41 |
| -5 | (1/64, 1/4) | 0.795464 | 0.95 |
| 10 | (1/64, 1/4) | 0.312528 | 0.39 |
| 15 | (1/64, 1/8) | 0.350614 | 0.52 |

TABLE 5
*Convergence results with one Jacobi pre- and postsmoothing and with nonconforming corrections.*

| $c$ | $(h_K, h_1)$ | $\overline{\delta}_v$ | $||E||^*$ |
|---|---|---|---|
| -5 | (1/16, 1/4) | 0.340037 | 0.41 |
| 10 | (1/16, 1/4) | 0.337256 | 0.39 |
| 15 | (1/16, 1/8) | 0.332164 | 0.38 |
| -5 | (1/32, 1/4) | 0.366170 | 0.47 |
| 10 | (1/32, 1/4) | 0.354337 | 0.41 |
| 15 | (1/32, 1/8) | 0.353695 | 0.41 |
| -5 | (1/64, 1/4) | 0.374882 | 0.57 |
| 10 | (1/64, 1/4) | 0.388482 | 0.55 |
| 15 | (1/64, 1/8) | 0.342782 | 0.42 |

with $v_{i+1} = Ev_i$, $1 \leq i \leq 50$, which gives some indication about the norm of $E$:

$$||E|| = \sup_{v \in V_K} \widehat{A}_K(Ev, Ev)/\widehat{A}_K(v, v).$$

In the cases where there is no convergence (denoted by NC in the tables), the coarsest levels in the multigrid iteration are not fine enough. This agrees with our earlier theory on the nonsymmetric and indefinite problem, where the coarsest levels need to be sufficiently fine. Overall, in the case where there is convergence, the Gauss–Seidel smoothing performs better than the Jacobi smoothing, and $\overline{\delta}_v$ and $||E||^*$ are quite small for both smoothers. When $c = 15$, the coarsest level needs to be finer. This is the case where the convection term becomes "bigger."

For comparison, we also demonstrate the results produced by using the first type of multigrid algorithm; i.e., all the coarse-grid corrections are defined on the nonconforming spaces instead of the conforming spaces. The results with one Jacobi and Gauss–Seidel presmoothing are presented in Tables 3 and 4, respectively. Evidently, the results with the Gauss–Seidel smoothing are much better than those with the Jacobi smoothing. As the finest level got higher (e.g., $h_K = 1/128$, not reported here), we observed that the average error reduction factor approaches 0.98 with the Jacobi smoothing. For this reason, we experimented with the first type of multigrid algorithm with one Jacobi and one Gauss–Seidel, both pre- and postsmoothing. The results are displayed in Tables 5 and 6, respectively. It appears that this type of algorithm needs at least two smoothing steps to have good results with the Jacobi smoothing. Finally, while there is no theoretical analysis for the $\mathcal{W}$-cycle algorithm

TABLE 6

*Convergence results with Gauss–Seidel pre- and postsmoothing and with nonconforming corrections.*

| $c$ | $(h_K, h_1)$ | $\overline{\delta}_v$ | $||E||^*$ |
|---|---|---|---|
| -5 | (1/16, 1/4) | 0.165309 | 0.18 |
| 10 | (1/16, 1/4) | 0.130279 | 0.16 |
| 15 | (1/16, 1/8) | 0.122040 | 0.15 |
| -5 | (1/32, 1/4) | 0.181391 | 0.23 |
| 10 | (1/32, 1/4) | 0.186105 | 0.23 |
| 15 | (1/32, 1/8) | 0.184373 | 0.24 |
| -5 | (1/64, 1/4) | 0.208130 | 0.26 |
| 10 | (1/64, 1/4) | 0.219439 | 0.28 |
| 15 | (1/64, 1/8) | 0.215479 | 0.28 |

TABLE 7

*Convergence results of the $\mathcal{W}$-cycle with Jacobi pre- and postsmoothing and with nonconforming corrections.*

| $c$ | $(h_K, h_1)$ | $\overline{\delta}_w$ | $||E||^*$ |
|---|---|---|---|
| -5 | (1/16, 1/4) | 0.330665 | 0.39 |
| 10 | (1/16, 1/4) | 0.330144 | 0.39 |
| 15 | (1/16, 1/8) | 0.326979 | 0.38 |
| -5 | (1/32, 1/4) | 0.345017 | 0.39 |
| 10 | (1/32, 1/4) | 0.344432 | 0.39 |
| 15 | (1/32, 1/8) | 0.343179 | 0.39 |
| -5 | (1/64, 1/4) | 0.325344 | 0.38 |
| 10 | (1/64, 1/4) | 0.325438 | 0.38 |
| 15 | (1/64, 1/8) | 0.325509 | 0.38 |

TABLE 8

*Convergence results of the $\mathcal{W}$-cycle with Gauss–Seidel pre- and postsmoothing and with nonconforming corrections.*

| $c$ | $(h_K, h_1)$ | $\overline{\delta}_w$ | $||E||^*$ |
|---|---|---|---|
| -5 | (1/16, 1/4) | 0.133058 | 0.16 |
| 10 | (1/16, 1/4) | 0.137478 | 0.18 |
| 15 | (1/16, 1/8) | 0.146508 | 0.14 |
| -5 | (1/32, 1/4) | 0.113530 | 0.15 |
| 10 | (1/32, 1/4) | 0.112054 | 0.15 |
| 15 | (1/32, 1/8) | 0.124663 | 0.18 |
| -5 | (1/64, 1/4) | 0.116710 | 0.15 |
| 10 | (1/64, 1/4) | 0.117474 | 0.15 |
| 15 | (1/64, 1/8) | 0.123881 | 0.15 |

for the nonsymmetric problem, we point out that the results generated by the $\mathcal{W}$-cycle algorithm are slightly better than those yielded by the $\mathcal{V}$-cycle algorithm, as shown in Tables 7 and 8.

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