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V-CYCLE MULTIGRID FOR CELL-CENTERED FINITE DIFFERENCES*

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Abstract. We introduce and analyze a V-cycle multigrid algorithm for cell-centered finite difference methods applied to second-order elliptic boundary value problems. Unlike conventional cell-centered multigrid algorithms that use the natural injection operator for prolongation, we use a new prolongation operator whose energy norm we prove is bounded by 1 in the constant coefficient case and $1+Ch$ in the nonconstant case. We are thus able to use general finite element multigrid theory to conclude that the V-cycle either converges well or serves as a reasonably good preconditioner, respectively. While our theory does not establish optimal performance, our numerical experiments do show that the resulting algorithm converges much faster than the conventional schemes. In fact, these results show that the energy norm convergence factor is small and remains bounded uniformly in the finest mesh size, while that of the conventional algorithm grows.

Key words. cell-centered method, finite volume method, multigrid method

AMS subject classifications. Primary, 65N30; Secondary, 65F10

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1. Introduction.

Cell-centered finite differences (CCFD) is one of the most popular methods for numerical solution of second-order elliptic boundary value problems [8], [13], [14], [15]. On the other hand, multigrid has become one of the most effective methods for solving large sparse algebraic systems resulting from discretizing elliptic boundary value problems. Multigrid algorithms for finite element methods are well developed [1], [2], [5], [6] and, in many cases, are known to converge with a reduction rate independent of the finest mesh size. However, existing multigrid algorithms for CCFD (e.g., [15], [10], [4]) do not perform as fast as their finite element counterparts and supporting analytical results are comparatively less available. There are two reasons for this scarcity of theory. First, the trial functions associated with CCFD are piecewise constants, which makes the analysis more difficult. Second, the bilinear form associated with the weak form is not inherited between two grids. Nevertheless, some analysis for CCFD is still possible. For example, the multigrid analysis for the finite elements in [6] can be applied to show that the W-cycle converges and a variable V-cycle is a good preconditioner. Such an application for CCFD using the natural injection operator for prolongation is carried out in [4]. But the numerical experiments in [4], [10] show that the condition number for the V-cycle grows with the number of levels. The reason for this behavior lies in the nonconforming nature of the space. In fact, the energy norm of this prolongation operator is greater than one, and so no conclusion can be drawn for the V-cycle using standard multigrid theory.

In this paper, we introduce a weighted prolongation operator for the multigrid algorithm applied to CCFD. We show that its energy norm is bounded by 1 in the constant coefficient case and by $1 + Ch$ in the nonconstant case. Thus the framework of [6] applies to show that the V-cycle with a fixed number of smoothing steps is convergent in the constant coefficient case and is a fairly good preconditioner in

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the nonconstant case. While our theory does not establish optimal performance of this scheme, our numerical results do show that it converges either directly or as a preconditioner for conjugate gradients with factors that are bounded uniformly in the mesh size. The results also show the superiority of our weighted interpolation scheme over the conventional approach based on injection.

The rest of the paper is organized as follows: In section 2, we define the CCFD method for solving a model elliptic problem. We then define its corresponding multigrid algorithm using a weighted prolongation operator. We estimate the energy norm of this operator and show the regularity and approximation property necessary to derive the convergence theory. In section 3, we report on some numerical experiments supporting our theoretical results.

2. Multigrid algorithm for the cell-centered method. In this section, we briefly describe the CCFD method and a corresponding V-cycle multigrid algorithm for a model problem. A weighted prolongation operator is then introduced and its energy norm is estimated. As a consequence, optimal V-cycle convergence is established.

First consider the following model problem:

$$-\nabla \cdot p\nabla \tilde{u} = f \text{ in } \Omega, \tag{2.1}$$

$$\tilde{u} = 0 \text{ on } \partial\Omega, \tag{2.2}$$

where Ω is the unit square. For $k = 1, 2, \ldots, J$, divide Ω uniformly into $n \times n$ axiparallel subsquares, where $n = 2^k$. Such subdivisions are denoted by $\{E^k\}$, and each subsquare in E^k is called a cell and denoted by E^k_{ij}, $i, j = 1, \ldots, 2^k$. Note that the cell E^k_{ij} is centered at the point $(x_i, y_j) \equiv ((i-1/2)2^{-k}, (j-1/2)2^{-k})$. For $k = 1, 2, \ldots, J$, let V_k denote the space of functions that are piecewise constant on each cell. CCFD is obtained by first integrating (2.1) formally against test functions ϕ_{ij} in V_k, where $\phi_{ij} = 1$ on E^k_{ij} and $\phi_{ij} = 0$ elsewhere. Integrating by parts, we obtain

$$-\int_{\partial E^k_{ij}} p\frac{\partial \tilde{u}}{\partial n} ds = \int_{E^k_{ij}} f dx \tag{2.3}$$

for $i, j = 1, \ldots, n$. Of course, for general $u \in V_k$, (2.3) does not make sense. Instead, we approximate (2.3) by replacing $p\partial\tilde{u}/\partial n$ on the common edge of $E^k_{i,j}$ and $E^k_{i,j+1}$ by

$$p_{i,j+1/2}\frac{u_{i,j+1} - u_{i,j}}{h}, \tag{2.4}$$

where $h = h_k = 1/2^k$, $p_{i,j+1/2} = p(x_i, y_{j+1/2})$, and $u_{i,j} = u(x_i, y_j)$ (and similarly for other subscripted u). Similarly, we use

$$p_{i+1/2,j}\frac{u_{i+1,j} - u_{i,j}}{h} \tag{2.5}$$

to approximate $p\partial\tilde{u}/\partial n$ on the common edge of $E^k_{i,j}$ and $E^k_{i+1,j}$. When one of the edges coincides with the boundary of Ω, we assign a fictitious value by reflection. For example, $u_{0,j}$ is taken as $-u_{1,j}$, and so $\partial\tilde{u}/\partial n$ at $x = 0$ is approximated by $-2u_{1,j}/h$. Similar rules apply to the other parts of the boundary. After dividing the resulting equation by h_k^2, we obtain a system of linear equation of the form

$$\bar{A}_k\bar{u} = \bar{f}, \tag{2.6}$$

where $\bar{A}_k$ is a typical sparse, $n \times n$, symmetric, positive definite matrix similar to those arising in the vertex finite difference method, $\bar{u}$ is the vector whose entries are $u_{i,j}$, and $\bar{f}$ is the vector whose entries are $f(x_i, y_j)$. Here and throughout the paper, we use $u, v, \ldots$ to denote functions in V_k while $\bar{u}, \bar{v}, \ldots$ denote their vector representations. We define a quadratic form A_k on $V_k \times V_k$ by

$$A_k(u,v) = h_k^2 \sum_{i,j}^{n} (\bar{A}_k \bar{u})_{i,j} v_{i,j} \quad \forall u, v \in V_k. \tag{2.7}$$

Then (2.6) is equivalent to the following problem: Find $u \in V_k$ satisfying

$$A_k(u, \phi) = (f, \phi) \quad \forall \phi \in V_k, \tag{2.8}$$

where $(\cdot, \cdot)$ is an L^2-inner product. The error analysis of CCFD is well known (cf. [8], [9]). Let $Q_k : L^2(\Omega) \to V_k$ denote the usual $L^2(\Omega)$ projection. If u is the solution of (2.8), then

$$A_k(u - Q_k\tilde{u}, u - Q_k\tilde{u}) \le C h_k^2 \|f\|^2, \tag{2.9}$$

where $\|\cdot\|$ is the usual L^2 norm.

To describe the multigrid algorithm for this problem, we need certain grid operators. Assuming we are given a certain prolongation operator $I_{k-1}^k : V_{k-1} \to V_k$, we define the restriction operator $I_k^{k-1} : V_k \to V_{k-1}$ as its adjoint with respect to $(\cdot, \cdot)$:

$$(I_k^{k-1} u, v) = (u, I_{k-1}^k v) \quad \forall u \in V_k, v \in V_{k-1}. \tag{2.10}$$

Since the space V_k can be viewed as the space of vectors having entries $u_{i,j}$, we also use I_{k-1}^k and I_k^{k-1} to denote their matrix representations.

Let $\bar{R}_k$ denote an approximate matrix inverse of $\bar{A}_k$ so that $\bar{x}^k \leftarrow \bar{x}^k + \bar{R}_k(\bar{f} - \bar{A}_k \bar{x}^k)$ represents one smoothing step. Let $\bar{R}_k^t$ denote the transpose of $\bar{R}_k$. Then one symmetric $V(1,1)$-cycle for solving (2.6) starting an initial guess $x^k \in V_k$ is defined in the following algorithm.

ALGORITHM $V(1,1)$. If $k = 1$, set $\bar{x}^k = \bar{A}_1^{-1} \bar{f}$. Otherwise:

1. Relax once:
$$\bar{x}^k \leftarrow \bar{x}^k + \bar{R}_k(\bar{f} - \bar{A}_k \bar{x}^k).$$
2. Beginning with initial guess $\bar{x}^{k-1} = 0$ in V_{k-1}, apply one $V(1,1)$ cycle to the coarse grid problem
$$\bar{A}_{k-1} \bar{x}^{k-1} = I_k^{k-1}(\bar{f} - \bar{A}_k \bar{x}^k).$$
Use the result $\bar{x}^{k-1}$ to correct the fine grid approximation:
$$\bar{x}^k \leftarrow \bar{x}^k + I_{k-1}^k \bar{x}^{k-1}.$$
3. Relax once:
$$\bar{x}^k \leftarrow \bar{x}^k + \bar{R}_k^t(\bar{f} - \bar{A}_k \bar{x}^k).$$

We now consider the prolongation operator. Since our spaces $V_1, \ldots, V_J$ are nested, a usual choice for the prolongation operator is the natural injection operator

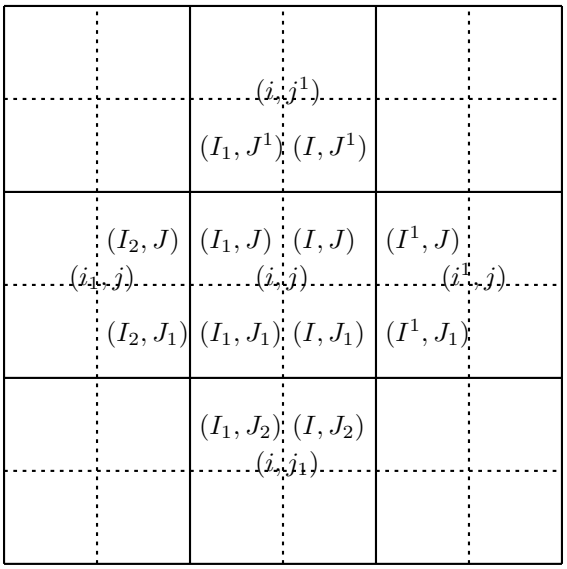

FIG. 1. *Element $E^k_{i,j}$ and its subdivision ($I_1 = I - 1, j^1 = j + 1, \ldots$).*

$I^k_{k-1} : V_{k-1} \to V_k$, but as shown in [4], the energy norm of I^n_k is $\sqrt{2}$, which violates the condition noted in [11] and, hence, we are forced to use either the variable V-cycle or W-cycle [7]. Indeed, the numerical experiments in [10], [4] show that the convergent factor for the V-cycle grows and may eventually lead to a divergent algorithm. On the other hand, Wesseling [15] and Wesseling and Khalil [12] considered the same prolongation operator together with a restriction operator based on the triangulation of the domain. These pair of operators are chosen to satisfy the following heuristic condition for the optimal multigrid convergence: $m_p + m_r > 2$, where m_p and m_r are one plus the orders of the prolongation and the restriction operators [7], respectively. However, such choices of prolongation and restriction do not fit well with the theory developed by Bramble, Pasciak, and Xu [6]; the symmetry of the algorithm plays a crucial role there and thus requires the restriction to be the transpose of the prolongation operator.

This motivates our use of a weighted prolongation operator similar to the transpose of the restriction used by Wesseling, but we instead choose the weight based on the rectangular partition. The restriction operator is chosen as its transpose to retain the symmetry of the algorithm. One could choose a prolongation operator based on linear or bilinear interpolation to obtain a similar algorithm, but our weighted operator seems to be the simplest for satisfying the conditions of the above framework.

Fix a square cell (i, j) at level $k-1$ and divide it into four subcells, labeling them counterclockwise as (I, J), $(I-1, J)$, $(I-1, J-1)$, $(I, J-1)$ (see Figure 1). For $v \in V_{k-1}$, define the weighted prolongation operator $\mathcal{I}^k_{k-1}$ as follows:

$$(\mathcal{I}^k_{k-1} v)_{I,J} = \frac{1}{4}(2v_{i,j} + v_{i,j+1} + v_{i+1,j}), \tag{2.11}$$

$$(\mathcal{I}^k_{k-1} v)_{I-1,J} = \frac{1}{4}(2v_{i,j} + v_{i,j+1} + v_{i-1,j}), \tag{2.12}$$

$$(\mathcal{I}^k_{k-1} v)_{I,J-1} = \frac{1}{4}(2v_{i,j} + v_{i+1,j} + v_{i,j-1}), \tag{2.13}$$

$$(\mathcal{I}_{k-1}^k v)_{I-1,J-1} = \frac{1}{4}(2v_{i,j} + v_{i-1,j} + v_{i,j-1}). \tag{2.14}$$

This operator is motivated by a geometric insight: the subcell $(I-1, J)$ shares two edges with subcells contained in the cell (i, j) and one each with cells in $(i, j+1)$ and $(i-1, j)$.

REMARK 2.1. *Note that $\mathcal{I}_{k-1}^k$ preserves piecewise constant functions, so it possesses an important approximation property that will be used later.*

We prove the following crucial energy norm estimate.

PROPOSITION 2.1. *For all $v \in V_{k-1}$, we have*

$$A_k(\mathcal{I}_{k-1}^k v, \mathcal{I}_{k-1}^k v) \le C(p) A_{k-1}(v, v), \tag{2.15}$$

where $C(p) = 1$ if the coefficient p is constant and $C(p) = 1 + O(h_k)$ if p is a general Lipschitz continuous function.

Proof. Fix $v \in V_{k-1}$. It is not hard to see that $A_{k-1}(v, v)$ is the square sum of differences of the function values across neighboring cells in E^{k-1}. Collecting the terms corresponding to the normal derivatives along the left walls and the top walls of each cell E_{ij}^{k-1}, we see that (see Figure 2 for indices)

$$\begin{aligned}
A_{k-1}(v,v) &= \sum_{i,j}^{n} v_{i,j} \Big[p_{ij-\frac{1}{2}}(v_{i,j} - v_{i,j-1}) + p_{i-\frac{1}{2}j}(v_{i,j} - v_{i-1,j}) \\
&\qquad + p_{i+\frac{1}{2}j}(v_{i,j} - v_{i+1,j}) + p_{ij+\frac{1}{2}}(v_{i,j} - v_{i,j+1}) \Big] \\
&= \sum_{i=2,j=1}^{n,n} p_{i-\frac{1}{2}j}(v_{i,j} - v_{i-1,j})^2 + \sum_{i=1,j=1}^{n,n-1} p_{ij+\frac{1}{2}}(v_{i,j+1} - v_{i,j})^2 \\
&\quad + 2\sum_{l=1}^{n} p_{l,\frac{1}{2}} v_{l,1}^2 + 2\sum_{l=1}^{n} p_{l,n+\frac{1}{2}} v_{l,n}^2 \\
&\quad + 2\sum_{m=1}^{n} p_{\frac{1}{2},m} v_{1,m}^2 + 2\sum_{m=1}^{n} p_{n+\frac{1}{2},m} v_{n,m}^2,
\end{aligned} \tag{2.16}$$

where $n = 2^{k-1}$.

Now we show that $A_k(\mathcal{I}_{k-1}^k v, \mathcal{I}_{k-1}^k v)$ can be bounded by (2.16). Set $u = \mathcal{I}_{k-1}^k v$. If we denote $I - \frac{1}{2}$ and $I + \frac{3}{2}$ by $I_{\frac{1}{2}}$ and $I^{\frac{3}{2}}$ (similarly for other terms), we have

$$\begin{aligned}
A_k(u,u) &= \sum_{I,J}^{2n} u_{I,J} \Big[p_{I,J_{\frac{1}{2}}}(u_{I,J} - u_{I,J_1}) + p_{I_{\frac{1}{2}},J}(u_{I,J} - u_{I_1,J}) \\
&\qquad + p_{I^{\frac{1}{2}},J}(u_{I,J} - u_{I^1,J}) + p_{I,J^{\frac{1}{2}}}(u_{I,J} - u_{I,J^1}) \Big] \\
&= \sum_{I,J}^{2n} \Big[p_{I_{\frac{3}{2}},J_1}(u_{I_1,J_1} - u_{I_2,J_1})^2 + p_{I_{\frac{3}{2}},J}(u_{I_1,J} - u_{I_2,J})^2 \\
&\qquad + p_{I_{\frac{1}{2}},J_1}(u_{I,J_1} - u_{I_1,J_1})^2 + p_{I_{\frac{1}{2}},J}(u_{I,J} - u_{I_1,J})^2 \\
&\qquad + p_{I_1,J_{\frac{1}{2}}}(u_{I_1,J} - u_{I_1,J_1})^2 + p_{I,J_{\frac{1}{2}}}(u_{I,J} - u_{I,J_1})^2 \\
&\qquad + p_{I_1,J^{\frac{1}{2}}}(u_{I_1,J^1} - u_{I_1,J})^2 + p_{I,J^{\frac{1}{2}}}(u_{I,J^1} - u_{I,J})^2 \Big] \\
&\qquad + \text{boundary terms}.
\end{aligned} \tag{2.17}$$

In the above equation, the first two terms come from the normal derivatives along the two left edges of E_{I_1,J_1} and $E_{I_1,J}$, while the second two come from the normal derivatives along two left edges of E_{I,J_1} and $E_{I,J}$. (We drop the superscript k here for simplicity.) Similarly, the next four come from the normal derivatives along the top edges of the same elements. Substituting (2.11) through (2.14) into (2.17), then, except for boundary terms, we have

$$A_k(u,u) = \frac{1}{16}\sum_{i,j} S_{i,j},$$

where

$$\begin{aligned} S_{i,j} \quad = \quad & p_{I_{\frac{3}{2}},J_1}(v_{i,j} - v_{i-1,j} + v_{i,j-1} - v_{i-1,j-1})^2 \\ & + p_{I_{\frac{3}{2}},J}(v_{i,j} - v_{i-1,j} + v_{i,j+1} - v_{i-1,j+1})^2 \\ & + p_{I_{\frac{1}{2}},J_1}(v_{i+1,j} - v_{i,j} + v_{i,j} - v_{i-1,j})^2 \\ & + p_{I_{\frac{1}{2}},J}(v_{i+1,j} - v_{i,j} + v_{i,j} - v_{i-1,j})^2 \\ & + p_{I_1,J_{\frac{1}{2}}}(v_{i,j+1} - v_{i,j} + v_{i,j} - v_{i,j-1})^2 \\ & + p_{I,J_{\frac{1}{2}}}(v_{i,j+1} - v_{i,j} + v_{i,j} - v_{i,j-1})^2 \\ & + p_{I_1,J^{\frac{1}{2}}}(v_{i,j+1} - v_{i,j} + v_{i-1,j+1} - v_{i-1,j})^2 \\ & + p_{I,J^{\frac{1}{2}}}(v_{i,j+1} - v_{i,j} + v_{i+1,j+1} - v_{i+1,j})^2. \end{aligned} \tag{2.18}$$

The Cauchy–Schwarz inequality implies that

$$\begin{aligned} S_{i,j} \le 2[& \phi_1(v_{i,j} - v_{i-1,j})^2 + \phi_2(v_{i,j-1} - v_{i-1,j-1})^2 \\ & + \phi_3(v_{i,j+1} - v_{i-1,j+1})^2 + \phi_4(v_{i+1,j} - v_{i,j})^2 \\ & + \phi_5(v_{i,j} - v_{i,j-1})^2 + \phi_6(v_{i,j+1} - v_{i,j})^2 \\ & + \phi_7(v_{i-1,j+1} - v_{i-1,j})^2 + \phi_8(v_{i+1,j+1} - v_{i+1,j})^2], \end{aligned}$$

where

$$\begin{aligned} & \phi_1 = p_{I_{\frac{3}{2}},J_1} + p_{I_{\frac{3}{2}},J} + p_{I_{\frac{1}{2}},J_1} + p_{I_{\frac{1}{2}},J}, \\ & \phi_2 = p_{I_{\frac{3}{2}},J_1}, \quad \phi_3 = p_{I_{\frac{3}{2}},J}, \\ & \phi_4 = p_{I_{\frac{1}{2}},J_1} + p_{I_{\frac{1}{2}},J}, \quad \phi_5 = p_{I_1,J_{\frac{1}{2}}} + p_{I,J_{\frac{1}{2}}}, \\ & \phi_6 = p_{I_1,J_{\frac{1}{2}}} + p_{I,J_{\frac{1}{2}}} + p_{I_1,J^{\frac{1}{2}}} + p_{I,J^{\frac{1}{2}}}, \\ & \phi_7 = p_{I_1,J^{\frac{1}{2}}}, \quad \phi_8 = p_{I,J^{\frac{1}{2}}}. \end{aligned}$$

We consider only terms of the form $(v_{i,j} - v_{i-1,j})^2$ and $(v_{i,j+1} - v_{i,j})^2$. First, S_{ij} has $2\phi_1(v_{i,j} - v_{i-1,j})^2$ and $2\phi_6(v_{i,j+1} - v_{i,j})^2$. Also, there are some contributions from neighboring cells: $2\phi_5(v_{i,j+1} - v_{i,j})^2$ and $2\phi_2(v_{i,j} - v_{i-1,j})^2$ come from $S_{i,j+1}$, $2\phi_3(v_{i,j} - v_{i-1,j})^2$ from $S_{i,j-1}$, $2\phi_4(v_{i,j} - v_{i-1,j})^2$, $2\phi_8(v_{i,j+1} - v_{i,j})^2$ from $S_{i-1,j}$, and finally $2\phi_7(v_{i,j+1} - v_{i,j})^2$ from $S_{i+1,j}$. Summing over all i, j, we see that

$$\begin{aligned} \frac{1}{16}\sum_{i,j}^{n} S_{ij} \quad \le \quad & \frac{1}{16}\sum_{i,j}^{n}(2\phi_1 + 2\phi_2 + 2\phi_3 + 2\phi_4)(v_{i,j} - v_{i-1,j})^2 \\ & + (2\phi_6 + 2\phi_5 + 2\phi_7 + 2\phi_8)(v_{i,j+1} - v_{i,j})^2. \end{aligned} \tag{2.19}$$

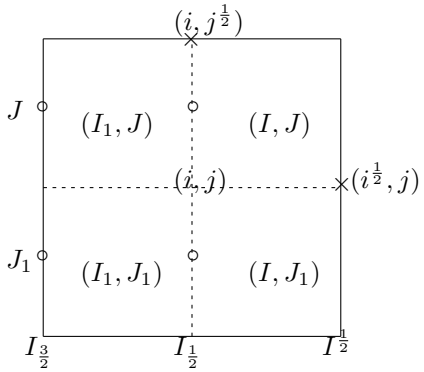

FIG. 2. *Edges of $E_{i,j}$ and $E_{I,J}$.*

If p is constant, (2.19) is obviously bounded by the first two sums of (2.16). If not, it is easy to see that

$$\frac{1}{16}(2\phi_1 + 2\phi_2 + 2\phi_3 + 2\phi_4) = \frac{1}{4}(p_{I_{\frac{3}{2}},J_1} + p_{I_{\frac{3}{2}},J} + p_{I_{\frac{1}{2}},J_1} + p_{I_{\frac{1}{2}},J}) = p_{i-\frac{1}{2},j} + O(h)$$

and

$$\frac{1}{16}(2\phi_5 + 2\phi_6 + 2\phi_7 + 2\phi_8) = p_{i,j+\frac{1}{2}} + O(h).$$

Hence, the bounding constant is $1 + O(h)$ in this case. Now we investigate boundary terms. Here we assume $p = 1$ for simplicity. For general p, the same argument as above applies. It suffices to look at the terms $u_{I,1}$. For I odd, we have

$$u_{I,1} = \frac{2v_{i,1} + v_{i-1,1} - v_{i,1}}{4} = \frac{v_{i,1} + v_{i-1,1}}{4},$$

and for I even, we have

$$u_{I,1} = \frac{2v_{i,1} + v_{i+1,1} - v_{i,1}}{4} = \frac{v_{i,1} + v_{i+1,1}}{4}.$$

Thus,

$$\begin{aligned} 2\sum_{l=1}^{2n} u_{l,1}^2 &= \frac{1}{8}\sum_{i=1}^{n}(v_{i,1} + v_{i-1,1})^2 + (v_{i,1} + v_{i+1,1})^2 \\ &\le \frac{1}{8}\sum_{i=1}^{n} 2(2v_{i,1}^2 + v_{i-1,1}^2 + v_{i+1,1}^2) \end{aligned} \tag{2.20}$$

so the coefficient of a typical term $v_{i,1}^2$ in the above sum is $(4 + 2 + 2)/8 = 1$. We now account for the contributions from $S_{i,1}$. Note that when $j = 1$, the first term in (2.18) vanishes, so

$$\begin{aligned} S_{i,1} \quad \le \quad & 2[3(v_{i,1} - v_{i-1,1})^2 \\ & + (v_{i,2} - v_{i-1,2})^2 + 2(v_{i+1,1} - v_{i,1})^2 \\ & + 2(v_{i,1} - v_{i,0})^2 + 4(v_{i,2} - v_{i,1})^2 \\ & + (v_{i-1,2} - v_{i-1,1})^2 + (v_{i+1,2} - v_{i+1,1})^2]. \end{aligned} \tag{2.21}$$

Hence, the contribution from $S_{i,1}$ is

$$4(v_{i,1} - v_{i,0})^2 = 16v_{i,1}^2$$

since $v_{i,0} = -v_{i,1}$. Thus, the sum of the coefficients is $1 + 16/16 = 2$, so the terms in (2.20) and (2.21) are bounded by the third sum of (2.16). Similarly, the other boundary terms can be bounded by the rest of the terms of (2.16). This completes the proof. ∎

The next ingredient for establishing multigrid convergence theory is the so-called regularity and approximation property: there exist constants $\alpha \in (0, 1]$ and C_α such that $\forall k = 1, \ldots, J$,

$$A_k((I - \mathcal{I}_{k-1}^k P_{k-1})u, u) \le C_\alpha^2 \left(\frac{\|A_k u\|^2}{\lambda_k} \right)^\alpha A_k(u, u)^{1-\alpha} \quad \forall u \in V_k. \tag{2.22}$$

Here, $A_k u$ denotes the function whose value at the cell (i, j) is $(\bar{A}_k \bar{u})_{i,j}$, λ_k is the largest eigenvalue of A_k, and P_{k-1} is the elliptic projection defined by

$$A_{k-1}(P_{k-1}u, v) = A_k(u, \mathcal{I}_{k-1}^k v) \quad \forall u \in V_k, v \in V_{k-1}. \tag{2.23}$$

We also need the following operator $\tilde{P}_{k-1} : V_k \to V_{k-1}$ defined by

$$A_{k-1}(\tilde{P}_{k-1}u, v) = A_k(u, v) \quad \forall u \in V_k, v \in V_{k-1}. \tag{2.24}$$

LEMMA 2.2. *Property* (2.22) *holds for* $\alpha = \frac{1}{2}$.

Proof. The proof is similar to the case of natural injection in [4]. Fix $u \in V_k$ and let w be the solution of the following boundary value problem:

$$\begin{aligned} -\nabla \cdot p \nabla w &= A_k u \qquad \text{in } \Omega, \\ w &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{2.25}$$

Note that u is the cell-centered approximation of w, so we have [8]

$$|A_k(u - Q_k w, u - Q_k w)| \le C h_k^2 \|A_k u\|^2. \tag{2.26}$$

By definition of $\mathcal{I}_{k-1}^k$, the left-hand side of (2.22) can be rewritten as

$$\begin{aligned} A_k((I - \mathcal{I}_{k-1}^k P_{k-1})u, u) &= A_k(u, u) - A_{k-1}(P_{k-1}u, P_{k-1}u) \\ &= A_k(u - Q_k w, u) \\ &\quad + A_{k-1}(Q_{k-1}w - P_{k-1}u, P_{k-1}u) \\ &\quad + A_k(Q_k w, u) - A_{k-1}(Q_{k-1}w, P_{k-1}u). \end{aligned} \tag{2.27}$$

Applying the Cauchy–Schwarz inequality to the first term, by (2.26) we have

$$\begin{aligned} |A_k(u - Q_k w, u)| &\le A_k(u - Q_k w, u - Q_k w)^{1/2} A_k(u, u)^{1/2} \\ &\le C h_k \|A_k u\| A_k(u, u)^{1/2}. \end{aligned} \tag{2.28}$$

The second term can be estimated similarly by noting that $\tilde{P}_{k-1}u$ is the cell-centered approximation of w in V_{k-1}. By (2.23) and (2.24), we have

$$\begin{aligned} A_{k-1}(Q_{k-1}w - P_{k-1}u, P_{k-1}u) &= A_{k-1}(Q_{k-1}w - \tilde{P}_{k-1}u, P_{k-1}u) \\ &\quad + A_{k-1}(\tilde{P}_{k-1}u - P_{k-1}u, P_{k-1}u) \\ &= A_{k-1}(Q_{k-1}w - \tilde{P}_{k-1}u, P_{k-1}u) \\ &\quad - A_k(u, (I - \mathcal{I}_{k-1}^k)P_{k-1}u). \end{aligned} \tag{2.29}$$

By similar arguments as above, we see that

$$|A_{k-1}(Q_{k-1}w - \tilde{P}_{k-1}u, P_{k-1}u)| \le Ch_k\|A_ku\|A_{k-1}(P_{k-1}u, P_{k-1}u)^{1/2}. \tag{2.30}$$

Since $\mathcal{I}_{k-1}^k$ preserves piecewise constant functions, we have

$$\begin{aligned} |A_k(u,(I-\mathcal{I}_{k-1}^k)P_{k-1}u)| &\le \|A_ku\|\cdot\|(I-\mathcal{I}_{k-1}^k)P_{k-1}u\| \\ &\le Ch_kA_{k-1}(P_{k-1}u,P_{k-1}u)^{1/2}\|A_ku\|. \end{aligned} \tag{2.31}$$

Note that

$$A_{k-1}(P_{k-1}u, P_{k-1}u) \le \frac{1}{2-C(p)}A_k(u,u) \tag{2.32}$$

if and only if (2.15) holds. Substituting (2.30) and (2.31) into (2.29), we see from (2.32) that the second term in (2.27) satisfies

$$\begin{aligned} |A_{k-1}(Q_{k-1}w - P_{k-1}u, P_{k-1}u)| &\le Ch_k\|A_ku\|A_{k-1}(P_{k-1}u,P_{k-1}u)^{1/2} \\ &\le Ch_k\|A_ku\|A_k(u,u)^{1/2}. \end{aligned} \tag{2.33}$$

We now estimate the third and fourth term in (2.27). Since $\mathcal{I}_{k-1}^kQ_{k-1}$ preserves piecewise constant functions, we have

$$\begin{aligned} |A_k(Q_kw,u) - A_{k-1}(Q_{k-1}w,P_{k-1}u)| &= |A_k(Q_kw,u) - A_k(\mathcal{I}_{k-1}^kQ_{k-1}w,u)| \\ &\le \|(I-\mathcal{I}_{k-1}^kQ_{k-1})w\|\,\|A_ku\| \\ &\le Ch_k\|w\|_1\|A_ku\|, \end{aligned}$$

where $\|\cdot\|_1$ is the Sobolev norm of order one. It remains to bound $\|w\|_1$ by $A_k(u,u)^{1/2}$, which can be done exactly the same way as in [4]. Thus, we have

$$|A_k(Q_kw,u) - A_{k-1}(Q_{k-1}w,P_{k-1}u)| \le Ch_k\|A_ku\|A_k(u,u)^{1/2}. \tag{2.34}$$

Combining estimates (2.28), (2.33), and (2.34), together with the obvious bound $\lambda_k \le Ch_k^{-2}$, we obtain (2.22). □

Now we can apply Theorems 1 and 8 of [6] to obtain multigrid convergence estimates. We first note that our $V(1,1)$-cycle algorithm has an error propagation operator of the form $E_k = I - B_kA_k$.

THEOREM 2.3. *We have the following.*

1. *If p is constant, then $V(1,1)$ converges according to*

$$0 \le A_k(E_ku,u) \le \delta_kA_k(u,u) \quad \forall u \in V_k, \tag{2.35}$$

where $\delta_k = \frac{Ck}{Ck+\sqrt{m}}$.

2. *If p is Lipschitz continuous, then*

$$\eta_0A_k(u,u) \le A_k(B_kA_ku,u) \le \eta_1A_k(u,u) \quad \forall u \in V_k, \tag{2.36}$$

where η_1 is independent of k and $\eta_0 \le 1-\delta_k$.

REMARK 2.2. *If* (2.36) *holds, then the linear iteration*

$$u^{\ell+1} \leftarrow u^\ell + B_k(f - A_ku^\ell), \quad \ell = 1,2,\ldots,$$

yields a reduction factor given by

$$\rho \equiv \|I - B_k A_k\|_A \leq \max\{\eta_1 - 1, 1 - \eta_0\},$$

where $\|\cdot\|_A$ *denotes the discrete energy norm induced by* $A_k(\cdot,\cdot)$. *For example, if* $\eta_1 - 1 = 1 - \eta_0$, *then* $\rho = (K-1)/(K+1)$, *where* $K = \eta_1/\eta_0$. *If we use the* $V(1,1)$ *cycle as a preconditioner for conjugate gradients, it is well known (see* [5], [3]*) that the resulting PCG method would converge with an asymptotic reduction factor bounded by*

$$\frac{\sqrt{K}-1}{\sqrt{K}+1}.$$

This is better than for the linear iteration because

$$\frac{\sqrt{K}-1}{\sqrt{K}+1} \leq \frac{K-1}{K+1}.$$

If (2.35) *holds, then* $\eta_1 = 1$ *and* $\eta_0 = 1 - \delta_k$ *and the above discussion would imply that the preconditioned conjugate gradient method has the reduction factor* $(1 - \sqrt{1-\delta_k})/(1+\sqrt{1+\delta_k}) \approx \delta_k/4$.

REMARK 2.3. *The same results hold for rectangular domains with nonuniform meshes, as long as the mesh lines are axi-parallel. Also, the results hold when the domain is nonrectangular but consists of rectangular pieces.*

3. Numerical experiment. Consider the following problem on the unit square:

$$\begin{aligned} -\nabla \cdot p \nabla \tilde{u} &= f \quad \text{in } \Omega = (0,1)^2, \\ \tilde{u} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

We study the performance of the $V(1,1)$-cycle with Gauss–Seidel relaxation (in lexicographic order before the coarse grid correction and in reverse order afterwards, corresponding to the use of $\bar{R}_k$ and $\bar{R}_k^t$). We compare our weighted interpolation version with the natural injection scheme. We report on the eigenvalues, condition numbers, and reduction factors of both multigrid algorithms. The power method (with origin shift for minimum eigenvalues) is used to estimate the eigenvalues of $B_k A_k$ and the iteration is stopped when the eigenvalues become stationary up to four decimal points. The reduction factor is computed as a geometric mean of reduction factor over 50 iterations, starting from a random initial guess applied to the homogeneous discrete problem. We also report on the reduction factors when the multigrid algorithms are used as preconditioners for conjugate gradients.

As a first example, we take $p = 1$. Table 1 shows the result with weighted interpolation. Note that the condition number in this case is almost the same as that of the conforming finite element methods. Table 2 shows that, with injection, the condition number deteriorates as the number of levels grows. As a second example, we take $p = p_j$, a discontinuous function:

$$p_j = \begin{cases} 10^j \ (j = 1,2) & \text{when } x > \frac{1}{2}, y > \frac{1}{2}, \\ 1 & \text{otherwise.} \end{cases}$$

In assembling the stiffness matrix when the coefficient is discontinuous, it is usual to take the harmonic averages of the coefficients on the interface [14]. However, to see when the multigrid algorithms diverge but still serve as a preconditioner, we just

TABLE 1
Weighted interpolation, $p = 1$.

h_J	λ_{min}	λ_{max}	K	δ
1/32	0.673	0.999	1.484	0.099
1/64	0.667	0.999	1.498	0.099
1/128	0.664	0.999	1.505	0.099
1/256	0.663	0.999	1.507	0.099

TABLE 2
Injection, $p = 1$.

h_J	λ_{min}	λ_{max}	K	δ
1/32	0.784	1.444	1.842	0.218
1/64	0.783	1.534	1.960	0.309
1/128	0.784	1.618	2.060	0.403
1/256	0.784	1.684	2.148	0.495

TABLE 3
Weighted interpolation, $p = p_1$.

h_J	λ_{min}	λ_{max}	K	δ
1/32	0.638	1.604	2.514	0.388
1/64	0.633	1.710	2.701	0.582
1/128	0.630	1.767	2.805	0.582
1/256	0.616	1.842	2.990	0.684

TABLE 4
Injection, $p = p_1$.

h_J	λ_{min}	λ_{max}	K	δ
1/32	0.755	2.810	3.722	divergent
1/64	0.728	2.987	4.103	divergent
1/128	0.727	3.140	4.319	divergent
1/256	0.722	3.266	4.526	divergent

TABLE 5
Weighted interpolation, $p = p_2$.

h_J	λ_{min}	λ_{max}	K	δ
1/32	0.488	8.92	18.28	divergent
1/64	0.448	9.77	21.81	divergent
1/128	0.423	10.44	24.68	divergent
1/256	0.395	11.03	27.92	divergent

TABLE 6
Injection, $p = p_2$.

h_J	λ_{min}	λ_{max}	K	δ
1/32	0.767	18.45	24.05	divergent
1/64	0.767	19.43	25.33	divergent
1/128	0.716	21.14	29.53	divergent
1/256	0.689	21.63	31.39	divergent

take one-sided limits to evaluate the function p. This results in a system that is more difficult to solve. Although our theory does not cover this case, Tables 3 and 4 show that when the jump is not severe, the weighted interpolation scheme converges well while the injection scheme diverges. When the jump is severe, both algorithms diverge as Tables 5 and 6 show. Table 7 shows that both algorithms perform well as preconditioners for the conjugate gradient method.

TABLE 7
Reduction factor with PCG, $p = 1$.

h_J	Weighted interp.	Injection
1/32	0.007	0.018
1/64	0.008	0.023
1/128	0.008	0.027
1/256	0.008	0.029

TABLE 8
Weighted interpolation with a $V(1,0)$ cycle, $p = 1$.

h_J	λ_{min}	λ_{max}	K	δ
1/32	0.555	1.270	2.309	0.199
1/64	0.538	1.330	2.472	0.205
1/128	0.530	1.331	2.511	0.207
1/256	0.525	1.332	2.537	0.209

TABLE 9
Injection with a $V(1,0)$ cycle, $p = 1$.

h_J	λ_{min}	λ_{max}	K	δ
1/32	0.558	2.405	4.301	divergent
1/64	0.568	2.965	5.211	divergent
1/128	0.565	3.627	6.419	divergent
1/256	0.557	4.421	7.937	divergent

Finally, we tested both algorithms using a simple $V(1,0)$-cycle (with no post smoothing). Tables 8 and 9 show that weighted interpolation still works well, while injection does not.

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