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An immersed weak Galerkin method for elliptic interface problems on polygonal meshes ☆

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ABSTRACT

In this paper we present an immersed weak Galerkin method for solving second-order elliptic interface problems on polygonal meshes, where the meshes do not need to be aligned with the interface. The discrete space consists of constants on each edge and broken linear polynomials satisfying the interface conditions in each element. For triangular meshes, such broken linear polynomials coincide with the basis functions in immersed finite element methods [33]. We establish some approximation properties of the broken linear polynomials and the discrete weak gradient of a certain projection of the solution on polygonal meshes. We then prove an optimal error estimate of our scheme in the discrete H^1-seminorm under some assumptions on the exact solution. Numerical experiments are provided to confirm our theoretical analysis.

1. Introduction

There are a wide range of physical and engineering problems that are governed by partial differential equations having an interface. For example, a second-order elliptic partial differential equation with a discontinuous coefficient is often used as a model problem in material sciences and porous media involving multiple materials or media. To solve such a problem, one can use some classical numerical schemes with interface-fitted meshes, such as finite element methods (FEMs), discontinuous Galerkin (DG) methods, etc. However, it is difficult and takes a lot of time to generate such fitted meshes when the domain boundary and the interface are geometrically complicated. Even worse, when the interface is moving, one needs to generate a new fitted mesh as time evolves.

To overcome such difficulties, researchers developed and studied some numerical methods using unfitted/structured meshes, such as cut finite element methods (CutFEMs) [3,15,25,26], extended finite element methods (XFEMs) [6,7,29,35,41], immersed finite element methods (IFEMs) [28,31,33,36,37], to name just a few. In particular, the IFEMs use basis functions that are modified so that they satisfy the interface conditions. The authors in [36,37] studied IFEMs using uniform triangular or rectangular grids. In [31,38], the performance of the IFEMs was improved by adding penalty terms that are commonly used in DG methods. Linear and bilinear nonconforming IFEMs were studied in [33,39]. The IFEM was also successfully applied to other interface problems: interface elasticity problems [32], elliptic eigenvalue interface problems [34], Stokes interface problems [1], etc.

On the other hand, several numerical methods using general polytopal meshes have been developed, such as hybrid high-order (HHO) methods [21–23], virtual element methods (VEMs) [2,4,11], weak Galerkin (WG) methods (or weak Galerkin finite element methods) [42,46,47], etc. Here we explain the WG methods in some detail. In WG methods, the discrete space consists of polynomials on an element interior and polynomials on its edges, and the differential operators are replaced by the so-called weak differential operators. Compared to the classical FEMs, the WG methods have several advantages. For example, WG methods can handle general polygonal and polyhedral meshes while the FEMs cannot. In addition, the WG methods can be generalized to higher orders directly. Due to such advantages, the WG methods were successfully applied to various problems: Darcy problems [47], Stokes equations [48], elasticity problems [51], Maxwell equations [43], etc. For more thorough survey, we refer to [27,40,42,46,49,50] and references therein.

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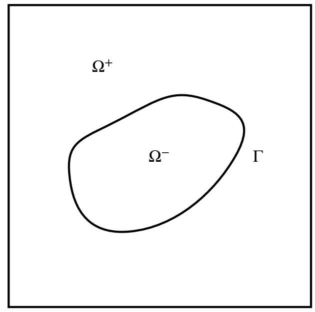

Fig. 1. A domain Ω with interface Γ.

Recently, some researchers developed numerical methods using unfitted polygonal meshes for solving interface problems. Using such meshes provides advantages of both polygonal meshes and unfitted meshes. For example, polygonal meshes enable us to implement the mesh generation process with great flexibility for complicated geometries. Unfitted meshes are easy to generate and useful for moving interfaces such as time-evolving interfaces. We also note that polygonal meshes have been used in many applications: adaptive locally refined meshes, non-matching meshes, hybrid meshes, etc (see, e.g., [13,30]). In [14,16], the authors proposed unfitted HHO methods for the elliptic interface problems. They used a Nitsche-type formulation and proved optimal error estimates in the H^1-norm. However, the methods double the degrees of freedom in the interface elements, and require some local cell-agglomeration procedures to ensure the assumptions on the interface elements. On the other hand, the Lagrange-type immersed VEMs for the elliptic interface problems were developed [17]. Unlike the classical Lagrange-type immersed FEM [31,38], the discrete space is conforming, and the method does not require the DG-type consistency terms. However, the authors only considered the triangular meshes, and their analysis cannot be generalized to the polygonal meshes. Meanwhile, an immersed WG method was proposed in [44], but it is also limited to the triangular meshes. Besides, the discrete bilinear form formulated in their method is different from the usual WG method; they use the usual gradient and DG-type consistency terms.

In this paper, we develop a new immersed WG method for the elliptic interface problems. Our method uses general polygonal meshes which allows the interface cut through the interior. We generalize the discrete weak gradient to the case when the coefficient is discontinuous, and use it to define the bilinear form. Our weak gradient coincides with the usual one [42] when the coefficient is constant. However, they are different from each other when the coefficient is non-constant. In addition, compared to the unfitted HHO method [14,16], our method has some advantages: the mesh assumption is less restrictive, that is, the local cell-agglomeration procedures are not necessary, and our method has fewer degrees of freedom on each interface element.

The rest of the paper is organized as follows. In the next section, we describe the model problem and summarize some preliminaries. In Section 3, we propose our immersed WG method for the model problem, and prove that the discrete problem is well-posed. In Section 4, we prove some technical inequalities and approximation properties of broken linear polynomials on polygonal elements. In Section 5, we derive an optimal error estimate in the discrete H^1-seminorm under some regularity assumptions on the exact solution. Finally, in Section 6, we present some numerical experiments that confirm our theoretical analysis.

2. Preliminaries

We follow the usual notation of Sobolev spaces, inner product, seminorms, and norms (see, for example, [20]). Let D be a bounded domain in $\mathbb{R}$ or $\mathbb{R}^2$. For $\sigma \geq 0$, we denote by $\|\cdot\|_{\sigma,D}$ and $|\cdot|_{\sigma,D}$ the usual norm and seminorm of the Sobolev space $H^\sigma(D)$, respectively. We also denote by $(\cdot,\cdot)_{0,D}$ the usual inner product in $L^2(D)$. We define $H^{-1/2}(D)$ as the dual space of $H^{1/2}(D)$ equipped with the norm given by

$$\|u\|_{-1/2,D} := \sup_{v\in H^{1/2}(D)} \frac{\langle u,v\rangle_D}{\|v\|_{1/2,D}},$$

where $\langle\cdot,\cdot\rangle_D$ is the duality pairing. For a nonnegative integer k, we denote by $\mathbb{P}_k(D)$ the space of all polynomials of degree $\leq k$ on D.

2.1. Model problem

Let Ω be a polygonal domain in $\mathbb{R}^2$, which is separated into two disjoint subdomains Ω^+ and Ω^- by an interface Γ, as in Fig. 1. Here we assume that Γ is a C^2-curve that is not self-intersecting. For any domain $D \subset \Omega$ and any function $u : D \to \mathbb{R}$, we define its jump across the portion of the interface $\Gamma \cap D$ as

$$[u]_{\Gamma\cap D} := u|_{D\cap\Omega^+} - u|_{D\cap\Omega^-}.$$

We consider the following elliptic interface problem: Given $f \in L^2(\Omega)$, find $u \in H_0^1(\Omega)$ such that

$$\begin{cases} -\nabla\cdot(\beta\nabla u) = f & \text{in } \Omega^+\cup\Omega^-, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

with the jump conditions on the interface

$$[u]_\Gamma = 0, \quad \left[\beta\frac{\partial u}{\partial \boldsymbol{n}}\right]_\Gamma = 0, \tag{2.2}$$

where β is a positive and piecewise $W^{1,\infty}$-function on $\overline{\Omega}$ bounded below and above by two positive constants β_* and β^* with $0 < \beta^- \leq \beta^+ < \infty$. That is,

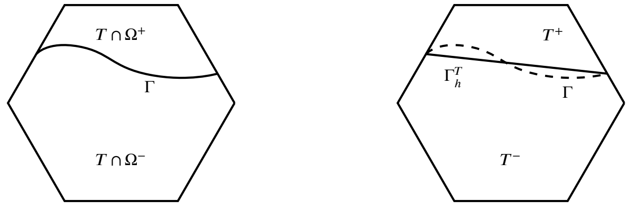

Fig. 2. An interface element T in $\mathcal{T}_h$.

$$\beta(\boldsymbol{x}) = \begin{cases} \beta^+(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \Omega^+, \\ \beta^-(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \Omega^-, \end{cases}$$

for some functions $\beta^+ \in W^{1,\infty}(\overline{\Omega^+}), \beta^- \in W^{1,\infty}(\overline{\Omega^-})$ such that $\beta_* \le \beta^s \le \beta^*$, $s = +, -$. A weak formulation of the model problem (2.1)-(2.2) is written as follows: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \beta \nabla u \cdot \nabla v \, \mathrm{d}\boldsymbol{x} = \int_\Omega f v \, \mathrm{d}\boldsymbol{x} \quad \forall v \in H_0^1(\Omega). \tag{2.3}$$

For any domain $D \subset \Omega$ and $3/2 < s \le 2$, let us introduce the space

$$\widetilde{H}^s(D) := \left\{ u \in H^1(D) : u|_{D\cap\Omega^s} \in H^s(D \cap \Omega^s), \ s = +, - \right\}$$

equipped with the following norm and seminorm:

$$\|u\|^2_{\widetilde{H}^s(D)} := \|u\|^2_{1,D} + |u|^2_{s,D\cap\Omega^+} + |u|^2_{s,D\cap\Omega^-},$$
$$|u|^2_{\widetilde{H}^s(D)} := |u|^2_{s,D\cap\Omega^+} + |u|^2_{s,D\cap\Omega^-}.$$

We also define

$$\widetilde{H}^s_\Gamma(D) := \left\{ u \in \widetilde{H}^s(D) : \left[\beta \frac{\partial u}{\partial \boldsymbol{n}} \right]_{\Gamma\cap D} = 0 \right\}.$$

Then we have the following regularity theorem for the problem (2.3); see [9,18,24].

Theorem 2.1. *Suppose that Ω is convex and $f \in L^2(\Omega)$. Then the problem (2.3) has a unique solution $u \in H_0^1(\Omega) \cap \widetilde{H}^2_\Gamma(\Omega)$ such that*

$$\|u\|_{\widetilde{H}^2(\Omega)} \le C \|f\|_{0,\Omega} \tag{2.4}$$

for some generic positive constant C.

Remark 2.2. Theorem 2.1 holds with lower regularity when Ω is nonconvex: the solution u is a piecewise H^s-function for some $3/2 < s \le 2$, where s depends on the angle of re-entrant corners of $\partial\Omega$ (see, e.g., [5,9,24]). However, since our analysis below can be carried out with minor change, we assume that $u \in \widetilde{H}^2_\Gamma(\Omega)$ for the simplicity of analysis.

2.2. Mesh assumptions

Let $\{\mathcal{T}_h\}_h$ be a family of decompositions (meshes) of Ω into finitely many nonoverlapping polygonal elements T with maximum diameter h. Let $\mathcal{E}_h$ be the set of all edges in $\mathcal{T}_h$. Let $\mathcal{E}_h^i$ and $\mathcal{E}_h^b$ denote the set of all interior and boundary edges in $\mathcal{T}_h$, respectively. For each $T \in \mathcal{T}_h$, let $\mathcal{E}_T$ be the set of all edges of T. For each $T \in \mathcal{T}_h$, we denote by $|T|$ the area of T, by h_T the diameter of T, and by $\boldsymbol{n}_T$ its exterior unit normal vector along the boundary ∂T. For each $e \in \mathcal{E}_h$, we denote by $|e|$ the length of e. For $e \in \mathcal{E}_h^i$, we define $\boldsymbol{n}_e$ by a unit normal vector of e with orientation fixed once and for all. For $e \in \mathcal{E}_h^b$, we define $\boldsymbol{n}_e$ by a unit normal vector on e in the outward direction with respect to Ω.

We call an element $T \in \mathcal{T}_h$ an interface element if the interface Γ passes through the interior of T; otherwise we call T a noninterface element. We denote by $\mathcal{T}_h^I$ the collection of all interface elements in $\mathcal{T}_h$, and by $\mathcal{T}_h^N$ the collection of all non-interface elements in $\mathcal{T}_h$. For an interface element $T \in \mathcal{T}_h$, we denote by Γ_h^T the line segment connecting the intersections of Γ and the edges of T. This line segment divides T into two parts T^+ and T^- with $\overline{T} = \overline{T^+ \cup T^-}$ (see, for example, Fig. 2). For any function $u : T \to \mathbb{R}$, we define its jump across $\Gamma_h^T \cap T$ as

$$[u]_{\Gamma_h^T} := u|_{T^+} - u|_{T^-}.$$

We assume that the following holds [4,33,47].

Assumption 2.3. $\{\mathcal{T}_h\}_h$ satisfies the following properties:

(i) there exists a constant $\rho > 0$ independent of h such that every element T of $\mathcal{T}_h$ is star-shaped with respect to a ball B_T with center $\boldsymbol{x}_T$ and radius ρh_T, and every edge of T has length larger than ρh_T;
(ii) the interface Γ meets the edges of an interface element at no more than two points;
(iii) the interface Γ meets each edge in $\mathcal{E}_h$ at most once, except possibly it passes through two vertices.

Remark 2.4. The assumptions (ii) and (iii) are reasonable if h is sufficiently small. Moreover, the assumption above is less restrictive than the one used in [14,16], since the methods in [14,16] require that both T^+ and T^- must contain balls with radius comparable to h_T. Note also that the assumption (i) implies the following properties [11]: there exists $N \in \mathbb{N}$ depending only on ρ such that any $T \in \mathcal{T}_h$ has at most N edges and vertices, and can be decomposed as at most N triangles, obtained by connecting the vertices of T to $\boldsymbol{x}_T$, such that the minimum angle of the triangles is controlled by ρ.

Throughout this paper, C will denote a generic positive constant independent of h, not necessarily the same in each occurrence.

3. Immersed weak Galerkin method

In this section, we describe an immersed WG method for the problem (2.3).

3.1. Broken polynomial space

Let $T \in \mathcal{T}_h$ be an interface element. We define the piecewise constant function $\overline{\beta}_T$ on the element T as follows:

$$\overline{\beta}_T(\boldsymbol{x}) = \begin{cases} \overline{\beta}^+ & \text{if } \boldsymbol{x} \in T^+, \\ \overline{\beta}^- & \text{if } \boldsymbol{x} \in T^-, \end{cases}$$

where $\overline{\beta}^s := \beta^s(\boldsymbol{x}^s)$ and $\boldsymbol{x}^s$ denotes the barycenter of T^s for $s = +, -$. We also let $\overline{\beta}$ be the piecewise constant function such that $\overline{\beta}|_T = \overline{\beta}_T$ on each $T \in \mathcal{T}_h$. The broken polynomial space $\widehat{\mathbb{P}}_1(T)$ of degree ≤ 1 is defined by

$$\widehat{\mathbb{P}}_1(T) := \left\{ q : q|_{T^+} \in \mathbb{P}_1(T^+),\ q|_{T^-} \in \mathbb{P}_1(T^-),\ [q]_{\Gamma_h^T} = 0,\ \left[\overline{\beta}_T \frac{\partial q}{\partial \boldsymbol{n}}\right]_{\Gamma_h^T} = 0 \right\}.$$

It is easy to see that $\dim \widehat{\mathbb{P}}_1(T) = 3$ (see, for example, [33, Theorem 2.2]), and the following piecewise polynomials form a basis of $\widehat{\mathbb{P}}_1(T)$:

$$\varphi_1(\boldsymbol{x}) = 1, \quad \varphi_2(\boldsymbol{x}) = \boldsymbol{t} \cdot (\boldsymbol{x} - \boldsymbol{x}_0), \quad \varphi_3(\boldsymbol{x}) = \overline{\beta}_T^{-1} \boldsymbol{n} \cdot (\boldsymbol{x} - \boldsymbol{x}_0),$$

where $\boldsymbol{x}_0$ is the midpoint of the line segment Γ_h^T, $\boldsymbol{n} = (n_1, n_2)$ is a unit vector normal to Γ_h^T pointing from T^+ to T^-, and $\boldsymbol{t} = (-n_2, n_1)$. Note that, since $\widehat{\mathbb{P}}_1(T) \subset H^1(T)$, the space $\nabla\widehat{\mathbb{P}}_1(T)$ is well-defined, and the vector-valued functions $\nabla\varphi_2$ and $\nabla\varphi_3$ form a basis of $\nabla\widehat{\mathbb{P}}_1(T)$.

For convenience, we set $\widehat{\mathbb{P}}_1(T) := \mathbb{P}_1(T)$ for any non-interface element $T \in \mathcal{T}_h$. Let

$$\widehat{\mathbb{P}}_1(\Omega) := \left\{ q \in L^2(\Omega) : q|_T \in \widehat{\mathbb{P}}_1(T)\ \forall T \in \mathcal{T}_h \right\}.$$

3.2. Weak Galerkin finite element space

We define the weak Galerkin finite element space V_h associated to $\mathcal{T}_h$ and its subspace $V_{h,0}$ as follows:

$$V_h := \left\{ v = \{v_0, v_\partial\} : v_0|_T \in \widehat{\mathbb{P}}_1(T)\ \forall T \in \mathcal{T}_h,\ v_\partial|_e \in \mathbb{P}_0(e)\ \forall e \in \mathcal{E}_h \right\},$$
$$V_{h,0} := \left\{ v \in V_h : v_\partial = 0 \text{ on } \partial\Omega \right\}.$$

Here we note that, for any $v = \{v_0, v_\partial\} \in V_h$, its second component v_∂ is a single-valued function on each edge $e \in \mathcal{E}_h$. Thus, the space V_h has 3 degrees of freedom on the interior of each element $T \in \mathcal{T}_h$ and 1 degree of freedom on each edge $e \in \mathcal{E}_h$.

For each element $T \in \mathcal{T}_h$, let Q_0 be the L^2-projection from $L^2(T)$ onto $\widehat{\mathbb{P}}_1(T)$. Similarly, for each edge $e \in \mathcal{E}_h$, let Q_∂ the L^2-projection from $L^2(e)$ onto $\mathbb{P}_0(e)$. We then define a projection operator $Q_h : H^1(\Omega) \to V_h$ by

$$Q_h v = \{Q_0 v, Q_\partial v\}, \quad v \in H^1(\Omega). \tag{3.1}$$

3.3. Discrete problem and well-posedness

For each $v_h = \{v_0, v_\partial\} \in V_h$, we define a discrete weak gradient $\nabla_w v_h$ of v_h as a vector-valued function satisfying $\nabla_w v_h|_T \in \nabla\widehat{\mathbb{P}}_1(T)$ and

$$\int_T \overline{\beta}_T \nabla_w v_h \cdot \nabla q \, d\boldsymbol{x} = \int_T \overline{\beta}_T \nabla v_0 \cdot \nabla q \, d\boldsymbol{x} - \int_{\partial T} (Q_\partial v_0 - v_\partial) \left(\overline{\beta}_T \nabla q \cdot \boldsymbol{n}_T \right) ds \quad \forall q \in \widehat{\mathbb{P}}_1(T), \tag{3.2}$$

for each element $T \in \mathcal{T}_h$.

We next introduce two bilinear forms on $V_h \times V_h$ as follows:

$$a(u_h, v_h) := \sum_{T \in \mathcal{T}_h} \int_T \overline{\beta}_T \nabla_w u_h \cdot \nabla_w v_h \, d\boldsymbol{x},$$

$$s(u_h, v_h) := \sum_{T \in \mathcal{T}_h} \lambda_T h_T^{-1} \int_{\partial T} (Q_\partial u_0 - u_\partial)(Q_\partial v_0 - v_\partial) \, ds,$$

for any $u_h = \{u_0, u_\partial\} \in V_h$ and $v_h = \{v_0, v_\partial\} \in V_h$, where λ_T is some positive constant independent of h. In the analysis, it suffices to choose $\lambda_T = 1$ for all $T \in \mathcal{T}_h$. In practice, there are some cases that the choice $\lambda_T = \max_{\boldsymbol{x} \in T} \overline{\beta}_T(\boldsymbol{x})$ exhibits more accurate results (see Section 6). The stabilization $a_s(\cdot,\cdot)$ of $a(\cdot,\cdot)$ is defined by

$$a_s(u_h, v_h) = a(u_h, v_h) + s(u_h, v_h) \quad \forall u_h, v_h \in V_h.$$

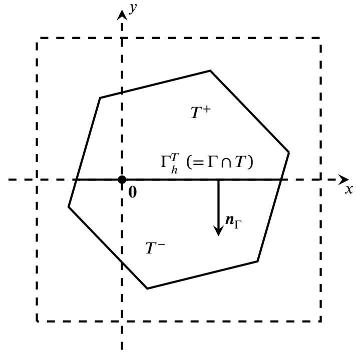

Fig. 3. Geometric assumptions on an interface element T.

We are now ready to formulate the immersed WG method for solving (2.3) as follows: Find $u_h \in V_{h,0}$ such that

$$a_s(u_h, v_h) = (f, v_0)_{0,\Omega}, \quad \forall v_h = \{v_0, v_\partial\} \in V_{h,0}. \tag{3.3}$$

We next analyze the well-posedness of the discrete problem (3.3). Define the energy-norm $|||\cdot|||$ by

$$|||v_h||| := \sqrt{a_s(v_h, v_h)} \quad \forall v_h \in V_h.$$

Clearly $|||\cdot|||$ is a seminorm on V_h. Moreover, $|||\cdot|||$ is a norm on $V_{h,0}$, as shown in the following lemma.

Lemma 3.1. *$|||\cdot|||$ is a norm on $V_{h,0}$.*

Proof. It suffices to show that $|||v_h||| = 0 \Rightarrow v_h \equiv 0$ for any $v_h \in V_{h,0}$. Suppose that $v_h = \{v_0, v_\partial\} \in V_{h,0}$ satisfies $|||v_h||| = 0$. Since

$$0 = |||v_h|||^2 = \sum_{T\in\mathcal{T}_h} \int_T \overline{\beta}_T |\nabla_w v_h|^2 \, d\boldsymbol{x} + \sum_{T\in\mathcal{T}_h} \lambda_T \sum_{e\subset\partial T} h_T^{-1} \int_e |Q_\partial v_0 - v_\partial|^2 \, ds$$

and since $0 < \beta_* < \overline{\beta}_T$ for any $T \in \mathcal{T}_h$, we obtain $\nabla_w v_h \equiv 0$ and $Q_\partial v_0 = v_\partial$ on each edge $e \in \mathcal{E}_h$. Then

$$\begin{aligned} 0 &= \int_T \overline{\beta}_T \nabla_w v_h \cdot \nabla v_0 \, d\boldsymbol{x} = \int_T \overline{\beta}_T \nabla v_0 \cdot \nabla v_0 \, d\boldsymbol{x} + \sum_{e\subset\partial T} \int_e (v_\partial - Q_\partial v_0) \left(\overline{\beta}_T \frac{\partial v_0}{\partial \boldsymbol{n}} \right) ds \\ &= \int_T \overline{\beta}_T |\nabla v_0|^2 \, d\boldsymbol{x} \geq \int_T \beta_* |\nabla v_0|^2 \, d\boldsymbol{x} \end{aligned}$$

for any $T \in \mathcal{T}_h$. This shows that $\nabla v_0 = 0$ on each $T \in \mathcal{T}_h$. Note that, for each $T \in \mathcal{T}_h$, $\nabla q = 0$ implies $q =$ constant for any $q \in \widehat{\mathbb{P}}_1(T)$. Since $v_0 \in \widehat{\mathbb{P}}_1(T)$ on each $T \in \mathcal{T}_h$, we obtain that v_0 is constant on each $T \in \mathcal{T}_h$. Since $Q_\partial v_0 = v_\partial$ on each $e \in \mathcal{E}_h$ and $v_\partial = 0$ on $\partial\Omega$, we conclude that $v_0 = v_\partial = 0$. □

The well-posedness of the discrete problem (3.3) directly follows from the lemma.

Corollary 3.2. *The discrete problem* (3.3) *is well-posed.*

Proof. From Lemma 3.1, the bilinear form $a_s(\cdot,\cdot)$ on $V_{h,0}$ is coercive and continuous with respect to the norm $|||\cdot|||$ on $V_{h,0}$. The conclusion follows from the Lax-Milgram Lemma. □

4. Some estimates on interface elements

In this section, we present some inequalities for the function spaces on the interface elements, which are needed for the error analysis of the immersed WG method.

4.1. Geometric assumptions on interface elements

Let $T \in \mathcal{T}_h$ be an interface element. Recall that Γ_h^T denotes the line segment connecting two intersection points of Γ and the edges of T. Although the analysis works for C^2-interface, we assume for the simplicity of presentation, that on each mesh element T, the portion $\Gamma \cap T$ is a line segment so that $\Gamma \cap T = \Gamma_h^T$ and $T^s = T \cap \Omega^s$ for $s = +, -$. In addition, we assume that $\Gamma \cap T$ aligns with the x-axis and the origin of the xy-plane is contained in T, so that

$$T^+ = T \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0\}, \quad T^- = T \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \leq 0\} \tag{4.1}$$

(see Fig. 3). Since $h_T = \mathrm{diam}(T)$, we have $T \subset [-h_T, h_T]^2$. Since $\beta^s \in W^{1,\infty}(\overline{\Omega^s})$ and $\overline{\beta}_T = \overline{\beta}^s$ on T^s for $s = +, -$, we have

$$\max_{\boldsymbol{x}\in T^s} |\beta(\boldsymbol{x}) - \overline{\beta}_T(\boldsymbol{x})| \leq C h_T, \quad \max_{\boldsymbol{x}\in e\cap\Omega^s} |\beta(\boldsymbol{x}) - \overline{\beta}_T(\boldsymbol{x})| \leq C h_T, \quad s = +, -, \tag{4.2}$$

where $e \subset \partial T$. Let $\boldsymbol{n}_\Gamma = (n_{1,h}, n_{2,h})$ be the unit vector normal to Γ pointing from T^+ to T^-, and let $\boldsymbol{t}_\Gamma = (-n_{2,h}, n_{1,h})$.

Remark 4.1. We briefly discuss the case when the interface is not piecewise linear, that is, $\Gamma \cap T \neq \Gamma_h^T$. Without loss of generality we assume that Γ_h^T aligns with the x-axis and T is contained in the box $I_x \times I_y$, where I_x and I_y are intervals with length not greater than $2h_T$. Since Γ is a regular C^2-curve, there exists a parametrization $t \mapsto (t, \gamma(t))$ of the curve $\Gamma \cap T$ for some $\gamma \in C^2(I_x)$, when h is sufficiently small. Then the unit normal vector $\boldsymbol{n}_\Gamma$ along $\Gamma \cap T$ pointing from Ω^+ to Ω^- is

$$\boldsymbol{n}_\Gamma(t, \gamma(t)) = \left(\frac{\gamma'(t)}{(1 + |\gamma'(t)|^2)^{1/2}}, \frac{-1}{(1 + |\gamma'(t)|^2)^{1/2}} \right), \quad t \in I_x.$$

Let us extend the vector-valued function $\boldsymbol{n}_\Gamma$ to the box $I_x \times I_y$ by setting $(t, y) \mapsto \boldsymbol{n}_\Gamma(t, \gamma(t))$. Then, since γ is C^2, we have

$$\sup_{\boldsymbol{x} \in T} \left| \boldsymbol{n}_\Gamma(\boldsymbol{x}) - \boldsymbol{n}_\Gamma^h \right| \leq C h_T, \tag{4.3}$$

where $\boldsymbol{n}_\Gamma^h$ is the unit normal vector along Γ_h^T pointing from T^+ to T^-. In addition, one can obtain a similar result for the tangential vector of $\Gamma \cap T$. Next, according to Lemma 2 in [8],

$$\|\nabla u\|_{0,T_r}^2 \leq C h_T^2 \sum_{s=+,-} \left(\|(\nabla u)|_{\Omega_s}\|_{0,\Gamma\cap T}^2 + h_T^2 |\nabla u|_{1,T\cap\Omega^s}^2 \right), \quad \forall u \in \widetilde{H}^2(T), \tag{4.4}$$

where T_r is a subset of T given by

$$T_r = T - (\Omega^+ \cap T^+) - (\Omega^- \cap T^-);$$

see Fig. 2. Note also that the first estimate in (4.2) is modified as follows:

$$\sup_{\boldsymbol{x} \in T^s \cap (T \cap \Omega^s)} |\beta(\boldsymbol{x}) - \overline{\beta}_T(\boldsymbol{x})| \leq C h_T, \quad s = +, -. \tag{4.5}$$

Using the estimates (4.3)-(4.5) and the standard trace inequality, all the results below can be derived with only minor modification. We leave the detailed analysis for a future investigation.

Lemma 4.2. *If h is sufficiently small, then either T^+ or T^- contains a ball with radius $\rho h_T/8$.*

Proof. Recall that T is star-shaped with respect to a ball B centered at $\boldsymbol{x}_T = (x_T, y_T)$ with radius ρh_T. First, assume that $|y_T| \leq \rho h_T/8$. Consider the ball B^+ centered at $(x_T, y_T + \rho h_T/2)$ with radius $\rho h_T/8$. Then $B^+ \subset B \cap T^+$.

One can show that, by the same argument, for the case $y_T \geq \rho h_T/8$ the set T^+ contains the ball centered at $(x_T, y_T + \rho h_T/2)$ with radius $\rho h_T/8$, and for the case $y_T \leq -\rho h_T/8$ the set T^- contains the ball centered at $(x_T, y_T - \rho h_T/2)$ with radius $\rho h_T/8$. □

4.2. Some inequalities for the broken polynomial space $\widehat{\mathbb{P}}_1$

Recall that, on each element $T \in \mathcal{T}_h$, the standard trace inequality holds:

$$h_T^{1/2} \|v\|_{0,\partial T} \leq C \left(\|v\|_{0,T} + h_T \|\nabla v\|_{0,T} \right) \quad \forall v \in H^1(T). \tag{4.6}$$

The following lemma provides a trace inequality for the space $\nabla\widehat{\mathbb{P}}_1$.

Lemma 4.3. *Let $T \in \mathcal{T}_h$ be an interface element. Then there exists a positive constant C depending only on ρ and β such that for any $q \in \widehat{\mathbb{P}}_1(T)$ and any edge e of T,*

$$\left\| \overline{\beta}_T \nabla q \right\|_{0,e} \leq C h_T^{-1/2} \left\| \overline{\beta}_T^{1/2} \nabla q \right\|_{0,T}, \tag{4.7}$$

Proof. Recall that the following piecewise polynomials form a basis of the space $\widehat{\mathbb{P}}_1(T)$:

$$\varphi_1(\boldsymbol{x}) = 1, \quad \varphi_2(\boldsymbol{x}) = \boldsymbol{t}_\Gamma \cdot (\boldsymbol{x} - \boldsymbol{x}_0), \quad \varphi_3(\boldsymbol{x}) = \overline{\beta}_T^{-1} \boldsymbol{n}_\Gamma \cdot (\boldsymbol{x} - \boldsymbol{x}_0), \quad \forall \boldsymbol{x} \in T,$$

where $\boldsymbol{x}_0$ is the midpoint of Γ_h^T. Let $q = a\varphi_1 + b\varphi_2 + c\varphi_3$ for $a, b, c \in \mathbb{R}$. Then

$$\nabla q = b\boldsymbol{t}_\Gamma + c\overline{\beta}_T^{-1} \boldsymbol{n}_\Gamma, \quad \nabla q \cdot \nabla q = b^2 + c^2 \overline{\beta}_T^{-2}.$$

By Assumption 2.3 (iii), we have

$$\left\| \overline{\beta}_T \nabla q \right\|_{0,e}^2 = \int_e \left| \overline{\beta}_T \nabla q \right|^2 \mathrm{d}s \leq ((\beta^*)^2 b^2 + c^2)|e| \leq C(\beta_*, \beta^*, \rho)(b^2 + c^2) h_T,$$

$$\left\| \overline{\beta}_T^{1/2} \nabla q \right\|_{0,T}^2 = \int_T \overline{\beta}_T |\nabla q|^2 \mathrm{d}\boldsymbol{x} \geq \beta_* (b^2 + c^2 (\beta^*)^{-2}) |T| \geq C(\beta_*, \beta^*, \rho)(b^2 + c^2) h_T^2.$$

Thus there exists a positive constant C depending only on ρ and β such that the inequality (4.7) holds. □

Note that we have the following inverse inequality holds (see, for example, (2.6) of [11]):

$$|q|_{1,T} \leq C h_T^{-1} \|q\|_{0,T} \quad \forall q \in \mathbb{P}_1(T), \qquad |q|_{1,B} \leq C h_T^{-1} \|q\|_{0,B} \quad \forall q \in \mathbb{P}_1(B), \tag{4.8}$$

where B is a ball in $\mathbb{R}^2$ with radius ρh_T and C is a positive constant depending only on ρ. The following lemma shows that the inverse inequality also holds for the space $\widehat{\mathbb{P}}_1$.

Lemma 4.4. *Let $T \in \mathcal{T}_h$ be an interface element. There exists a positive constant C depending only on ρ and β such that*

$$|q|_{1,T} \le C h_T^{-1} \|q\|_{0,T} \quad \forall q \in \widehat{\mathbb{P}}_1(T).$$

Proof. By Lemma 4.2, we may assume that T^+ contains a ball B^+ with radius $\rho h_T/8$. As in the proof of the previous lemma, consider the basis $\{\varphi_1, \varphi_2, \varphi_3\}$ of $\widehat{\mathbb{P}}_1(T)$ and let $q = a\varphi_1 + b\varphi_2 + c\varphi_3$ for $a, b, c \in \mathbb{R}$, and define

$$q_+ := a + b\boldsymbol{t}_\Gamma \cdot (\boldsymbol{x} - \boldsymbol{x}_0) + c\left(\overline{\beta}^+\right)^{-1} \boldsymbol{n}_\Gamma \cdot (\boldsymbol{x} - \boldsymbol{x}_0).$$

Then $q = q_+$ on T^+. By (4.8),

$$|q_+|_{1,B^+} \le C h_T^{-1} \|q_+\|_{0,B^+} = C h_T^{-1} \|q\|_{0,B^+} \le C h_T^{-1} \|q\|_{0,T}. \tag{4.9}$$

Since $\boldsymbol{t}_\Gamma \cdot \boldsymbol{n}_\Gamma = 0$,

$$\begin{aligned} |q_+|^2_{1,B^+} &= \int_{B^+} \left| b\boldsymbol{t}_\Gamma + c\left(\overline{\beta}^+\right)^{-1} \boldsymbol{n}_\Gamma \right|^2 \mathrm{d}\boldsymbol{x} = \int_{B^+} \left(b^2 + \left(\overline{\beta}^+\right)^{-2} c^2\right) \mathrm{d}\boldsymbol{x} \\ &\ge \frac{\pi \rho^2 h_T^2}{64} C(\beta_*, \beta^*)(b^2 + c^2), \end{aligned} \tag{4.10}$$

$$\begin{aligned} |q|^2_{1,T} &= \int_{T^+} \left| b\boldsymbol{t}_\Gamma + c\left(\overline{\beta}^+\right)^{-1} \boldsymbol{n}_\Gamma \right|^2 \mathrm{d}\boldsymbol{x} + \int_{T^-} \left| b\boldsymbol{t}_\Gamma + c\left(\overline{\beta}^-\right)^{-1} \boldsymbol{n}_\Gamma \right|^2 \mathrm{d}\boldsymbol{x} \\ &= \int_{T^+} \left(b^2 + \left(\overline{\beta}^+\right)^{-2} c^2\right) \mathrm{d}\boldsymbol{x} + \int_{T^-} \left(b^2 + \left(\overline{\beta}^-\right)^{-2} c^2\right) \mathrm{d}\boldsymbol{x} \\ &\le C(\beta_*, \beta^*) h_T^2 (b^2 + c^2). \end{aligned} \tag{4.11}$$

Combining the inequalities (4.9)-(4.11), we obtain

$$|q|_{1,T} \le \frac{8}{\sqrt{\pi}\rho} C(\beta_*, \beta^*) |q_+|_{1,B^+} \le C(\beta_*, \beta^*, \rho) h_T^{-1} \|q\|_{0,T}.$$

This completes the proof of the lemma. □

4.3. Approximation properties of the broken polynomial space $\widehat{\mathbb{P}}_1$

In this subsection, we derive some approximation properties of the broken linear polynomial space $\widehat{\mathbb{P}}_1(T)$.

It is well-known that, on each non-interface element $T \in \mathcal{T}_h$, for any $u \in H^2(T)$ there exists $q \in \mathbb{P}_1$ such that

$$\|u - q\|_{0,T} + h_T |u - q|_{1,T} \le C_\rho h_T^2 \|u\|_{2,T}, \tag{4.12}$$

where C_ρ is a positive constant depending only on ρ [12, Lemma 4.3.8].

Theorem 4.5. *Let $u \in \widetilde{H}^2_\Gamma(\Omega)$. Then there exists $q \in \widehat{\mathbb{P}}_1(\Omega)$ such that*

$$\|u - q\|_{0,\Omega} + h|u - q|_{1,\Omega} \le C h^2 \|u\|_{\widetilde{H}^2(\Omega)},$$

where C is a positive constant depending only on ρ and β.

Proof. Let $T \in \mathcal{T}_h$ be an interface element. Then we have

$$\nabla u = (\nabla u \cdot \boldsymbol{t}_\Gamma)\boldsymbol{t}_\Gamma + (\nabla u \cdot \boldsymbol{n}_\Gamma)\boldsymbol{n}_\Gamma \tag{4.13}$$

on T. We note that $\nabla u \cdot \boldsymbol{t}_\Gamma \in H^1(T)$ and $\beta \nabla u \cdot \boldsymbol{n}_\Gamma \in H^1(T)$. Thus, from (4.12), there exist $c_t, c_n \in \mathbb{R}$ such that

$$\|\nabla u \cdot \boldsymbol{t}_\Gamma - c_t\|_{0,T} \le C_\rho h_T |\nabla u \cdot \boldsymbol{t}_\Gamma|_{1,T}, \quad \|\beta \nabla u \cdot \boldsymbol{n}_\Gamma - c_n\|_{0,T} \le C_\rho h_T |\beta \nabla u \cdot \boldsymbol{n}_\Gamma|_{1,T}.$$

Note that

$$|\nabla u \cdot \boldsymbol{t}_\Gamma|_{1,T} \le C \|u\|_{\widetilde{H}^2(T)}, \quad |\nabla u \cdot \boldsymbol{n}_\Gamma|_{1,T} \le C \|u\|_{\widetilde{H}^2(T)}. \tag{4.14}$$

Thus

$$\|\nabla u \cdot \boldsymbol{t}_\Gamma - c_t\|_{0,T} \le C h_T \|u\|_{\widetilde{H}^2(T)}, \quad \|\beta \nabla u \cdot \boldsymbol{n}_\Gamma - c_n\|_{0,T} \le C h_T \|u\|_{\widetilde{H}^2(T)}. \tag{4.15}$$

Let

$$\boldsymbol{r} := c_t \boldsymbol{t}_\Gamma + \overline{\beta}_T^{-1} c_n \boldsymbol{n}_\Gamma.$$

Then $\boldsymbol{r} \in \nabla \widehat{\mathbb{P}}_1(T)$. By (4.13), (4.15), and (4.2),

$$\|\nabla u-\boldsymbol{r}\|_{0,T}\le\|\nabla u\cdot\boldsymbol{t}_\Gamma-c_t\|_{0,T}+\beta_*^{-1}\|c_n-\overline{\beta}_T\nabla u\cdot\boldsymbol{n}_\Gamma\|_{0,T}$$
$$\le Ch_T\|u\|_{\widetilde{H}^2(T)}+\beta_*^{-1}\|c_n-\beta\nabla u\cdot\boldsymbol{n}_\Gamma\|_{0,T}+\beta_*^{-1}\|(\beta-\overline{\beta}_T)\nabla u\|_{0,T}$$
$$\le Ch_T\|u\|_{\widetilde{H}^2(T)}. \tag{4.16}$$

Since $\boldsymbol{r}\in\nabla\widehat{\mathbb{P}}_1(T)$, there exists $q\in\widehat{\mathbb{P}}_1(T)$ such that $\nabla q=\boldsymbol{r}$ and $\int_T q\,\mathrm{d}\boldsymbol{x}=\int_T u\,\mathrm{d}\boldsymbol{x}$. Then (4.16) and Poincaré-Friedrichs inequality (cf. [10]) imply that

$$\|u-q\|_{0,T}\le Ch_T|u-q|_{1,T}\le Ch_T^2\|u\|_{\widetilde{H}^2(T)}.$$

This completes the proof of the theorem. □

As a corollary, we obtain the estimate for the L^2-projection Q_0 onto the space $\widehat{\mathbb{P}}_1$ as follows.

Corollary 4.6. *There exists a positive constant C, depending only on ρ and β, such that*

$$\|u-Q_0u\|_{0,\Omega}+h|u-Q_0u|_{1,\Omega}\le Ch^2\|u\|_{\widetilde{H}^2(\Omega)}\quad\forall u\in\widetilde{H}^2_\Gamma(\Omega).$$

Proof. Let $T\in\mathcal{T}_h$ be an interface element. By Theorem 4.5, there exists $q'\in\widehat{\mathbb{P}}_1(T)$ such that

$$\|u-q'\|_{0,T}+h_T|u-q'|_{1,T}\le Ch_T^2\|u\|_{\widetilde{H}^2(T)}, \tag{4.17}$$

where C is a positive constant depending only on ρ and β. Since $\|Q_0v\|_{0,T}\le\|v\|_{0,T}$ for any $v\in H^1(T)$ and $Q_0q=q$ for any $q\in\widehat{\mathbb{P}}_1(T)$, we obtain

$$\|u-Q_0u\|_{0,T}\le\|u-q'\|_{0,T}+\|Q_0q'-Q_0u\|_{0,T}\le Ch_T^2\|u\|_{\widetilde{H}^2(T)}.$$

By Lemma 4.4,

$$|u-Q_0u|_{1,T}\le|u-q'|_{1,T}+|Q_0q'-Q_0u|_{1,T}\le|u-q'|_{1,T}+h_T^{-1}\|Q_0q'-Q_0u\|_{0,T}$$
$$\le|u-q'|_{1,T}+h_T^{-1}\|q'-u\|_{0,T}\le Ch_T\|u\|_{\widetilde{H}^2(T)}.$$

This completes the proof. □

The following lemma gives the L^2-norm estimate of $\beta\nabla u-\overline{\beta}_T\nabla(Q_0u)$ on each mesh edge (see Proposition 5.2 in [31]).

Lemma 4.7. *There exists a positive constant C independent of h such that*

$$\sum_{T\in\mathcal{T}_h}\left\|\beta\nabla u-\overline{\beta}_T\nabla(Q_0u)\right\|^2_{0,\partial T}\le Ch\|u\|^2_{\widetilde{H}^2(\Omega)}\quad\forall u\in\widetilde{H}^2_\Gamma(\Omega).$$

Proof. Let $T\in\mathcal{T}_h$ be an interface element. Let $q=Q_0u$, and let $e\subset\partial T$. As in (4.13), we have

$$\nabla u=(\nabla u\cdot\boldsymbol{t}_\Gamma)\boldsymbol{t}_\Gamma+(\nabla u\cdot\boldsymbol{n}_\Gamma)\boldsymbol{n}_\Gamma,\quad\nabla q=(\nabla q\cdot\boldsymbol{t}_\Gamma)\boldsymbol{t}_\Gamma+(\nabla q\cdot\boldsymbol{n}_\Gamma)\boldsymbol{n}_\Gamma \tag{4.18}$$

on T. Since $u\in\widetilde{H}^2(T)$, we have $\nabla u\cdot\boldsymbol{t}_\Gamma\in H^1(T)$ and $\beta\nabla u\cdot\boldsymbol{n}_\Gamma\in H^1(T)$. Note also that $\overline{\beta}_T\nabla q\cdot\boldsymbol{n}_\Gamma$ and $\nabla q\cdot\boldsymbol{t}_\Gamma$ are constants on T. Then, by (4.2),

$$\left\|\beta\nabla u-\overline{\beta}_T\nabla q\right\|_{0,e}\le\left\|\beta\nabla u-\overline{\beta}_T\nabla u\right\|_{0,e}+\left\|\overline{\beta}_T\nabla u-\overline{\beta}_T\nabla q\right\|_{0,e}$$
$$\le Ch_T\|\nabla u\|_{0,e}+C\|\nabla u-\nabla q\|_{0,e}. \tag{4.19}$$

By the trace inequality (4.6) and (4.14),

$$\|\nabla u\|_{0,e}\le\|\nabla u\cdot\boldsymbol{t}_\Gamma\|_{0,e}+\beta_*^{-1}\|\beta\nabla u\cdot\boldsymbol{n}_\Gamma\|_{0,e}$$
$$\le Ch_T^{-1/2}\left(\|\nabla u\cdot\boldsymbol{t}_\Gamma\|_{0,T}+h_T|\nabla u\cdot\boldsymbol{t}_\Gamma|_{1,T}\right)+Ch_T^{-1/2}\left(\|\beta\nabla u\cdot\boldsymbol{n}_\Gamma\|_{0,T}+h_T|\beta\nabla u\cdot\boldsymbol{n}_\Gamma|_{1,T}\right)$$
$$\le Ch_T^{-1/2}|u|_{1,T}+Ch_T^{1/2}\|u\|_{\widetilde{H}^2(T)}\le Ch_T^{-1/2}\|u\|_{\widetilde{H}^2(T)}. \tag{4.20}$$

By (4.18) and (4.2),

$$\|\nabla u-\nabla q\|_{0,e}\le\|\nabla u\cdot\boldsymbol{t}_\Gamma-\nabla q\cdot\boldsymbol{t}_\Gamma\|_{0,e}+\beta_*^{-1}\|\overline{\beta}_T\nabla u\cdot\boldsymbol{n}_\Gamma-\overline{\beta}_T\nabla q\cdot\boldsymbol{n}_\Gamma\|_{0,e}$$
$$\le\|\nabla u\cdot\boldsymbol{t}_\Gamma-\nabla q\cdot\boldsymbol{t}_\Gamma\|_{0,e}+\beta_*^{-1}\|(\overline{\beta}_T-\beta)\nabla u\cdot\boldsymbol{n}_\Gamma\|_{0,e}$$
$$+\beta_*^{-1}\|\beta\nabla u\cdot\boldsymbol{n}_\Gamma-\overline{\beta}_T\nabla q\cdot\boldsymbol{n}_\Gamma\|_{0,e}$$
$$\le\|\nabla u\cdot\boldsymbol{t}_\Gamma-\nabla q\cdot\boldsymbol{t}_\Gamma\|_{0,e}+C\beta_*^{-1}h_T\|\nabla u\|_{0,e}$$
$$+\beta_*^{-1}\|\beta\nabla u\cdot\boldsymbol{n}_\Gamma-\overline{\beta}_T\nabla q\cdot\boldsymbol{n}_\Gamma\|_{0,e}. \tag{4.21}$$

By the trace inequality (4.6), Corollary 4.6, and (4.2),

$$\begin{aligned}\|\nabla u\cdot \boldsymbol{t}_\Gamma-\nabla q\cdot \boldsymbol{t}_\Gamma\|_{0,e}&\le Ch_T^{-1/2}\|\nabla u\cdot \boldsymbol{t}_\Gamma-\nabla q\cdot \boldsymbol{t}_\Gamma\|_{0,T}+Ch_T^{1/2}|\nabla u\cdot \boldsymbol{t}_\Gamma-\nabla q\cdot \boldsymbol{t}_\Gamma|_{1,T}\\&\le Ch_T^{-1/2}\|\nabla u\cdot \boldsymbol{t}_\Gamma-\nabla q\cdot \boldsymbol{t}_\Gamma\|_{0,T}+Ch_T^{1/2}|\nabla u\cdot \boldsymbol{t}_\Gamma|_{1,T}\\&\le Ch_T^{1/2}\|u\|_{\widetilde{H}^2(T)},\end{aligned}\tag{4.22}$$

$$\begin{aligned}\|\beta\nabla u\cdot \boldsymbol{n}_\Gamma-\overline{\beta}_T\nabla q\cdot \boldsymbol{n}_\Gamma\|_{0,e}&\le Ch_T^{-1/2}\|\beta\nabla u\cdot \boldsymbol{n}_\Gamma-\overline{\beta}_T\nabla q\cdot \boldsymbol{n}_\Gamma\|_{0,T}+Ch_T^{1/2}|\beta\nabla u\cdot \boldsymbol{n}_\Gamma-\overline{\beta}_T\nabla q\cdot \boldsymbol{n}_\Gamma|_{1,T}\\&\le Ch_T^{-1/2}\|\beta\nabla u\cdot \boldsymbol{n}_\Gamma-\overline{\beta}_T\nabla q\cdot \boldsymbol{n}_\Gamma\|_{0,T}+Ch_T^{1/2}|\beta\nabla u\cdot \boldsymbol{n}_\Gamma|_{1,T}\\&\le Ch_T^{-1/2}\|(\beta-\overline{\beta}_T)\nabla u\|_{0,T}+Ch_T^{-1/2}\|\nabla u-\nabla q\|_{0,T}+Ch_T^{1/2}\|u\|_{\widetilde{H}^2(T)}\\&\le Ch_T^{1/2}\|u\|_{\widetilde{H}^2(T)}.\end{aligned}\tag{4.23}$$

Now the conclusion follows from the inequalities (4.19)-(4.23). □

The following lemma gives the L^2-norm estimate of $\nabla_w(Q_hu)-\nabla(Q_0u)$ on each element in $\mathcal{T}_h$.

Lemma 4.8. *There exists a positive constant C independent of h such that*

$$\|\nabla_w(Q_hu)-\nabla(Q_0u)\|_{0,\Omega}\le Ch\|u\|_{\widetilde{H}^2(\Omega)}\quad u\in \widetilde{H}^2_\Gamma(\Omega).$$

Proof. Let T be an interface element. By the definition of the discrete weak gradient (3.2), we have

$$\int_T \overline{\beta}_T\left(\nabla_w(Q_hu)-\nabla(Q_0u)\right)\cdot\nabla q\,\mathrm{d}\boldsymbol{x}=-\int_{\partial T}(Q_\partial(Q_0u)-Q_\partial u)\left(\overline{\beta}_T\frac{\partial q}{\partial \boldsymbol{n}}\right)\mathrm{d}s\quad \forall q\in\widehat{\mathbb{P}}_1(T).$$

Let $q\in\widehat{\mathbb{P}}_1(T)$ satisfy $\nabla q=\nabla_w(Q_hu)-\nabla(Q_0u)$. By the trace inequality (4.6), Lemma 4.3, Poincaré-Friedrichs inequality, and Corollary 4.6, we obtain

$$\begin{aligned}\|\nabla_w(Q_hu)-\nabla(Q_0u)\|^2_{0,\Omega}&\le C\sum_{T\in\mathcal{T}_h}\|u-Q_0u\|_{0,\partial T}\left\|\overline{\beta}_T\nabla q\right\|_{0,\partial T}\\&\le C\sum_{T\in\mathcal{T}_h}\left(h_T^{-1}\|u-Q_0u\|_{0,T}+|u-Q_0u|_{1,T}\right)\left\|\overline{\beta}_T^{1/2}\nabla q\right\|_{0,T}\\&\le C\sum_{T\in\mathcal{T}_h}|u-Q_0u|_{1,T}\left\|\overline{\beta}_T^{1/2}\nabla q\right\|_{0,T}\le Ch\|u\|_{\widetilde{H}^2(\Omega)}\|\nabla_w(Q_hu)-\nabla(Q_0u)\|_{0,\Omega},\end{aligned}$$

and this completes the proof. □

5. Error analysis

In this section, we present the error estimate in the discrete H^1-seminorm for the scheme (3.3).

5.1. Discrete H^1-seminorm

We introduce a discrete H^1-seminorm as follows:

$$|v_h|^2_{1,h}:=\sum_{T\in\mathcal{T}_h}\|\nabla v_0\|^2_{0,T}+\lambda_T h_T^{-1}\|Q_\partial v_0-v_\partial\|^2_{0,\partial T},\quad v_h=\{v_0,v_\partial\}\in V_h.$$

The following lemma shows that two seminorms $|||\cdot|||$ and $|\cdot|_{1,h}$ on V_h are equivalent.

Lemma 5.1. *There exist two positive constants C_1 and C_2 independent of h such that*

$$C_1|v_h|_{1,h}\le |||v_h|||\le C_2|v_h|_{1,h}\quad \forall v_h\in V_h.$$

Proof. The proof is similar to the proof of Lemma 5.3 in [42]. Let $v_h=\{v_0,v_\partial\}\in V_h$. By the definition of the discrete weak gradient (3.2), we have

$$\int_T\overline{\beta}_T\nabla_w v_h\cdot\nabla q\,\mathrm{d}\boldsymbol{x}=\int_T\overline{\beta}_T\nabla v_0\cdot\nabla q\,\mathrm{d}\boldsymbol{x}+\int_{\partial T}\left(v_\partial-Q_\partial v_0\right)\left(\overline{\beta}_T\nabla q\cdot\boldsymbol{n}_T\right)\mathrm{d}s\quad\forall q\in\widehat{\mathbb{P}}_1(T).\tag{5.1}$$

Let $q\in\widehat{\mathbb{P}}_1(\Omega)$ satisfy $\nabla q=\nabla_w v_h$ on each $T\in\mathcal{T}_h$. Then, by Lemma 4.3,

$$\begin{aligned}\left\|\overline{\beta}^{1/2}\nabla_w v_h\right\|^2_{0,\Omega}&=\sum_{T\in\mathcal{T}_h}\left(\int_T\overline{\beta}_T\nabla v_0\cdot\nabla_w v_h\,\mathrm{d}\boldsymbol{x}+\int_{\partial T}\left(v_\partial-Q_\partial v_0\right)\left(\overline{\beta}_T\nabla_w v_h\cdot\boldsymbol{n}_T\right)\mathrm{d}s\right)\\&\le C\sum_{T\in\mathcal{T}_h}\left(\|\nabla v_0\|_{0,T}\left\|\overline{\beta}_T^{1/2}\nabla_w v_h\right\|_{0,T}+\|Q_\partial v_0-v_\partial\|_{0,\partial T}\|\overline{\beta}_T\nabla_w v_h\|_{0,\partial T}\right)\\&\le C\sum_{T\in\mathcal{T}_h}\left(\|\nabla v_0\|_{0,T}\left\|\overline{\beta}_T^{1/2}\nabla_w v_h\right\|_{0,T}+Ch^{-1/2}\|Q_\partial v_0-v_\partial\|_{0,\partial T}\left\|\overline{\beta}_T^{1/2}\nabla_w v_h\right\|_{0,T}\right)\\&\le C|v_h|_{1,h}\left\|\overline{\beta}^{1/2}\nabla_w v_h\right\|_{0,\Omega}.\end{aligned}$$

Thus we have $\|\overline{\beta}^{-1/2}\nabla_w v_h\|_{0,\Omega} \le C|v_h|_{1,h}$. Since $s(v_h,v_h) \le |v_h|_{1,h}^2$, we have

$$|||v_h|||^2 = \left\|\overline{\beta}^{1/2}\nabla_w v_h\right\|_{0,\Omega}^2 + s(v_h,v_h) \le C|v_h|_{1,h}^2.$$

On the other hand, let $q \in \widehat{\mathbb{P}}_1(\Omega)$ satisfy $\nabla q = \nabla v_0$ on each $T \in \mathcal{T}_h$. Then, by (5.1) and Lemma 4.3 we have

$$\begin{aligned}
\|\nabla v_0\|_{0,\Omega}^2 &\le C \sum_{T\in\mathcal{T}_h} \int_T \overline{\beta}_T \nabla v_0 \cdot \nabla v_0 \,\mathrm{d}\boldsymbol{x} \\
&= C \sum_{T\in\mathcal{T}_h} \left(\int_T \overline{\beta}_T \nabla_w v_h \cdot \nabla v_0 \,\mathrm{d}\boldsymbol{x} - \int_{\partial T} (v_\partial - Q_\partial v_0)\left(\overline{\beta}_T \nabla v_0 \cdot \boldsymbol{n}_T\right) \mathrm{d}s \right) \\
&\le C \sum_{T\in\mathcal{T}_h} \left(\left\|\overline{\beta}_T^{-1/2}\nabla_w v_h\right\|_{0,T} \|\nabla v_0\|_{0,T} + \|v_\partial - Q_\partial v_0\|_{0,\partial T} \left\|\overline{\beta}_T \nabla v_0 \cdot \boldsymbol{n}_T\right\|_{0,\partial T} \right) \\
&\le C \sum_{T\in\mathcal{T}_h} \left(\left\|\overline{\beta}_T^{-1/2}\nabla_w v_h\right\|_{0,T} \|\nabla v_0\|_{0,T} + h_T^{-1/2}\|v_\partial - Q_\partial v_0\|_{0,\partial T} \|\nabla v_0\|_{0,T} \right) \\
&\le C|||v_h||| \|\nabla v_0\|_{0,\Omega}.
\end{aligned}$$

Thus $\|\nabla v_0\|_{0,\Omega} \le C|||v_h|||$. Since $s(v_h,v_h) \le |||v_h|||^2$, we obtain

$$|v_h|_{1,h}^2 = \|\nabla v_0\|_{0,\Omega}^2 + s(v_h,v_h) \le C|||v_h|||^2.$$

Hence we have proved the lemma. □

5.2. Error equation

The error equation presented in the following lemma will be used to derive the error estimate.

Lemma 5.2. *Let $u \in H_0^1(\Omega)$ be the solution of (2.3) with $f \in L^2(\Omega)$, and let $u_h \in V_{h,0}$ be the solution of (3.3). Then we have*

$$\begin{aligned}
a_s(Q_h u - u_h, v_h) = s(Q_h u, v_h) &+ \sum_{T\in\mathcal{T}_h} \int_T (\overline{\beta}_T \nabla_w (Q_h u) - \beta\nabla u)\cdot \nabla v_0 \,\mathrm{d}\boldsymbol{x} \\
&+ \sum_{T\in\mathcal{T}_h} \int_{\partial T} \left(Q_\partial v_0 - v_\partial\right)\left(\overline{\beta}_T \nabla_w (Q_h u) - \beta\nabla u\right)\cdot \boldsymbol{n}_T \,\mathrm{d}s \\
&+ \sum_{T\in\mathcal{T}_h} \int_{\partial T} \left(v_0 - Q_\partial v_0\right) \beta \frac{\partial u}{\partial \boldsymbol{n}} \,\mathrm{d}s, \quad \forall v_h \in V_{h,0}.
\end{aligned} \tag{5.2}$$

Proof. Note that, for any $v_h \in V_{h,0}$,

$$\begin{aligned}
a_s(Q_h u - u_h, v_h) &= a_s(Q_h u, v_h) - (f, v_0)_{0,\Omega} \\
&= a_s(Q_h u, v_h) - \sum_{T\in\mathcal{T}_h} \left(\int_T \beta\nabla u\cdot\nabla v_0 \,\mathrm{d}\boldsymbol{x} - \int_{\partial T} \beta \frac{\partial u}{\partial \boldsymbol{n}} v_0 \,\mathrm{d}s \right) \\
&= \sum_{T\in\mathcal{T}_h} \int_T \overline{\beta}_T \nabla_w (Q_h u)\cdot\nabla_w v_h \,\mathrm{d}\boldsymbol{x} \\
&\quad + s(Q_h u, v_h) - \sum_{T\in\mathcal{T}_h} \left(\int_T \beta\nabla u\cdot\nabla v_0 \,\mathrm{d}\boldsymbol{x} - \int_{\partial T} \beta \frac{\partial u}{\partial \boldsymbol{n}} v_0 \,\mathrm{d}s \right) \\
&= \sum_{T\in\mathcal{T}_h} \int_T \overline{\beta}_T \nabla_w (Q_h u)\cdot\nabla v_0 \,\mathrm{d}\boldsymbol{x} - \sum_{T\in\mathcal{T}_h} \int_{\partial T} (v_0 - v_\partial)\left(Q_\partial\left(\overline{\beta}_T \nabla_w (Q_h u)\cdot \boldsymbol{n}_T\right)\right) \mathrm{d}s \\
&\quad + s(Q_h u, v_h) - \sum_{T\in\mathcal{T}_h} \left(\int_T \beta\nabla u\cdot\nabla v_0 \,\mathrm{d}\boldsymbol{x} - \int_{\partial T} \beta \frac{\partial u}{\partial \boldsymbol{n}} v_0 \,\mathrm{d}s \right) \\
&= s(Q_h u, v_h) + \sum_{T\in\mathcal{T}_h} \int_T (\overline{\beta}_T \nabla_w (Q_h u) - \beta\nabla u)\cdot\nabla v_0 \,\mathrm{d}\boldsymbol{x} \\
&\quad - \sum_{T\in\mathcal{T}_h} \int_{\partial T} (v_0 - v_\partial)\left(Q_\partial\left(\overline{\beta}_T \nabla_w (Q_h u)\cdot \boldsymbol{n}_T\right)\right) \mathrm{d}s + \sum_{T\in\mathcal{T}_h} \int_{\partial T} \beta \frac{\partial u}{\partial \boldsymbol{n}} v_0 \,\mathrm{d}s.
\end{aligned}$$

Since $\left[\beta \frac{\partial u}{\partial \boldsymbol{n}}\right]_e = 0$ for each interior edge e and $v_\partial|_e = 0$ for each boundary edge e, we obtain

$$
\begin{aligned}
\sum_{T\in\mathcal{T}_h}\int_{\partial T}\beta\frac{\partial u}{\partial \boldsymbol{n}}v_0\,\mathrm{d}s &= \sum_{T\in\mathcal{T}_h}\int_{\partial T}\beta\frac{\partial u}{\partial \boldsymbol{n}}(v_0-v_\partial)\,\mathrm{d}s\\
&=\sum_{T\in\mathcal{T}_h}\int_{\partial T}(v_0-v_\partial)\beta\frac{\partial u}{\partial \boldsymbol{n}}\,\mathrm{d}s-\sum_{T\in\mathcal{T}_h}\int_{\partial T}(v_0-v_\partial)(Q_\partial(\beta\nabla u\cdot\boldsymbol{n}_T))\,\mathrm{d}s\\
&\quad+\sum_{T\in\mathcal{T}_h}\int_{\partial T}(v_0-v_\partial)(Q_\partial(\beta\nabla u\cdot\boldsymbol{n}_T))\,\mathrm{d}s\\
&=\sum_{T\in\mathcal{T}_h}\int_{\partial T}(v_0-Q_\partial v_0)\beta\frac{\partial u}{\partial \boldsymbol{n}}\,\mathrm{d}s+\sum_{T\in\mathcal{T}_h}\int_{\partial T}(v_0-v_\partial)(Q_\partial(\beta\nabla u\cdot\boldsymbol{n}_T))\,\mathrm{d}s.
\end{aligned}
$$

Using the above equation we obtain

$$
\begin{aligned}
a_s(Q_hu-u_h,v_h)&=s(Q_hu,v_h)+\sum_{T\in\mathcal{T}_h}\int_T(\overline{\beta}_T\nabla_w(Q_hu)-\beta\nabla u)\cdot\nabla v_0\,\mathrm{d}\boldsymbol{x}\\
&\quad+\sum_{T\in\mathcal{T}_h}\int_{\partial T}\left(Q_\partial v_0-v_\partial\right)\left(\overline{\beta}_T\nabla_w(Q_hu)-\beta\nabla u\right)\cdot\boldsymbol{n}_T\,\mathrm{d}s\\
&\quad+\sum_{T\in\mathcal{T}_h}\int_{\partial T}\left(v_0-Q_\partial v_0\right)\beta\frac{\partial u}{\partial \boldsymbol{n}}\,\mathrm{d}s.
\end{aligned}
$$

This completes the proof of the lemma. □

The following lemma can be found in [19].

Lemma 5.3. *Let $T\in\mathcal{T}_h$ and $e\subset\partial T$. There exists a positive constant C independent of h such that*

$$\|u-\overline{u}_e\|_{-1/2,e}\le Ch|u|_{1,T}\quad\forall u\in H^1(T),$$

where $\overline{u}_e=\frac{1}{|e|}\int_e u\,\mathrm{d}s$.

5.3. Error estimate

Now we prove the error estimates in the energy norm and the discrete H^1-seminorm.

Theorem 5.4. *Suppose that $u\in\widetilde{H}^2(\Omega)\cap H_0^1(\Omega)$ is the solution of (2.3) with $f\in L^2(\Omega)$. Suppose further that $\beta\nabla u\in H^1(\Omega)$. Let $u_h\in V_{h,0}$ be the solution of (3.3). Then there exists a positive constant C independent of h such that*

$$|||Q_hu-u_h|||\le Ch\|u\|_{\widetilde{H}^2(\Omega)}.$$

Proof. Let $v_h=Q_hu-u_h$. From the error equation (5.2), we have

$$
\begin{aligned}
|||Q_hu-u_h|||^2 &= a_s(Q_hu-u_h,v_h)\\
&= s(Q_hu,v_h)+\sum_{T\in\mathcal{T}_h}\int_T(\overline{\beta}_T\nabla_w(Q_hu)-\beta\nabla u)\cdot\nabla v_0\,\mathrm{d}\boldsymbol{x}\\
&\quad+\sum_{T\in\mathcal{T}_h}\int_{\partial T}\left(Q_\partial v_0-v_\partial\right)\left(\overline{\beta}_T\nabla_w(Q_hu)-\beta\nabla u\right)\cdot\boldsymbol{n}_T\,\mathrm{d}s\\
&\quad+\sum_{T\in\mathcal{T}_h}\int_{\partial T}\left(v_0-Q_\partial v_0\right)\beta\frac{\partial u}{\partial \boldsymbol{n}}\,\mathrm{d}s\\
&=:\ I_1+I_2+I_3+I_4.
\end{aligned}
\tag{5.3}
$$

By the trace inequality (4.6), Poincaré-Friedrichs inequality, and Corollary 4.6,

$$
\begin{aligned}
|I_1|&\le C\sum_{T\in\mathcal{T}_h}h_T^{-1/2}\|u-Q_0u\|_{0,\partial T}h_T^{-1/2}\|v_\partial-Q_\partial v_0\|_{0,\partial T}\\
&\le C\sum_{T\in\mathcal{T}_h}\left(h_T^{-1}\|u-Q_0u\|_{0,T}+|u-Q_0u|_{1,T}\right)h_T^{-1/2}\|v_\partial-Q_\partial v_0\|_{0,\partial T}\\
&\le C\sum_{T\in\mathcal{T}_h}|u-Q_0u|_{1,T}h_T^{-1/2}\|v_\partial-Q_\partial v_0\|_{0,\partial T}\le Ch\|u\|_{\widetilde{H}^2(\Omega)}|||v_h|||.
\end{aligned}
\tag{5.4}
$$

From (4.2), Lemma 4.8, and Lemma 5.1,

$$
\begin{aligned}
|I_2|&\le\sum_{T\in\mathcal{T}_h}\left(\|\overline{\beta}_T\nabla_w(Q_hu)-\overline{\beta}_T\nabla u\|_{0,T}+\|(\overline{\beta}_T-\beta)\nabla u\|_{0,T}\right)\|\nabla v_0\|_{0,T}\\
&\le Ch\|u\|_{\widetilde{H}^2(\Omega)}|||v_h|||.
\end{aligned}
\tag{5.5}
$$

Since $\nabla_w(Q_h u), \nabla(Q_0 u) \in \widehat{\mathbb{P}}_1(T)$ on each $T \in \mathcal{T}_h$, using Lemma 4.3, Lemma 4.7, and Lemma 4.8, we have

$$\begin{aligned}
|I_3| &\le \sum_{T\in\mathcal{T}_h} h_T^{-1/2}\|v_\partial - Q_\partial v_0\|_{0,\partial T} h_T^{1/2}\left(\left\|\overline{\beta}_T\left(\nabla_w(Q_h u) - \nabla(Q_0 u)\right)\right\|_{0,\partial T} + \left\|\overline{\beta}_T\nabla(Q_0 u) - \beta\nabla u\right\|_{0,\partial T}\right)\\
&\le C|||v_h|||\left(\sum_{T\in\mathcal{T}_h} h_T\left(\left\|\overline{\beta}_T\left(\nabla_w(Q_h u) - \nabla(Q_0 u)\right)\right\|^2_{0,\partial T} + \left\|\overline{\beta}_T\nabla(Q_0 u) - \beta\nabla u\right\|^2_{0,\partial T}\right)\right)^{1/2}\\
&\le C|||v_h|||\left(\sum_{T\in\mathcal{T}_h}\left(\left\|\overline{\beta}_T^{1/2}\left(\nabla_w(Q_h u) - \nabla(Q_0 u)\right)\right\|^2_{0,T} + h_T\left\|\overline{\beta}_T\nabla(Q_0 u) - \beta\nabla u\right\|^2_{0,\partial T}\right)\right)^{1/2}\\
&\le Ch\|u\|_{\widetilde{H}^2(\Omega)}|||v_h|||.
\end{aligned} \tag{5.6}$$

Let $T \in \mathcal{T}_h$ and $e \subset \partial T$. Since $\beta\nabla u \in H^1(\Omega)$, by Lemma 5.3 and the trace theorem, we have

$$\left|\int_e (v_0 - Q_\partial v_0)\beta\frac{\partial u}{\partial \boldsymbol{n}}\,\mathrm{d}s\right| \le \|\beta\nabla u\|_{1/2,e}\|v_0 - Q_\partial v_0\|_{-1/2,e} \le Ch\|u\|_{\widetilde{H}^2(T)}|\nabla v_0|_{1,T}.$$

Thus we obtain from Remark 2.4 that

$$|I_4| \le \sum_{T\in\mathcal{T}_h}\sum_{e\subset\partial T}\left|\int_e (v_0 - Q_\partial v_0)\beta\frac{\partial u}{\partial \boldsymbol{n}}\,\mathrm{d}s\right| \le Ch\sum_{T\in\mathcal{T}_h}\|u\|_{\widetilde{H}^2(T)}|\nabla v_0|_{1,T} \le Ch|||v_h|||\|u\|_{\widetilde{H}^2(\Omega)}. \tag{5.7}$$

Now combining the inequalities (5.3)-(5.7) we have

$$|||u_h - Q_h u|||^2 \le Ch\|u\|_{\widetilde{H}^2(\Omega)}|||v_h||| = Ch\|u\|_{\widetilde{H}^2(\Omega)}|||u_h - Q_h u|||.$$

This concludes the proof of the theorem. □

Using Lemma 5.1 and Theorem 5.4, we immediately obtain the discrete H^1-seminorm error estimate.

Corollary 5.5. *Suppose that $u \in \widetilde{H}^2(\Omega)\cap H_0^1(\Omega)$ is the solution of (2.3) with $f \in L^2(\Omega)$. Suppose further that $\beta\nabla u \in H^1(\Omega)$. Let $u_h \in V_{h,0}$ be the solution of (3.3). Then there exists a positive constant C independent of h such that*

$$|u_h - Q_h u|_{1,h} \le Ch\|u\|_{\widetilde{H}^2(\Omega)}.$$

6. Numerical examples

In this section, we report several numerical results. We solve the problem (2.1)-(2.2) with $\Omega = (0,1)^2$ partitioned into two different families of meshes as follows:

(i) M1: uniform triangular meshes with $h = 1/2^3, 1/2^4, \cdots, 1/2^7$,
(ii) M2: unstructured polygonal meshes with $h = 1/2^3, 1/2^4, \cdots, 1/2^7$.

Some examples of the meshes are shown in Fig. 4. The unstructured polygonal meshes are generated from PolyMesher [45]. Let u be the exact solution and let $u_h = \{u_0, u_\partial\}$ be the solution of our immersed WG method. Here, the parameter λ_T is chosen as $\lambda_T = \lambda_T^{(1)} := \max_{\boldsymbol{x}\in T}\overline{\beta}_T(\boldsymbol{x})$ or $\lambda_T = \lambda_T^{(2)} := 1$ for each $T \in \mathcal{T}_h$. We compute errors in the discrete H^1-seminorm and L^2-norm, which are given by

$$|u_h - Q_h u|_{1,h}, \quad \|u_0 - Q_0 u\|_{0,\Omega},$$

respectively. For the examples below, we plot the error curves versus h in Figs. 6 to 10. We observe that the discrete H^1-seminorm error converges with order $O(h)$, which agrees with our theoretical result. We also observe that error in the L^2-norm converges with order $O(h^2)$. Moreover, in Fig. 6 and 7, the choice $\lambda_T = \max_T \overline{\beta}_T$ gives better performance than $\lambda_T \equiv 1$.

We also plot the error curves versus the number of degrees of freedom, say N, in Figs. 11 to 15, where λ_T is chosen as $\lambda_T = \lambda_T^{(1)}$. Since N is roughly proportional to $h^{-1/2}$, the errors in the H^1-seminorm and the L^2-norm are expected to converge with order $O(N^{-1/2})$ and $O(N^{-1})$, respectively. As shown in Figs. 11 to 15, the errors converge as expected. We also observe that the choice of the mesh does not significantly affect the performance of our scheme.

Example 6.1 *(Circular interface).* Take a circle centered at $(0.5, 0.5)$ with radius $r_0 = 1/6$ as an interface, and choose the following exact solution

$$u(x,y) = \begin{cases} \frac{1}{\beta^+}(r^2 - r_0^2)^3 & \text{if } (x,y) \in \Omega^+, \\ \frac{1}{\beta^-}(r^2 - r_0^2)^3 & \text{if } (x,y) \in \Omega^-, \end{cases}$$

where β^+ and β^- are constants and $r = \sqrt{(x-0.5)^2 + (y-0.5)^2}$. Here we consider two cases when $(\beta^+, \beta^-) = (100, 1)$ and $(10000, 1)$.

Example 6.2 *(Sharp edge).* In this example, we consider an interface with sharp edge. Let $L(x,y) = -(2y-1)^2 + ((2x-2)\tan\theta)^2(2x-1)$ be the level-set function, with $\theta = 10°$, and

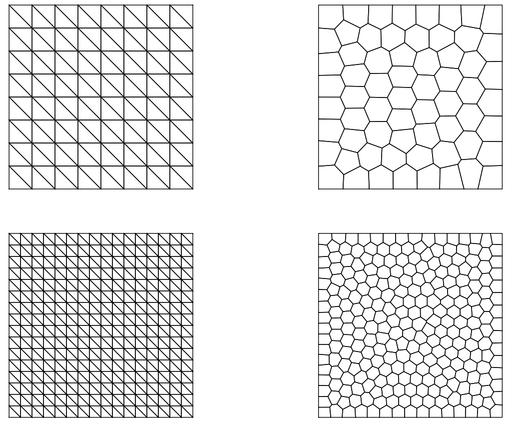

Fig. 4. The meshes M1 (left) and M2 (right).

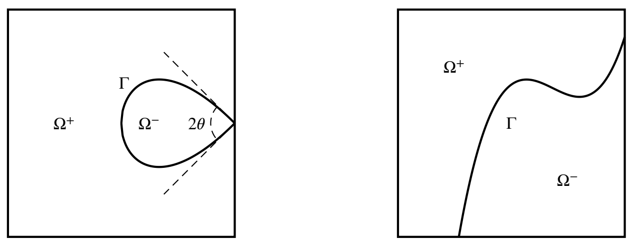

Fig. 5. The interfaces in Example 6.2 (left) and Example 6.4 (right).

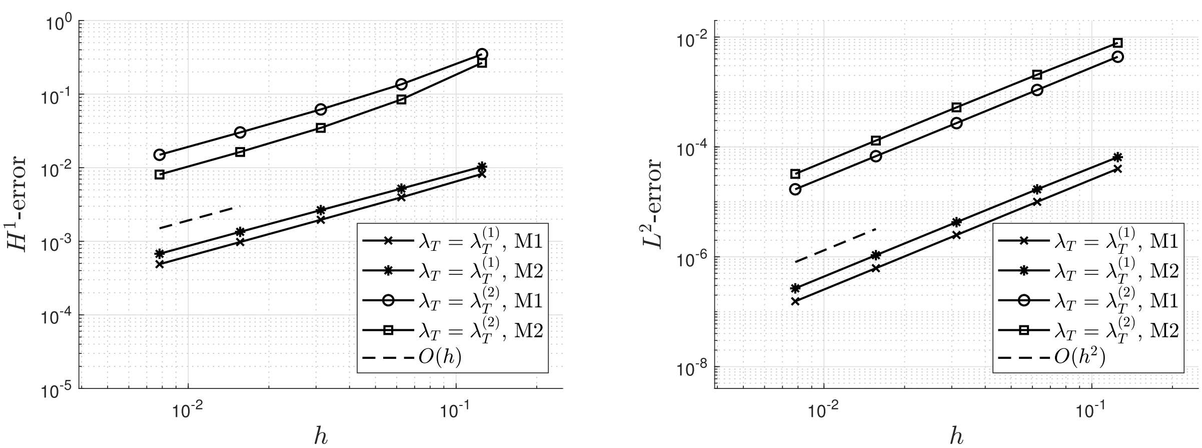

Fig. 6. The error curves versus h of Example 6.1 with $(\beta^+, \beta^-) = (100, 1)$.

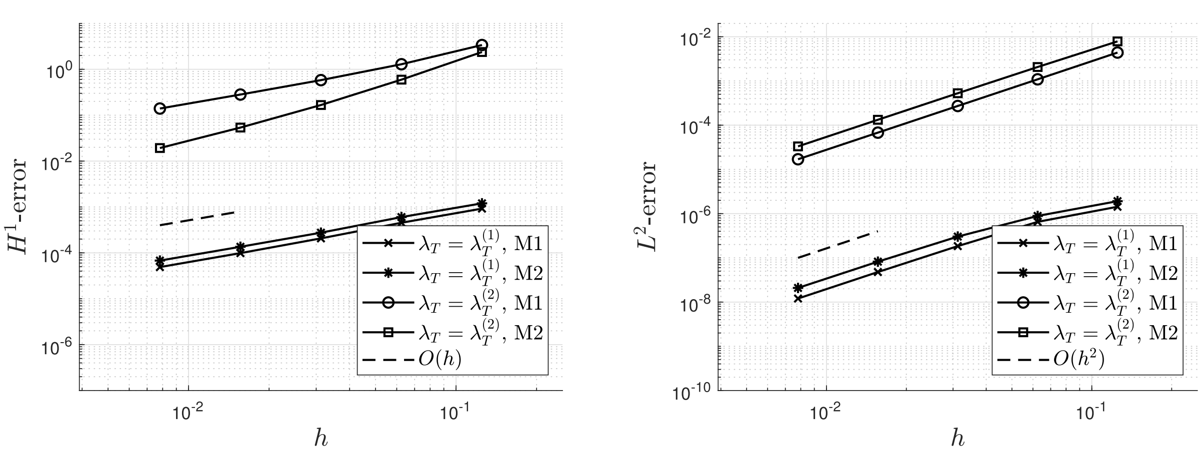

Fig. 7. The error curves versus h of Example 6.1 with $(\beta^+, \beta^-) = (10000, 1)$.

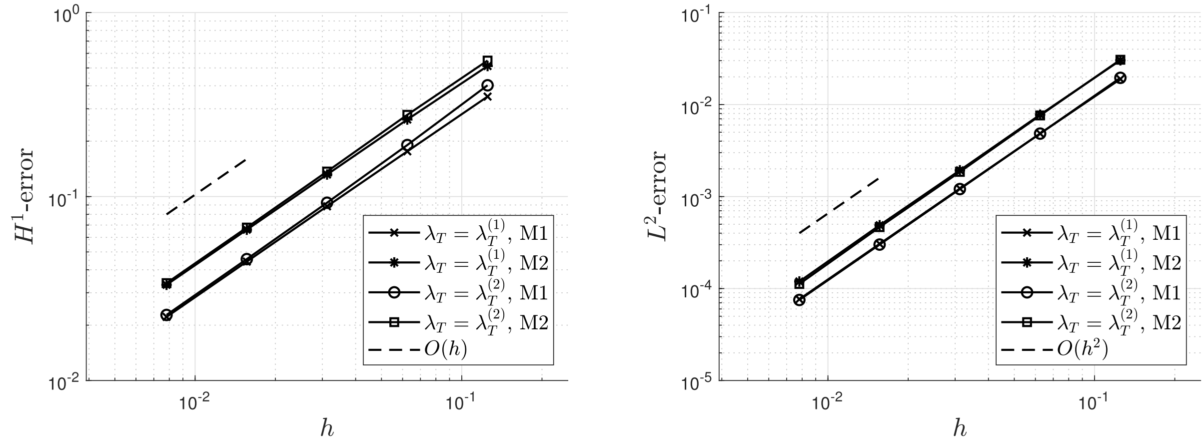

Fig. 8. The error curves versus h of Example 6.2.

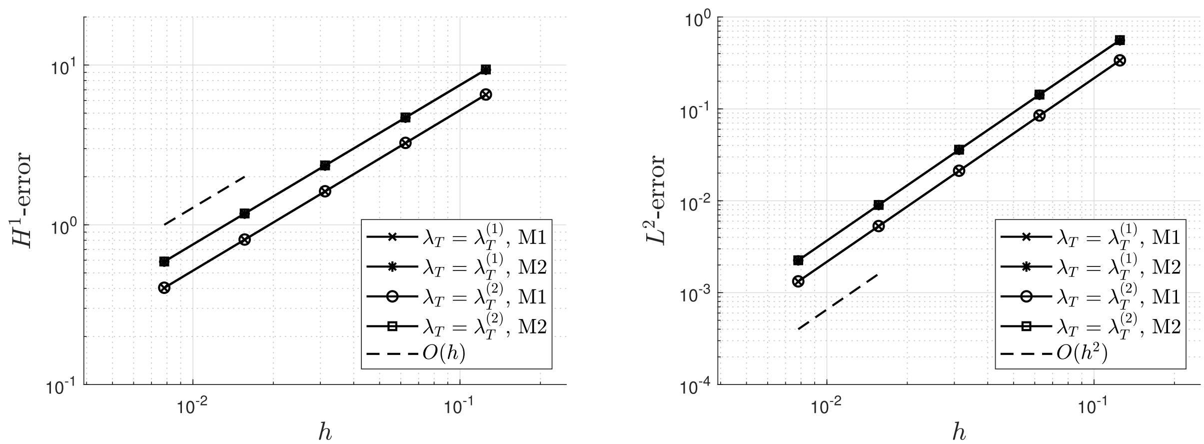

Fig. 9. The error curves versus h of Example 6.3.

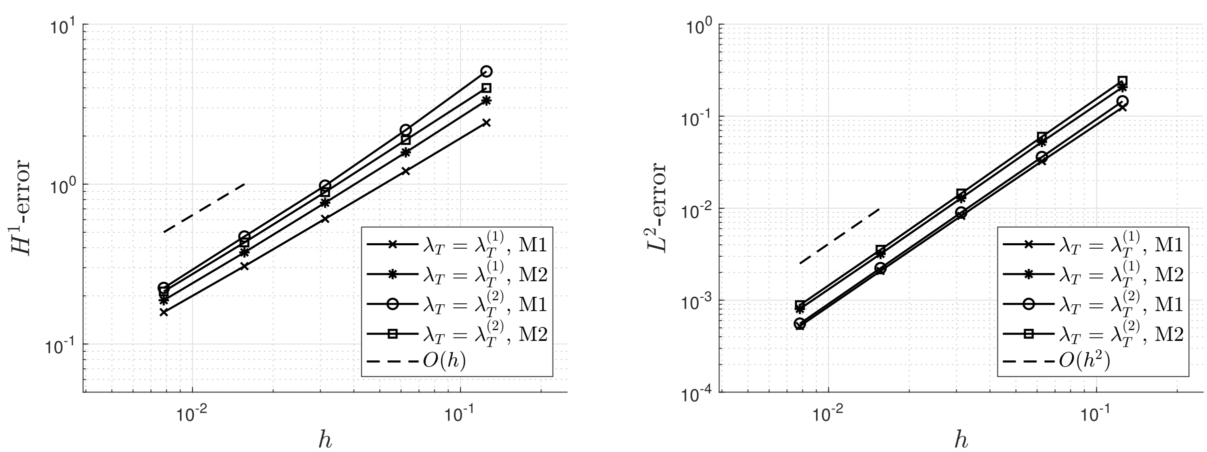

Fig. 10. The error curves versus h of Example 6.4.

$$\Gamma = \{(x,y) \in \Omega : L(x,y) = 0\}, \quad \Omega^+ = \{(x,y) \in \Omega : L(x,y) > 0\}, \quad \Omega^- = \{(x,y) \in \Omega : L(x,y) < 0\}.$$

Then the interface Γ has a sharp corner at $(1, 0.5)$ (see Fig. 5). The exact solution is chosen as $u = L/\beta$, where $\beta^+ = 10$ and $\beta^- = 1$.

Example 6.3 *(Variable coefficient)*. In this example, we take the level set of $L(x,y) = (x-0.5)^2/r_1^2 + (y-0.5)^2/r_2^2 - 1$ with $r_1 = 0.25$ and $r_2 = 0.125$ as an interface, that is, we set

$$\Gamma = \{(x,y) \in \Omega : L(x,y) = 0\}, \quad \Omega^+ = \{(x,y) \in \Omega : L(x,y) > 0\}, \quad \Omega^- = \{(x,y) \in \Omega : L(x,y) < 0\}.$$

The exact solution is chosen as $u = L/\beta$, where

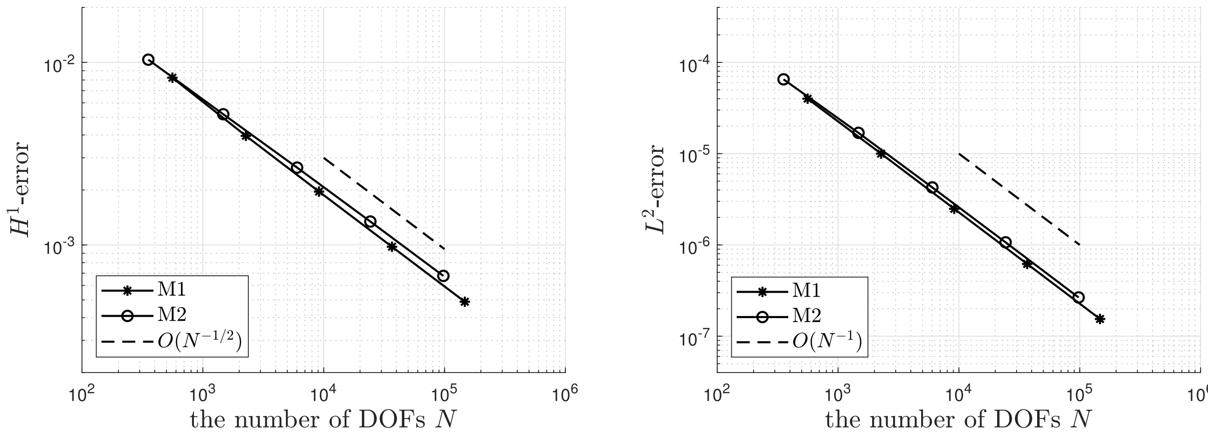

Fig. 11. The error curves versus the number of DOFs of Example 6.1 with $(\beta^+, \beta^-) = (100, 1)$ and $\lambda_T = \lambda_T^{(1)}$.

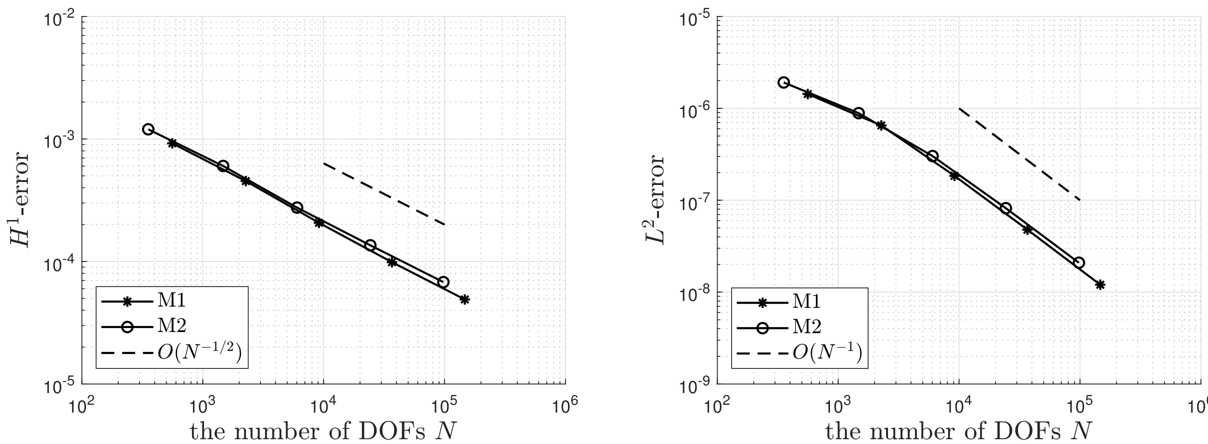

Fig. 12. The error curves versus the number of DOFs of Example 6.1 with $(\beta^+, \beta^-) = (10000, 1)$ and $\lambda_T = \lambda_T^{(1)}$.

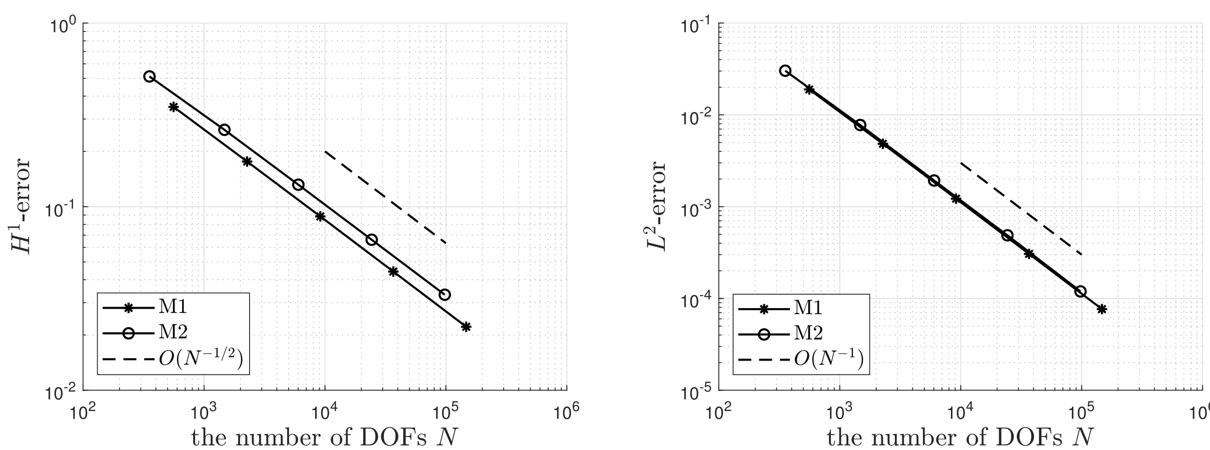

Fig. 13. The error curves versus the number of DOFs of Example 6.2 with $\lambda_T = \lambda_T^{(1)}$.

$$\beta(x,y) = \begin{cases} 1 & \text{if } (x,y) \in \Omega^+, \\ 1 + 0.5(2x-1)^2 - (2x-1)(2y-1) + (2y-1)^2 & \text{if } (x,y) \in \Omega^-. \end{cases}$$

Example 6.4 *(Cubic curve)*. In this example, we consider a cubic curve. Let $L(x,y) = (2y-1) - 3(2x-1)(2x-1.3)(2x-1.8) - 0.34$ be the level-set function and

$$\Gamma = \{(x,y) \in \Omega : L(x,y) = 0\}, \quad \Omega^+ = \{(x,y) \in \Omega : L(x,y) > 0\}, \quad \Omega^- = \{(x,y) \in \Omega : L(x,y) < 0\};$$

see Fig. 5. The exact solution is chosen as $u = L/\beta$, where $\beta^+ = 100$ and $\beta^- = 1$.

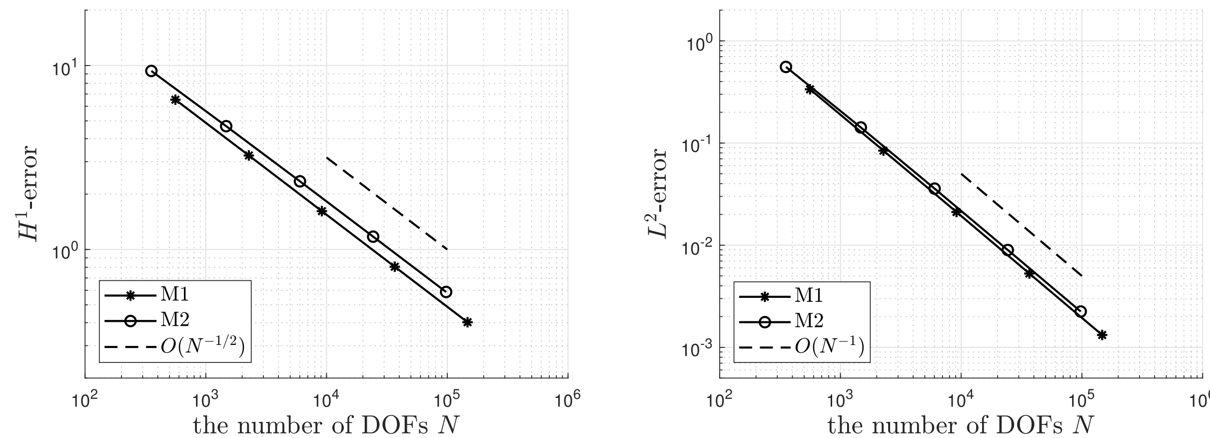

Fig. 14. The error curves versus the number of DOFs of Example 6.3 with $\lambda_T = \lambda_T^{(1)}$.

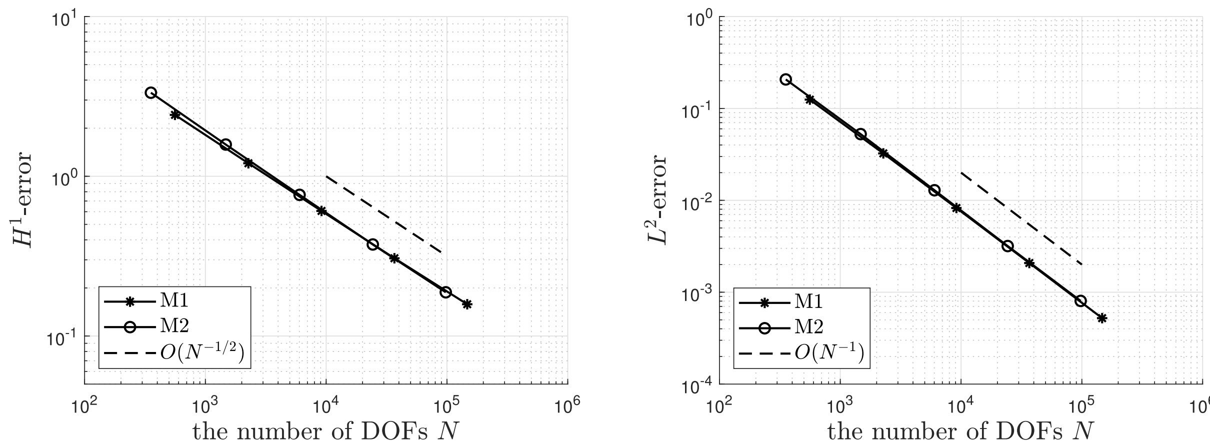

Fig. 15. The error curves versus the number of DOFs of Example 6.4 with $\lambda_T = \lambda_T^{(1)}$.

7. Conclusion

We introduce an immersed WG method for the elliptic interface problems on general unfitted polygonal meshes. The discrete space consists of constant functions on the mesh edges and piecewise linear functions in the mesh elements, satisfying the interface conditions. We prove an optimal-order convergence in the discrete H^1-seminorm under some assumptions on the exact solution.

Data availability

No data was used for the research described in the article.

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