Numer. Math. Theor. Meth. Appl. Vol. **13**, No. 2, pp. 281-295
doi: 10.4208/nmtma.OA-2019-0034 May 2020

A Stabilized Low Order Finite Element Method for Three Dimensional Elasticity Problems

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Received 25 February 2019; Accepted (in revised version) 11 June 2019

Abstract. We introduce a low order finite element method for three dimensional elasticity problems. We extend Kouhia-Stenberg element [12] by using two nonconforming components and one conforming component, adding stabilizing terms on the associated bilinear form to ensure the discrete Korn's inequality. Using the second Strang's lemma, we show that our scheme has optimal convergence rates in L^2 and piecewise H^1-norms even when Poisson ratio ν approaches $1/2$. Even though some efforts have been made to design a low order method for three dimensional problems in [11, 16], their method uses some higher degree basis functions. Our scheme is the first true low order method. We provide three numerical examples which support our analysis. We compute two examples having analytic solutions. We observe the optimal L^2 and H^1 errors for many different choice of Poisson ratios including the nearly incompressible cases. In the last example, we simulate the driven cavity problem. Our scheme shows non-locking phenomena for the driven cavity problems also.

AMS subject classifications: 65N12, 65N30
Key words: Elasticity equation, low order finite element, Kouhia-Stenberg element, locking free, Korn's inequality.

1. Introduction

We consider the following type of elasticity equation in a convex polyhedral domain Ω in $\mathbb{R}^3$:

$$-\operatorname{div}\boldsymbol{\sigma}(\mathbf{u}) = \mathbf{f} \qquad \text{in} \quad \Omega, \tag{1.1a}$$

$$\mathbf{u} = 0 \qquad \text{in} \quad \partial\Omega, \tag{1.1b}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the displacement variable and $\mathbf{f} \in [L^2(\Omega)]^3$ is an external force. We may also consider the pure traction problems, but we choose the Dirichlet

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boundary conditions just for simplicity of presentation. Here, the strain tensor $\boldsymbol{\epsilon}(\mathbf{u})$ and the stress tensor $\boldsymbol{\sigma}(\mathbf{u})$ are as usual,

$$\boldsymbol{\epsilon}_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad \boldsymbol{\sigma}(\mathbf{u}) = 2\mu\boldsymbol{\epsilon}(\mathbf{u}) + \lambda \mathrm{tr}(\boldsymbol{\epsilon}(\mathbf{u}))\mathbf{I},$$

where $\mathbf{I}$ is 3×3 by identity matrix. The Lamé constants μ and λ are given in terms of modulus of elasticity $E > 0$ and Poisson's ratio $0 < \nu < 1/2$,

$$\mu = \frac{E}{2(1+\nu)}, \qquad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}.$$

We note that as $\nu \to 1/2$, the parameter λ goes to infinity, as in incompressible case.

It is well known that conforming low order finite elements for solving elasticity problem usually yields a locking phenomena as the Poisson ratio approaches to 1/2 [15]. For nonconforming elements, the associated bilinear form fails to satisfy the discrete Korn's inequality [9]. Hence the coerciveness does not hold. The analyses regarding the locking phenomena in [9, 15] are developed in two dimensional problems. However, by restricting to R^2, we can see that low order methods are also locking in R^3. Thus, to obtain optimal convergence rates using Lagrangian type of finite element methods, one must use polynomial of order ≥ 4, when the material is nearly incompressible [15]. However, some nonconforming elements of degree ≥ 2 converges uniformly as the Poisson ratio approaches $1/2$ [9].

Some efforts have been made to avoid locking phenomena using lower order nonconforming methods. One often uses reduced integration or macro element technique [4, 9]. Some people apply the mixed methods [5] to elasticity equations (see [14]). Other approaches are to design the finite element (FE) space or to modify the bilinear form to satisfy the discrete Korn's inequality. Kouhia-Stenberg (KS) [12] used conforming-nonconforming pair for two dimensional problems, while Hansbo, et al. [10] used nonconforming pair with stability terms to enforce coerciveness. However, it was shown [11, 16] that a straightforward extension of KS element to three dimensional case is impossible. For example, the pair (P_n^1, P_n^1, P_c^1) would not satisfy the Korn's inequality if we restrict it to the first two components. The authors in [16] used Q_2-conforming space in one of the components while the authors in [11] used bubble functions of degree 3 in one of the components.

In this paper, we present a simple extension of KS element to three dimensional elasticity problems using the pair (P_n^1, P_n^1, P_c^1). Instead we add stability terms on the first two component, which yields the smaller number of degrees of freedom than the spaces introduced in [11, 16]. The concept of adding stabilizing term on the bilinear form is motivated by [10, 13]. With the aid of the stabilizing term, we were able to prove that our scheme is stable, i.e., the bilinear form is coercive with respect to broken H^1-norm. In this way, we obtain a new extension of KS method to 3D element using only piecewise linear functions, while the number of unknowns is about 69 percent of (P_n^1, P_n^1, P_n^1) elements (see Example 4.2). We provide optimal error estimates in the energy norm and L^2-norm. We show that our scheme converges uniformly as ν approaches 1/2, hence it is non-locking.

We provide numerical results which supports our analysis. The numerical results show that our scheme is optimal in convergence both in L^2 and H^1-norms regardless of the Poisson ratio. In [10], the P_1-nonconforming methods with a stabilizing terms was developed for two dimensional elasticity problems while the 3D extension of their work using (P_n^1, P_n^1, P_n^1) was discussed without numerical tests. We also report the performance of (P_n^1, P_n^1, P_c^1) element compared to that of [10](both with stability terms).

We introduce some notations. Let D be any open domain. We let $H^m(D)$ be the Sobolev space with (semi)-norms denoted by $|\cdot|_{m,D}$ and $\|\cdot\|_{m,D}$. We define $H_0^1(\Omega)$ to be the set of functions in $H^1(\Omega)$ with vanishing traces. The following theorem is well known [6].

Theorem 1.1. *There exists an unique* $\mathbf{u} \in [H_0^1(\Omega)]^3$*, satisfying* (1.1a)-(1.1b).

To the author's knowledge, the regularity of the solution $\mathbf{u}$ in three dimensional elasticity problems is in general not known. Thus, we assume the following result for our analysis.

(A.1) For all $\mathbf{f} \in L^2(\Omega)$, there exists a solution $\mathbf{u} \in [H_0^1(\Omega)]^3 \cap [H^2(\Omega)]^3$ and a constant $C > 0$ independent of λ such that

$$\|\mathbf{u}\|_{H^2(\Omega)} + \lambda \|\mathrm{div}\mathbf{u}\|_{H^1(\Omega)} \le C\|\mathbf{f}\|_{L^2(\Omega)}.$$

We note that the above assumption is an extension of the L^2-stability theorem for two dimensional elasticity problem [1].

2. Three dimensional generalization of KS element

Let $\mathcal{T}_h$ be any regular partitioning of Ω by tetrahedra described in [7]. In other words, there exists a constant $\eta > 0$ such that $h_T/\rho_T \le \eta$ for all $T \in \mathcal{T}_h$, where $h_T = \mathrm{diam}(T)$ and ρ_T is the radius of maximum sphere contained in T. We let $h = \max_{T\in\mathcal{T}_h} h_T$. We denote the set of interior faces of $\mathcal{T}_h$ by $\mathcal{E}_h$.

We define local spaces on each element $T \in \mathcal{T}_h$. Let $S_h(T)$ be the space of the linear functions of the form $a + bx + cy + dz$ on T with vertex degrees of freedom (dof) and $N_h(T)$ be the space of the similar linear functions defined by face average dof. With the local space given by $\mathbf{U}_h(T) := N_h(T) \times N_h(T) \times S_h(T)$, the global space is defined by

$$\mathbf{U}_h(\Omega) = \left\{ \begin{array}{l} \boldsymbol{\phi}_h = (\phi_{h,1}, \phi_{h,2}, \phi_{h,3}) \in \mathbf{U}_h(T) \quad \text{for any } T \in \mathcal{T}_h, \\ \displaystyle\int_F \phi_{h,1}|_{T_1} = \int_F \phi_{h,1}|_{T_2}, \text{ where } F \text{ is comman face of } T_1 \text{ and } T_2, \\ \displaystyle\int_F \phi_{h,2}|_{T_1} = \int_F \phi_{h,2}|_{T_2}, \text{ where } F \text{ is comman face of } T_1 \text{ and } T_2, \\ \phi_{h,3} \ \text{ is continuous on each vertex of } T \in \mathcal{T}_h, \\ \displaystyle\int_F \phi_{h,1} = \int_F \phi_{h,2} = \phi_{h,3}|_F = 0, \text{ where } F \text{ is part of } \partial\Omega. \end{array} \right\}$$

For each face $F \in \mathcal{E}_h$, we associate the unit normal vector $\mathbf{n}_F$ at F. We define a jump $[\mathbf{u}]_F$ for $\mathbf{u} \in \mathbf{H}_h(\Omega) := \mathbf{U}_h(\Omega) + [H_0^1(\Omega)]^3$ as

$$[\mathbf{u}]_F(x) := \lim_{\delta \to 0+} (\mathbf{u}(x - \delta \mathbf{n}_F) - \mathbf{u}(x + \delta \mathbf{n}_F)).$$

Let $a_h(\cdot, \cdot)$ be the bilinear form on $\mathbf{H}_h(\Omega)$ defined by

$$\begin{aligned} a_h(\mathbf{u}, \mathbf{v}) := & \sum_{T \in \mathcal{T}_h} \int_T 2\mu \boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{v}) + \sum_{T \in \mathcal{T}_h} \int_T \lambda \mathrm{div}\mathbf{u} \mathrm{div}\mathbf{v} \\ & + \sum_{F \in \mathcal{E}_h} \frac{2\mu\tau}{|F|^{\frac{1}{2}}} \int_F [\mathbf{P}\mathbf{u}]_F \cdot [\mathbf{P}\mathbf{v}]_F, \end{aligned} \tag{2.1}$$

where τ is a positive constant. Here, $\mathbf{P} : [L^2(\Omega)]^3 \to [L^2(\Omega)]^2$ is a projection operator defined by, $\mathbf{P}(\mathbf{u}) = (u_1, u_2)$ for $\mathbf{u} = (u_1, u_2, u_3)$. By dividing (2.1) by 2μ and replacing $\lambda/2\mu$ by λ, we may assume $2\mu = 1$ from now on. Our methods reads: Find $\mathbf{u}_h \in \mathbf{U}_h(\Omega)$ such that,

$$\forall \mathbf{v}_h \in \mathbf{U}_h(\Omega), \qquad a_h(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h). \tag{2.2}$$

3. Error analysis

We define piecewise Sobolev (semi)-norms $\|\cdot\|_{1,h}$ and energy like norms $\|\cdot\|_{E_h(\Omega)}$ for $\mathbf{u} \in \mathbf{H}_h(\Omega)$,

$$\|\mathbf{u}\|_{1,h}^2 = \sum_{T \in \mathcal{T}_h} \|\mathbf{u}\|_{1,T}^2, \qquad |\mathbf{u}|_{1,h}^2 = \sum_{T \in \mathcal{T}_h} |\mathbf{u}|_{1,T}^2,$$

$$\|\mathbf{u}\|_{E_h}^2 := \sum_{T \in \mathcal{T}_h} \left(\|\boldsymbol{\epsilon}(\mathbf{u})\|_{0,T}^2 + \|\sqrt{\lambda} \mathrm{div}\mathbf{u}\|_{0,T}^2 + \sum_{F \in \partial T \cap \mathcal{E}_h} \frac{1}{|F|^{\frac{1}{2}}} \|[\mathbf{P}\mathbf{u}]_F\|_{0,F}^2 \right).$$

It is easy to see that $\sqrt{a_h(\cdot, \cdot)}$ and $\|\cdot\|_{E_h}$ are equivalent, i.e., there exists some $C > 0$ such that

$$\frac{1}{C} \|\mathbf{u}\|_{E_h}^2 \le a_h(\mathbf{u}, \mathbf{u}) \le C \|\mathbf{u}\|_{E_h}^2, \tag{3.1}$$

for all $\mathbf{u} \in \mathbf{H}_h(\Omega)$. We define interpolation operator $\mathbf{I}_h : \mathbf{u} \in [H^2(T)]^3 \to \mathbf{U}_h(T)$ by

$$\int_{F_i} (\mathbf{I}_h \mathbf{u})_1 = \int_{F_i} \mathbf{u}_1, \qquad \int_{F_i} (\mathbf{I}_h \mathbf{u})_2 = \int_{F_i} \mathbf{u}_2, \qquad (\mathbf{I}_h \mathbf{u})_3(A_i) = \mathbf{u}_3(A_i),$$

where F_i, $(i = 1, 2, 3, 4)$ and A_i, $(i = 1, 2, 3, 4)$ are faces and nodes of T respectively. We extend the definition of $\mathbf{I}_h$ for $\mathbf{u} \in [H^2(\Omega)]^3$ by $(\mathbf{I}_h \mathbf{u})|_T = \mathbf{I}_h(\mathbf{u}|_T)$ for each $T \in \mathcal{T}_h$. The following interpolation property are obvious

$$\|\mathbf{u} - I_h \mathbf{u}\|_{L^2(\Omega)} + h \|\mathbf{u} - I_h \mathbf{u}\|_{1,h} \le C h^2 \|\mathbf{u}\|_{H^2(\Omega)}. \tag{3.2}$$

Also, we will prove the following interpolation property with respect to $\|\cdot\|_{E_h}$.

Proposition 3.1. *There exists $C > 0$ independent of λ such that*

$$\|\mathbf{u} - I_h\mathbf{u}\|_{E_h} \leq Ch\left(\|\mathbf{u}\|_{H^2(\Omega)} + \lambda\|\text{div}\mathbf{u}\|_{H^1(\Omega)}\right) \tag{3.3}$$

for all $\mathbf{u} \in [H^2(\Omega)]^3$.

For the proof, we follow the lines of Lemma 4.3 in [13]. We simplify the notation as $X(T) = [H^2(T)]^3$ for $T \in \mathcal{T}_h$. For any $\mathbf{u} = (u_1, u_2, u_3) \in X(T)$, we define norms:

$$\|\mathbf{u}\|^2_{X(T)} := \|\mathbf{u}\|^2_{1,T} + |\mathbf{u}|^2_{2,T} + \|\sqrt{\lambda}\text{div}\mathbf{u}\|^2_{1,T},$$
$$|\!|\!|\mathbf{u}|\!|\!|^2_{2,T} := |\mathbf{u}|^2_{2,T} + |\sqrt{\lambda}\text{div}\mathbf{u}|^2_{1,T} + \sum_{i=1}^{4} |\overline{u_1}|_{F_i}|^2 + \sum_{i=1}^{4} |\overline{u_2}|_{F_i}|^2 + \sum_{i=1}^{4} |u_3(A_i)|^2,$$

where F_i's $(i = 1, 2, 3, 4)$ and A_i's $(i = 1, 2, 3, 4)$ are faces and nodes of T, and $\overline{u_j}|_{F_i}$ are the average of u_j's $(j = 1, \cdots, 4)$ over F_i, $(i = 1, \cdots, 4)$. First, we need the following Lemma.

Lemma 3.1. $|\!|\!|\mathbf{u}|\!|\!|^2_{2,T}$ *is a norm on the space $X(T)$ and is equivalent to $\|\cdot\|_{X(T)}$.*

Proof. To show that $|\!|\!|\cdot|\!|\!|_{2,T}$ is a norm, assume that $|\!|\!|\mathbf{u}|\!|\!|_{2,T} = 0$. Then, by $|\mathbf{u}|_{2,T} = 0$, $\mathbf{u}$ is linear on T. However, since

$$\sum_{i=1}^{4} |\overline{u_1}|_{F_i}|^2 + \sum_{i=1}^{4} |\overline{u_2}|_{F_i}|^2 + \sum_{i=1}^{4} |u_3(A_i)|^2 = 0,$$

we see that $\mathbf{u} = 0$ on T.

Now we show that $|\!|\!|\cdot|\!|\!|_{2,T}$ and $\|\cdot\|_{X(T)}$ are equivalent. By Sobolev embedding theorem, $X(T)$ is embedded in $C^0(T)$ and there exists some constant $C(T) > 0$ such that for all $u \in X(T)$, we have

$$\sum_{i=1}^{4} |\overline{u_1}|_{F_i}| + \sum_{i=1}^{4} |\overline{u_2}|_{F_i}| + \sum_{i=1}^{4} |u_3(A_i)| \leq 12\|\mathbf{u}\|_{L^\infty(T)} \leq 12C(T)\|\mathbf{u}\|_{X(T)}. \tag{3.4}$$

Thus, we see that

$$|\!|\!|\mathbf{u}|\!|\!|_{2,T} \leq 12C(T)\|\mathbf{u}\|_{X(T)}. \tag{3.5}$$

Now suppose that the converse

$$\|\mathbf{u}\|_{X(T)} \leq C|\!|\!|\mathbf{u}|\!|\!|_{2,T}, \qquad \forall \mathbf{u} \in X(T),$$

fails for any $C > 0$. Then there exists a sequence $\{\mathbf{u}_k\}$ in $X(T)$ with

$$\|\mathbf{u}_k\|_{X(T)} = 1, \qquad |\!|\!|\mathbf{u}_k|\!|\!|_{2,T} \leq \frac{1}{k}, \qquad k = 1, 2, \cdots. \tag{3.6}$$

By Rellich compactness theorem, there exists a subsequence of $\{\mathbf{u}_k\}$ which converges in $[H^1(T)]^3$. Without loss of generality, we can assume that the sequence itself converges. Then $\{\mathbf{u}_k\}$ is Cauchy sequence in $[H^1(T)]^3$. We claim that $\{\mathbf{u}_k\}$ is Cauchy sequence in $X(T)$. By the definition of the norm $|\!|\!|\cdot|\!|\!|_{2,T}$, we have $|\mathbf{u}_k|_{2,T}+|\sqrt{\lambda}\text{div}\mathbf{u}_k|_{1,T} \le |\!|\!|\mathbf{u}_k|\!|\!|_{2,T} \le 1/k$. Combining the fact that $\mathbf{u}_k$ is Cauchy in $[H^1(T)]^3$ we see that,

$$\begin{aligned}\|\mathbf{u}_k-\mathbf{u}_l\|^2_{X(T)} =&\|\mathbf{u}_k-\mathbf{u}_l\|^2_{1,T}+|\mathbf{u}_k-\mathbf{u}_l|^2_{2,T}+\|\sqrt{\lambda}\text{div}(\mathbf{u}_k-\mathbf{u}_l)\|^2_{1,T}\\ \le&\|\mathbf{u}_k-\mathbf{u}_l\|^2_{1,T}+\|\sqrt{\lambda}\text{div}(\mathbf{u}_k-\mathbf{u}_l)\|^2_{0,T}\\ &\quad+|\mathbf{u}_k|^2_{2,T}+|\mathbf{u}_l|^2_{2,T}+|\sqrt{\lambda}\text{div}\mathbf{u}_k|^2_{1,T}+|\sqrt{\lambda}\text{div}\mathbf{u}_l|^2_{1,T}\\ \le&(1+\lambda)\|\mathbf{u}_k-\mathbf{u}_l\|^2_{1,T}+1/k^2+1/l^2\to 0,\end{aligned}$$

as $k,l\to\infty$. Since $X(T)$ is complete space, $\mathbf{u}_k$ converges to some element $\mathbf{u}^*\in X(T)$.

By (3.5) and (3.6), we have

$$|\!|\!|\mathbf{u}^*|\!|\!|_{2,T}\le |\!|\!|u^*-u_k|\!|\!|_{2,T}+|\!|\!|u_k|\!|\!|_{2,T}\le 12C(T)\|u^*-u_k\|_{X(T)}+\frac{1}{k}\to 0,$$

which implies that $\mathbf{u}^*=0$. This is a contradiction, since as a limit of the sequence satisfying $\|\mathbf{u}_k\|_{X(T)}=1$, we must have $\|\mathbf{u}^*\|_{X(T)}=1$. □

Now we show the interpolation error estimate with respect to energy like norm. To show it we need a reference element. Let $\hat{T}$ be a reference element and $F(\hat{\mathbf{x}}) = B\hat{\mathbf{x}}+\mathbf{b}:\hat{T}\to T$ be the usual affine mapping. We let $X(\hat{T})$ be the space of all functions defined by $\hat{\mathbf{u}}:=\mathbf{u}\circ F(\hat{\mathbf{x}})$, $\mathbf{u}\in X(T)$. We need a lemma from [7].

Lemma 3.2. *There exists some $C>0$ independent of $v\in H^m(T)$, such that*

$$|\hat{v}|_{m,\hat{T}}\le C\|B\|^m|det\,B|^{-1/2}|v|_{m,T},\qquad \forall v\in H^m(T), \tag{3.7}$$

and

$$|v|_{m,T}\le C\|B^{-1}\|^m|det\,B|^{1/2}|\hat{v}|_{m,\hat{T}},\qquad \forall \hat{v}\in H^m(\hat{T}). \tag{3.8}$$

Lemma 3.3. *There exists some $C>0$ independent of λ, h and $\mathbf{u}\in X(T)$ such that*

$$\|\mathbf{u}-I_h\mathbf{u}\|_{1,T}+\|\sqrt{\lambda}\text{div}(\mathbf{u}-I_h\mathbf{u})\|_{0,T}\le Ch(|\!|\!|\mathbf{u}|\!|\!|_{2,T}+\lambda\|\text{div}\mathbf{u}\|_{1,T})$$

for all $T\in\mathcal{T}_h$ and $\mathbf{u}\in X(T)$.

Proof. By the definition of $\|\cdot\|_{X(T)}$ ($\hat{T}$ in place of T), we have

$$\|\hat{\mathbf{u}}-I_h\hat{\mathbf{u}}\|_{1,\hat{T}}+\|\sqrt{\lambda}\text{div}(\hat{\mathbf{u}}-I_h\hat{\mathbf{u}})\|_{0,\hat{T}}\le\|\hat{\mathbf{u}}-I_h\hat{\mathbf{u}}\|_{X(\hat{T})}. \tag{3.9}$$

Also, by definition of I_h and $|\!|\!|\cdot|\!|\!|_{2,T}$, we see that

$$|\!|\!|\hat{\mathbf{u}}-I_h\hat{\mathbf{u}}|\!|\!|_{2,\hat{T}}=|\hat{\mathbf{u}}|_{2,\hat{T}}+|\sqrt{\lambda}\text{div}\hat{\mathbf{u}}\;|_{1,\hat{T}}. \tag{3.10}$$

The desired inequality follows by the equivalence of the norms $\|\cdot\|_{X(\hat{T})}$ and $\|\cdot\|_{2,\hat{T}}$ (Lemma 3.1 applied to $\hat{T}$) and the standard scaling argument as follows:

$$\begin{aligned}
&\|\mathbf{u}-I_h\mathbf{u}\|_{1,T}+\|\sqrt{\lambda}\mathrm{div}(\mathbf{u}-I_h\mathbf{u})\|_{0,T}\\
\le&\|\hat{\mathbf{u}}-I_h\hat{\mathbf{u}}\|_{1,\hat{T}}+\|\sqrt{\lambda}\mathrm{div}(\hat{\mathbf{u}}-I_h\hat{\mathbf{u}})\|_{0,\hat{T}} \quad \text{(by (3.8))}\\
\le&\tilde{C}_1h^{1/2}\|\hat{\mathbf{u}}-I_h\hat{\mathbf{u}}\|_{X(\hat{T})} \quad \text{(by (3.9))}\\
=&\tilde{C}_1C(\hat{T})h^{1/2}(|\hat{\mathbf{u}}|_{2,\hat{T}}+|\sqrt{\lambda}\mathrm{div}\hat{\mathbf{u}}\;|_{1,\hat{T}}) \quad \text{(by Lemma 3.1 and (3.10))}\\
\le&\tilde{C}_1C(\hat{T})h^{1/2}\cdot\tilde{C}_2h^{1/2}(\|\mathbf{u}\|_{2,T}+\lambda\|\mathrm{div}\mathbf{u}\|_{1,T}), \quad \text{(by (3.7))}
\end{aligned}$$

where $\tilde{C}_1$ and $\tilde{C}_2$ are constants arising from the scaling between the T and $\hat{T}$. We see that the constants are independent of λ, h and $\mathbf{u}\in X(T)$. □

Now we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. By Lemma 3.3, we see that

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h}\Big(\|\boldsymbol{\epsilon}(\mathbf{u}-I_h\mathbf{u})\|_{0,T}+\|\sqrt{\lambda}\mathrm{div}(\mathbf{u}-I_h\mathbf{u})\|_{0,T}\Big)\\
\le&Ch(\|\mathbf{u}\|_{H^2(\Omega)}+\lambda\|\mathrm{div}\mathbf{u}\|_{H^1(\Omega)}).
\end{aligned}$$

By the trace inequality and (3.2), we have

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h}\sum_{F\in\partial T\cap\mathcal{E}_h}\frac{1}{|F|^{\frac{1}{2}}}\|[\mathbf{P}(\mathbf{u}-I_h\mathbf{u})]_F\|_{0,F}^2\\
\le&Ch^{-2}\sum_{T\in\mathcal{T}_h}(\|\mathbf{u}-I_h\mathbf{u}\|_{0,T}+h\|\mathbf{u}-I_h\mathbf{u}\|_{1,T})^2\\
\le&Ch^2\|\mathbf{u}\|_{H^2(\Omega)}^2.
\end{aligned}$$

Thus, we complete the proof. □

3.1. H^1-error estimate

To show that $a_h(\cdot,\cdot)$ is coercive we need the discrete Poincaré and the discrete Korn's inequality for the space $\mathbf{U}_h(\Omega)$. Since $\mathbf{U}_h(\Omega)$ is a subspace of P_1-nonconforming space (P_n^1,P_n^1,P_n^1), the standard discrete Poincaré inequality for nonconforming space holds (see the inequality (1.5) of [2]):

Lemma 3.4. *There exists a constant $C>0$ such that for any $\mathbf{v}_h\in\mathbf{U}_h(\Omega)$,*

$$C\|\mathbf{v}_h\|_{L^2(\Omega)}^2\le|\mathbf{v}_h|_{1,h}^2.$$

Next, we need the discrete Korn's inequality. Since $\mathbf{U}_h(\Omega)$ is a piecewise H^1-vector fields, we can apply the inequality (1.18) of [3]:

Lemma 3.5. *There exists a constant $C>0$ such that*

$$|\mathbf{v}_h|_{1,h}^2 \leq C \sum_{T\in\mathcal{T}_h} \left(\|\boldsymbol{\epsilon}(\mathbf{v}_h)\|_{0,T}^2 + \|Q(\mathbf{v}_h)\|_{0,T}^2\right) + \sum_{F\in\mathcal{E}_h} \int_F \frac{1}{\text{diam(F)}} [\mathbf{v}_h]_F^2 ds,$$

for all $\mathbf{v}_h \in \mathbf{U}_h(\Omega)$, where

$$Q(\mathbf{v}_h) := \mathbf{v}_h - \frac{1}{|T|}\int_T \mathbf{v}_h dx.$$

Corollary 3.1. *There exists a constant $C>0$ independent of λ, such that when $h>0$ is sufficiently small*

$$\|\mathbf{v}_h\|_{1,h}^2 \leq C\|\mathbf{v}_h\|_{E_h}^2 \tag{3.11}$$

holds for all $\mathbf{v}_h \in \mathbf{U}_h(\Omega)$.

Proof. By Bramble-Hilbert lemma, there exists a constant $C>0$ independent of $\mathbf{v}_h \in \mathbf{U}_h(T)$ such that the following holds:

$$\|Q(\mathbf{v}_h)\|_{0,T}^2 \leq Ch^2|\mathbf{v}_h|_{1,T}^2.$$

Hence, by Lemma 3.5, we have

$$|\mathbf{v}_h|_{1,h}^2 \leq C \sum_{T\in\mathcal{T}_h} \left(\|\boldsymbol{\epsilon}(\mathbf{v}_h)\|_{0,T}^2 + Ch^2|\mathbf{v}_h|_{1,T}^2\right) + \sum_{F\in\mathcal{E}_h} \int_F \frac{1}{\text{diam (F)}} [\mathbf{v}_h]_F^2 ds.$$

However, by the regularity assumption on the partitioning $\mathcal{T}_h$, there exists some $C>0$ such that

$$\sum_{F\in\mathcal{E}_h} \int_F \frac{1}{\text{diam (F)}} [\mathbf{v}_h]_F^2 ds \leq C \sum_{F\in\mathcal{E}_h} \int_F \frac{1}{|F|^{\frac{1}{2}}} [\mathbf{v}_h]_F^2 ds = C \sum_{F\in\mathcal{E}_h} \int_F \frac{1}{|F|^{\frac{1}{2}}} [\mathbf{P}\mathbf{v}_h]_F^2 ds.$$

Note that we used the projection $\mathbf{P}$ onto the first two component to stabilize the bilinear form. By Lemma 3.4 and the definition of $\|\cdot\|_{E_h}$, we see (3.11) holds for all sufficiently small $h>0$. □

Now we estimate the consistency error.

Theorem 3.1. *Let $\mathbf{u}$ and $\mathbf{u}_h$ be the solutions of* (1.1) *and* (2.2)*, respectively. There exists $C>0$ such that*

$$|a_h(\mathbf{u}-\mathbf{u}_h, \mathbf{v}_h)| \leq Ch\left(\|\mathbf{u}\|_{H^2(\Omega)} + \lambda\|\text{div}\mathbf{u}\|_{H^1(\Omega)}\right)\|\mathbf{v}_h\|_{E_h}. \tag{3.12}$$

Proof. By definition of the bilinear form in (2.1) and by $\mathbf{u}_h$ is a solution of (2.2), we have

$$
\begin{aligned}
&|a_h(\mathbf{u}-\mathbf{u}_h,\mathbf{v}_h)| = |a_h(\mathbf{u},\mathbf{v}_h)-(\mathbf{f},\mathbf{v}_h)| \\
=&\Bigg|\sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{\epsilon}(\mathbf{u}):\boldsymbol{\epsilon}(\mathbf{v}_h)+\sum_{T\in\mathcal{T}_h}\int_T \lambda \mathrm{div}\mathbf{u}\cdot \mathrm{div}\mathbf{v}_h \\
&\quad + \sum_{F\in\mathcal{E}_h}\frac{\tau}{|F|^{\frac{1}{2}}}\int_F [\mathbf{Pu}]_F\cdot[\mathbf{Pv}_h]_F+\sum_{T\in\mathcal{T}_h}\int_T \mathrm{div}\boldsymbol{\sigma}(\mathbf{u})\cdot\mathbf{v}_\mathrm{h}\Bigg|.
\end{aligned}
$$

However, by applying integration by parts to the last term and the fact that $[\mathbf{Pu}]_F=0$ on each $F\in\mathcal{E}_h$, we have

$$
\begin{aligned}
&|a_h(\mathbf{u}-\mathbf{u}_h,\mathbf{v}_h)| \\
=&\Bigg|\sum_{T\in\mathcal{T}_h}\int_T \boldsymbol{\epsilon}(\mathbf{u}):\boldsymbol{\epsilon}(\mathbf{v}_h)+\sum_{T\in\mathcal{T}_h}\int_T \lambda \mathrm{div}\mathbf{u}\mathrm{div}\mathbf{v}_h \\
&\quad -\sum_{T\in\mathcal{T}_h}\left(\int_T \boldsymbol{\epsilon}(\mathbf{u}):\boldsymbol{\epsilon}(\mathbf{v}_h)+\int_T \lambda\mathrm{div}\mathbf{u}\mathrm{div}\mathbf{v}_h-\int_{\partial T}(\boldsymbol{\sigma}(\mathbf{u})\cdot\mathbf{n})\cdot\mathbf{v}_h\right)\Bigg| \\
=&\left|\sum_{T\in\mathcal{T}_h}\sum_{F\in\partial T}\int_F(\boldsymbol{\sigma}(\mathbf{u})\cdot\mathbf{n})\cdot[\mathbf{v}_h]_F\right|.
\end{aligned}
$$

Since $\int_F [\mathbf{v}_h]_F = 0$, we see that

$$
\begin{aligned}
&\left|\sum_{T\in\mathcal{T}_h}\sum_{F\in\partial T}\int_F(\boldsymbol{\sigma}(\mathbf{u})\cdot\mathbf{n})\cdot[\mathbf{v}_h]_F\right| \\
=&\left|\sum_{T\in\mathcal{T}_h}\sum_{F\in\partial T}\int_F\left(\boldsymbol{\sigma}(\mathbf{u})\cdot\mathbf{n}-\overline{\boldsymbol{\sigma}(\mathbf{u})\cdot\mathbf{n}}\right)\cdot[\mathbf{v}_h]_F\right|,
\end{aligned}
$$

where $\overline{\boldsymbol{\sigma}(\mathbf{u})\cdot\mathbf{n}}$ is average of $\boldsymbol{\sigma}(\mathbf{u})\cdot\mathbf{n}$ over F.

We can use a similar technique as in [8] (Lemma 3, page 41) to show that

$$
\begin{aligned}
&\left|\sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{E}_h}\int_F\left(\boldsymbol{\sigma}(\mathbf{u})\cdot\mathbf{n}-\overline{\boldsymbol{\sigma}(\mathbf{u})\cdot\mathbf{n}}\right)\cdot[\mathbf{v}_h]_F\right| \\
\le& Ch\sum_{T\in\mathcal{T}_h}\|\boldsymbol{\sigma}(\mathbf{u})\|_{1,T}|\mathbf{v}_h|_{1,T} \\
\le& Ch\left(\|\mathbf{u}\|_{H^2(\Omega)}+\lambda\|\mathrm{div}\mathbf{u}\|_{H^1(\Omega)}\right)|\mathbf{v}_h|_{1,h}.
\end{aligned}
$$

We have the conclusion by Corollary 3.1. □

Now we prove an optimal error estimate in energy like norm.

Theorem 3.2. *Let* $\mathbf{u}$ *and* $\mathbf{u}_h$ *be the solutions of* (1.1) *and* (2.2) *respectively. Then we have*

$$\|\mathbf{u}-\mathbf{u}_h\|_{E_h} \leq Ch\left(\|\mathbf{u}\|_{H^2(\Omega)}+\lambda\|\mathrm{div}\mathbf{u}\|_{H^1(\Omega)}\right) \leq Ch\|\mathbf{f}\|_{L^2(\Omega)},$$

uniformly as $\lambda \to \infty$.

Proof. By the triangle inequality, we have

$$\|\mathbf{u}-\mathbf{u}_h\|_{E_h} \leq \|\mathbf{u}_h - I_h\mathbf{u}\|_{E_h} + \|\mathbf{u}-I_h\mathbf{u}\|_{E_h}. \tag{3.13}$$

By the norm equivalence (3.1) and by (3.12), we have

$$\begin{aligned}
&\frac{1}{C}\|\mathbf{u}_h - I_h\mathbf{u}\|^2_{E_h} \leq a_h(\mathbf{u}_h - I_h\mathbf{u}, \mathbf{u}_h - I_h\mathbf{u}) \\
=&a_h(\mathbf{u}-I_h\mathbf{u}, \mathbf{u}_h - I_h\mathbf{u}) + a_h(\mathbf{u}_h - \mathbf{u}, \mathbf{u}_h - I_h\mathbf{u}) \\
\leq&C\|\mathbf{u}-I_h\mathbf{u}\|_{E_h}\|\mathbf{u}_h - I_h\mathbf{u}\|_{E_h} + Ch\left(\|\mathbf{u}\|_{H^2(\Omega)}+\lambda\|\mathrm{div}\mathbf{u}\|_{H^1(\Omega)}\right)\|\mathbf{u}_h - I_h\mathbf{u}\|_{E_h}.
\end{aligned} \tag{3.14}$$

By dividing the both sides by $\|\mathbf{u}_h - I_h\mathbf{u}\|_{E_h}$, we have

$$\|\mathbf{u}_h - I_h\mathbf{u}\|_{E_h} \leq C\|\mathbf{u}-I_h\mathbf{u}\|_{E_h} + Ch\left(\|\mathbf{u}\|_{H^2(\Omega)}+\lambda\|\mathrm{div}\mathbf{u}\|_{H^1(\Omega)}\right). \tag{3.15}$$

By combining (3.13), (3.14) and (3.15), we have

$$\|\mathbf{u}-\mathbf{u}_h\|_{E_h} \leq C\|\mathbf{u}-I_h\mathbf{u}\|_{E_h} + Ch\left(\|\mathbf{u}\|_{H^2(\Omega)}+\lambda\|\mathrm{div}\mathbf{u}\|_{H^1(\Omega)}\right).$$

Finally, by the interpolation property (3.3), we have

$$\|\mathbf{u}-\mathbf{u}_h\|_{E_h} \leq Ch\left(\|\mathbf{u}\|_{H^2(\Omega)}+\lambda\|\mathrm{div}\mathbf{u}\|_{H^1(\Omega)}\right).$$

The second inequality follows from (A.1). □

We remark that we can also obtain the optimal L^2-error estimate by the standard duality argument.

4. Numerical results

In this section, we provide some numerical experiments. The domain Ω is $[-0.5,0.5]^3$ in Example 4.1 and Example 4.2 and $[-1,1]^3$ in Example 4.3. Our $\mathcal{T}_h$ is obtained by dividing Ω using uniform cubes of size $h=2^{-k}$, $k=1,\cdots,5$, and each cube is then divided into 6 tetrahedra.

The analytic solutions are known in Example 4.1 and Example 4.2. We present the numerical results with various Poisson ratio including the nearly incompressible case. We set $\mu=1$ without loss of generality. In Example 4.2, we compare our schemes with the P_1-nonconforming methods with stabilizing terms introduced in [10]. In Example 4.3, we simulate the driven cavity problem.

In all examples, we see that the proposed scheme is locking free.

Example 4.1. We choose the exact solution $\mathbf{u} = ((x^2+y^2+z^2-1)(y-z), (x^2+y^2+z^2-1)(z-x), (x^2+y^2+z^2-1)(x-y))$. We tests four cases of increasing values of ν, case 1: $\nu = 0.3$, case 2: 0.49, case 3: 0.499, case 4: 0.4999.

The errors in L^2 and piecewise H^1-norms are given in Table 1 under the parameter setting $\tau = 5$. We observe $\mathcal{O}(h^2)$ in L^2 and $\mathcal{O}(h)$ in H^1-norm for all cases as proved in Section 3. Hence, our methods yield optimal convergence uniformly as $\nu \to 1/2$.

For the comparison's sake, we set $\tau = 0.5$ and report the computation results for $\nu = 0.4999$ in Table 2. We see that the errors are similar to the previous case when $\tau = 5$.

Example 4.2. We choose the exact solution $\mathbf{u} = (x(z-y)\sin(x+y+z), y(x-z)\sin(x+y+z), z(y-x)\sin(x+y+z))$. We only consider the incompressible case $\nu = 0.4999$. We set $\tau = 5$.

Table 1: L^2 and piecewise H^1-norm errors with respect to the different Poisson ratio for Example 4.1 ($\tau = 5$).

$\nu = 0.3$ $1/h$	$\\|\mathbf{u}-\mathbf{u}_h\\|_{L_2(\Omega)}$	order	$\\|\mathbf{u}-\mathbf{u}_h\\|_{1,h}$	order
2	1.024×10^{-1}		8.079×10^{-1}	
4	2.790×10^{-2}	1.875	4.573×10^{-1}	0.821
8	7.148×10^{-3}	1.965	2.358×10^{-1}	0.956
16	1.800×10^{-3}	1.989	1.190×10^{-1}	0.987
32	4.510×10^{-4}	1.997	5.968×10^{-2}	0.995

$\nu = 0.49$ $1/h$	$\\|\mathbf{u}-\mathbf{u}_h\\|_{L_2(\Omega)}$	order	$\\|\mathbf{u}-\mathbf{u}_h\\|_{1,h}$	order
2	1.031×10^{-1}		8.410×10^{-1}	
4	2.979×10^{-2}	1.792	4.777×10^{-1}	0.816
8	7.709×10^{-3}	1.951	2.442×10^{-1}	0.968
16	1.948×10^{-3}	1.985	1.229×10^{-1}	0.991
32	4.882×10^{-4}	1.996	6.158×10^{-2}	0.996

$\nu = 0.499$ $1/h$	$\\|\mathbf{u}-\mathbf{u}_h\\|_{L_2(\Omega)}$	order	$\\|\mathbf{u}-\mathbf{u}_h\\|_{1,h}$	order
2	1.041×10^{-1}		8.546×10^{-1}	
4	3.040×10^{-2}	1.776	4.819×10^{-1}	0.827
8	7.852×10^{-3}	1.953	2.454×10^{-1}	0.974
16	1.982×10^{-3}	1.986	1.233×10^{-1}	0.993
32	4.964×10^{-4}	1.997	6.178×10^{-2}	0.997

$\nu = 0.4999$ $1/h$	$\\|\mathbf{u}-\mathbf{u}_h\\|_{L_2(\Omega)}$	order	$\\|\mathbf{u}-\mathbf{u}_h\\|_{1,h}$	order
2	1.043×10^{-1}		8.569×10^{-1}	
4	3.048×10^{-2}	1.775	4.824×10^{-1}	0.829
8	7.869×10^{-3}	1.953	2.455×10^{-1}	0.974
16	1.986×10^{-3}	1.986	1.234×10^{-1}	0.993
32	4.974×10^{-4}	1.997	6.181×10^{-2}	0.997

Table 2: L^2 and piecewise H^1-norm errors with Poisson ratio 0.4999 for Example 4.1 ($\tau = 0.5$).

$\nu = 0.4999$ $1/h$	$\|\mathbf{u} - \mathbf{u}_h\|_{L_2(\Omega)}$	order	$\|\mathbf{u} - \mathbf{u}_h\|_{1,h}$	order
2	9.964×10^{-2}		7.933×10^{-1}	
4	2.826×10^{-2}	1.818	4.342×10^{-1}	0.870
8	7.508×10^{-3}	1.912	2.213×10^{-1}	0.972
16	1.915×10^{-3}	1.971	1.112×10^{-1}	0.993
32	4.813×10^{-4}	1.992	5.569×10^{-2}	0.998

For easiness of presentation, we denote the scheme (P_n^1, P_n^1, P_c^1) by P_{nnc}^1 and the scheme (P_n^1, P_n^1, P_n^1) by P_{nnn}^1. First we compare the number of degrees of freedom. The degrees of freedom used in P_{nnn}^1 is $36 \cdot 2^{3k} + 18 \cdot 2^{2k}$ where meshsize is $h = 2^{-k}$ while that of P_{nnc}^1 is $25 \cdot 2^{3k} + 15 \cdot 2^{2k} + 3 \cdot 2^k + 1$. For example, degrees of freedom of our scheme is about 69% of P_{nnn}^1 when $h = 2^{-5}$.

As for the absolute errors, we report L^2 and H^1-errors for both schemes in Table 3. We see that both the schemes are locking free for the incompressible case. The errors of P_{nnc}^1 and P_{nnn}^1 are compared in a $\log_2$ scale in Fig. 1. We see that errors of P_{nnc}^1 are relatively higher than P_{nnn}^1.

Finally, we compare the computational complexity of the two schemes. The diagonally preconditioned conjugate gradient method (PCG) was used with the stopping criteria,

$$\frac{\|b - Ax\|}{\|b\|} < 10^{-12},$$

where $Ax = b$ stands for the discretized systems arising from either scheme. We show the PCG iteration numbers and CPU time to reach the stopping criteria for both the discretization in Table 4. We see that the CPU time of the proposed method is about 47% of that of P_{nnn}^1 when $h = 2^{-5}$ while the L^2 error of the P_{nnn}^1 is about 57% of that of the proposed method. Thus, we may believe that our method is comparable to P_{nnn}^1 since there is a reasonable trade-off between the accuracy and the computational costs.

Example 4.3 (Driven cavity)**.** We impose the boundary condition; $\mathbf{u} = (0, 0, 0)$ on $z = -1$ or $x = -1$ or $x = 1$ or $y = -1$ or $y = 1$ and $\mathbf{u} = (1, 0, 0)$ on $z = 1$. We present the velocity fields restricted to plane $y = 0$ for the case of $\nu = 0.49$ and $\nu = 0.4999$. For all cases, we set $\tau = 1$.

We show the numerical results by our scheme in Fig. 2 and the results by P_1-nonconforming methods with stabilizing term [10] in Fig. 3 and the results by a linear element in Fig. 4. We see that both our scheme and P_1-nonconforming methods with stabilizing term [10] give the stable results for compressible ($\nu = 0.49$) and incompressible case ($\nu = 0.4999$). On the other hand, the numerical solutions by linear element show locking phenomena when $\nu = 0.4999$, i.e., the shape of numerical solutions become distorted when $\nu = 0.4999$.

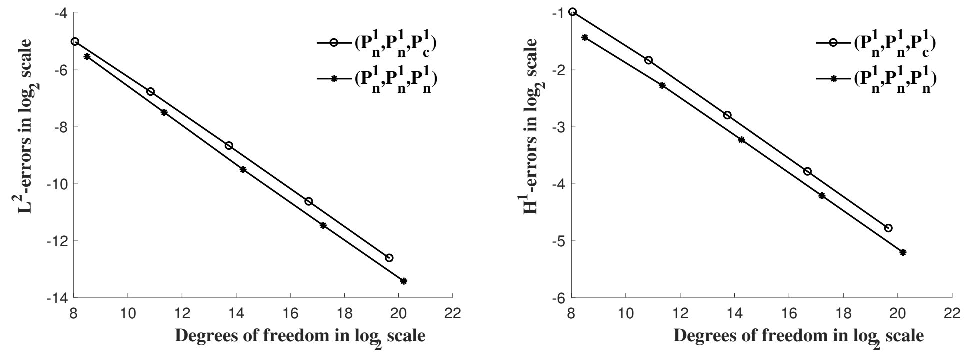

Figure 1: Comparison of L^2 and H^1-errors of (P^1_n, P^1_n, P^1_c) and (P^1_n, P^1_n, P^1_n) in a $\log_2$ scale for Example 4.2 ($\nu = 0.4999$). The x-axis shows the number of degrees of freedom in a $\log_2$ scale.

Table 3: L^2 and piecewise H^1-norm errors for Example 4.2 ($\nu = 0.4999$).

(P^1_n, P^1_n, P^1_c) $1/h$	$\\|\mathbf{u}-\mathbf{u}_h\\|_{L_2(\Omega)}$	order	$\\|\mathbf{u}-\mathbf{u}_h\\|_{1,h}$	order
2	3.026×10^{-2}		4.991×10^{-1}	
4	8.951×10^{-3}	1.757	2.771×10^{-1}	0.849
8	2.408×10^{-3}	1.894	1.422×10^{-1}	0.962
16	6.213×10^{-4}	1.954	7.177×10^{-2}	0.987
32	1.571×10^{-4}	1.983	3.602×10^{-2}	0.995

(P^1_n, P^1_n, P^1_n) $1/h$	$\\|\mathbf{u}-\mathbf{u}_h\\|_{L_2(\Omega)}$	order	$\\|\mathbf{u}-\mathbf{u}_h\\|_{1,h}$	order
2	2.120×10^{-2}		3.672×10^{-1}	
4	5.468×10^{-3}	1.955	2.052×10^{-1}	0.840
8	1.361×10^{-3}	2.006	1.058×10^{-1}	0.955
16	3.503×10^{-4}	1.958	5.362×10^{-2}	0.981
32	9.024×10^{-5}	1.957	2.698×10^{-2}	0.991

Table 4: CPU time for solving the discretized systems arising from (P^1_n, P^1_n, P^1_c) and (P^1_n, P^1_n, P^1_n) for Example 4.2 ($\nu = 0.4999$).

(P^1_n, P^1_n, P^1_c) $1/h$	Iteration	CPU time (sec)	(P^1_n, P^1_n, P^1_n) $1/h$	Iteration	CPU time (sec)
2	255	0.03	2	238	0.05
4	1455	1.18	4	1313	2.22
8	3690	31.21	8	4444	71.49
16	8558	617.84	16	10476	1447.45
32	18161	10804.80	32	21820	24803.51

5. Conclusions

We propose a true low order FEM for 3D elasticity problems. We extend KS scheme for 3D by using two nonconforming components and one conforming component with

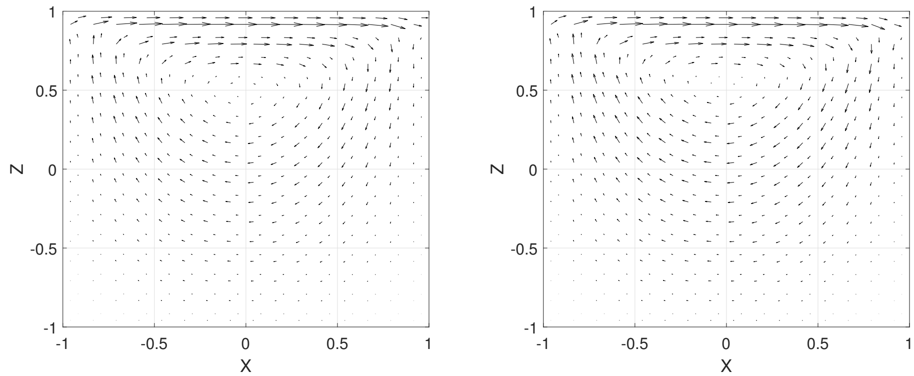

Figure 2: Example 4.3: Solutions by the proposed scheme for the case of $\nu = 0.49$ (left) and $\nu = 0.4999$ (right).

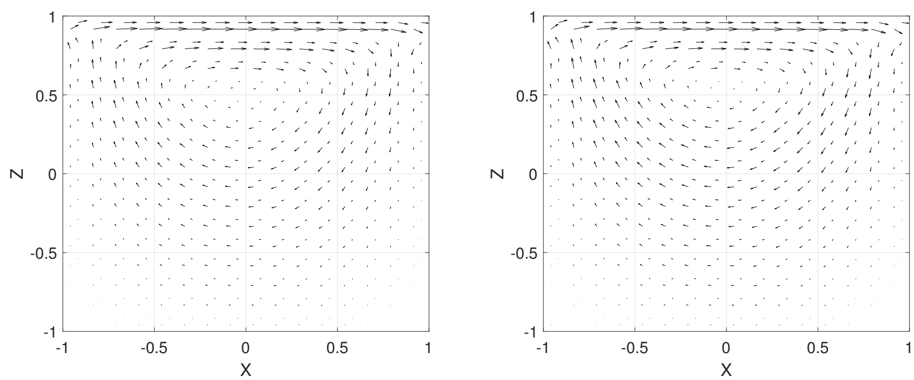

Figure 3: Example 4.3: Solutions by P_1-nonconforming methods with stabilizing term for the case of $\nu = 0.49$ (left) and $\nu = 0.4999$ (right).

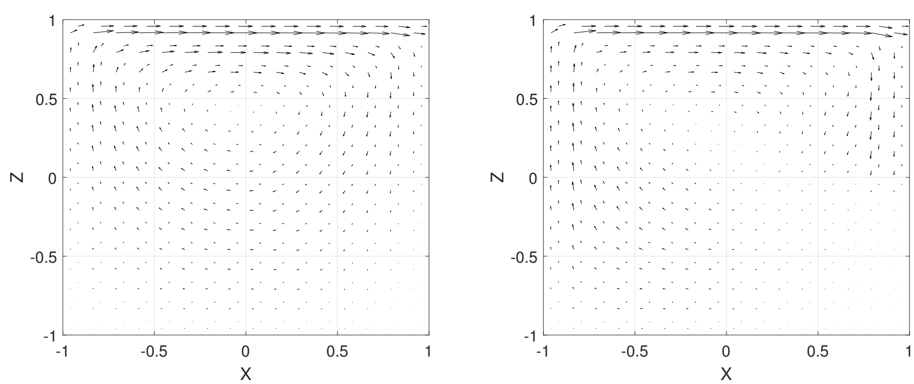

Figure 4: Example 4.3: Solutions by linear element for the case of $\nu = 0.49$ (left) and $\nu = 0.4999$ (right).

stabilizing terms in the bilinear form to satisfy the discrete Korn's inequality. We prove that our methods have optimal convergence rates in L^2 and piecewise H^1-norm even for the nearly incompressible case. The numerical results show that our methods are robust.

Acknowledgements This work is supported by national research foundation of Korea, contract No. 2017R1D1A1B03032765.

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