New curl conforming finite elements on parallelepiped

J. H. Kim · Do Y. Kwak

Abstract In this paper, we introduce new curl conforming elements of higher order \((k \geq 1)\) on parallelepiped. This element has smaller number of degrees of freedom than the well known Nedelec spaces do, however, one can still obtain the same convergence order. To prove error estimate for curl conforming finite element methods, we also define the corresponding divergence conforming elements. Some efficient way of implementing our element is discussed in Remarks 2 and 3.

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1 Introduction

Finite element methods for solving three dimensional electromagnetics problems are considered in this paper. In these methods, vector basis functions play a central role and finite elements are built using these functions. The most useful of elements are curl or divergence conforming element, which have continuous tangential or normal components across adjacent elements, respectively. In general, curl conforming elements
in $H($curl$)$ space are appropriate for discretizations of electric field, while divergence conforming element in $H($div$)$ space are appropriate for discretizations of magnetic field.

Curl conforming elements were introduced by Nedelec [15, 16], and used in various applications [1, 2]. Divergence conforming elements were introduced by Raviart and Thomas [17], Brezzi et al. [5, 6] in two dimensions and by Nedelec [15, 16] and Brezzi et al. [4, 7] in three dimensions. For more thorough understanding of finite element method for Maxwell’s equation see [3, 8, 10–14].

In this paper, we introduce new curl conforming elements of higher order $(k \geq 1)$ on parallelepiped with fewer degrees of freedom than the existing ones. To construct such elements, we eliminate some redundant degrees of freedom from Nedelec element [15] and provide proper degrees of freedom to ensure curl conformity. To prove error estimate for curl conforming finite element methods, we also need divergence conforming elements of higher order $(k \geq 1)$, which satisfy the commutative de Rham diagram [3].

The organization of this paper is as follows: In the next section, we give some preliminaries. In Sects. 3 and 4, we introduce and analyze new curl conforming element and divergence conforming element, respectively. Finally, we prove the error estimates of curl conforming finite element methods in Sect. 5.

2 Preliminaries

For simplicity, we shall restrict ourselves to a regular hexahedron $\Omega$ with edges parallel to the coordinate axis. We assume that the domain $\Omega$ is covered with regular parallelepipeds of maximum diameter $h$ to form a mesh $\mathcal{T}_h$. Let the reference element $\hat{K}$ be a unit cube and let $K$ be any cubic element. Then there is an affine mapping $F_K : \hat{K} \rightarrow K$ such that $F_K(\hat{K}) = K$ and $F_K(\hat{x}) = B_K(\hat{x}) + b_K$, where $B_K$ is a non-singular matrix and $b_K$ is a vector. A scalar function $\hat{p}$ on $\hat{K}$ is transformed to a scalar function $p$ on $K$ by $p \circ F_K = \hat{p}$. However, vector functions must be transformed in a more careful way to conserve special properties. Those are described in the following lemmas, see [7, 13, 14], for example.

**Lemma 1** Suppose that $\hat{w} \in H($div$,$ $\hat{K})$. If $\hat{w}$ and $w$ are related by

$$w \circ F_K = \frac{1}{J_K}B_K\hat{w}, \ J_K = det(B_K),$$

then $w \in H($div$,$ $K)$.

**Lemma 2** Suppose that $\hat{v} \in H($curl$,$ $\hat{K})$. If $\hat{v}$ and $v$ are related by

$$v \circ F_K = (B_K^T)^{-1}\hat{v},$$

then $v \in H($curl$,$ $K)$.

We now recall a few examples. A well known example for $H($div$)$ conforming finite element space is the Raviart-Thomas-Nedelec (RTN$_{[k]}$) space of index $k \geq 0$ which is given by
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\[ Q_{k+1,k,k}(\hat{\mathbf{k}}) \times Q_{k,k+1,k}(\hat{\mathbf{k}}) \times Q_{k,k,k+1}(\hat{\mathbf{k}}), \]

where \( Q_{r,s,t} \) denotes the space of polynomial functions of degree at most \( r, s \) and \( t \) in \( \hat{x}, \hat{y} \) and \( \hat{z} \), respectively. Another example is Brezzi-Douglas-Duran-Fortin (BDDF) space of index \( k \geq 1 \) which is defined by

\[ P_k(\hat{\mathbf{k}}) + \text{Span} \left[ \sum_{i=0}^{k} r_i \text{curl} (0, 0, \hat{x}^{\hat{y}^{k+1} \hat{z}^{k-i}}), \sum_{i=0}^{k} s_i \text{curl} (\hat{x}^{k-i} \hat{y}^{k+1}, 0, 0), \sum_{i=0}^{k} t_i \text{curl} (0, \hat{x}^{k+1} \hat{y}^{k-i} \hat{z}, 0) \right], \]

where \( P_k \) is the space of polynomial of total degree \( k \). On the other hand, a well known \( H(\text{curl}) \) conforming finite element space is the Nedelec space (Nedelec) of index \( k \geq 0 \) which is defined by

\[ Q_{k,k+1,k+1}(\hat{\mathbf{k}}) \times Q_{k+1,k,k+1}(\hat{\mathbf{k}}) \times Q_{k+1,k+1,k}(\hat{\mathbf{k}}). \]

3 Construction of new curl conforming elements

In this section, we introduce new family of \( H(\text{curl}) \) conforming finite element spaces which will be used to discretize the electric field.

Definition 1 We first consider the vectors

\[
\{ \hat{\alpha}_{i,j} \}_{j=1,2}, \quad \hat{\beta}_i (i = 1, 2, 3), \quad \hat{\gamma}_i, \quad \text{and} \quad \hat{\delta}.
\]

(3)

defined by the Table 1. We let \( \hat{\mathbf{V}}(\hat{\mathbf{k}})(k \geq 1) \) be the subspace of \( \text{Ned}_{[k]} \), where the elements in the set \( \{ \hat{\alpha}_{i,j} \}_{j=1,2} \) are replaced by the element \( \hat{\beta}_i \), for \( i = 1, 2, 3 \) and the three elements \( \hat{\gamma}_i \) are replaced by the single element \( \hat{\delta} \).

With abuse of notation, we may write

\[
\hat{\alpha}_{11} + \hat{\alpha}_{12} \Rightarrow \hat{\beta}_1, \quad \hat{\alpha}_{21} + \hat{\alpha}_{22} \Rightarrow \hat{\beta}_2, \quad \hat{\alpha}_{31} + \hat{\alpha}_{32} \Rightarrow \hat{\beta}_3 \quad \text{and} \quad \hat{\gamma}_1 + \hat{\gamma}_2 + \hat{\gamma}_3 \Rightarrow \hat{\delta}.
\]

The dimension of \( \hat{\mathbf{V}}(\hat{\mathbf{k}}) \) is \( 3(k+1)(k+2)^2 - 3(k+1) - 2 = 3k^3 + 15k^2 + 21k + 7 \).

We give an example for \( k = 1 \):

\[
\hat{\mathbf{V}}(\hat{\mathbf{k}}) = \begin{pmatrix} Q_{1,1,1} + Q_{0,2,2} \setminus Q_{0,1,1} \\ Q_{1,1,1} + Q_{2,0,2} \setminus Q_{1,0,1} \\ Q_{1,1,1} + Q_{2,2,0} \setminus Q_{1,1,0} \end{pmatrix} \oplus \begin{pmatrix} \hat{x}^2 \hat{y}^2 \\ \hat{x}^2 \hat{y}^2 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} \hat{x}^2 \hat{y}^2 \\ \hat{x}^2 \hat{y}^2 \\ \hat{x}^2 \hat{y}^2 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ \hat{y}^2 \hat{z}^2 \\ \hat{y}^2 \hat{z}^2 \end{pmatrix} \oplus \begin{pmatrix} \hat{x}^2 \hat{z}^2 \\ \hat{x}^2 \hat{z}^2 \\ \hat{x}^2 \hat{z}^2 \end{pmatrix} \oplus \begin{pmatrix} \hat{x}^2 \hat{y}^2 \hat{z}^2 \\ \hat{x}^2 \hat{y}^2 \hat{z}^2 \\ \hat{x}^2 \hat{y}^2 \hat{z}^2 \end{pmatrix}.
\]
The Definition 3.1. Consider the vectors defined by the Table 2:

Table 1: The vectors \( \hat{\alpha}_{i,j} \) and \( \hat{\gamma}_j \) are replaced by \( \hat{\beta}_i \) and \( \hat{\delta} \), respectively.

<table>
<thead>
<tr>
<th>Vectors in ( \text{Ned}_k )</th>
<th>Replaced vectors in the New space</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\alpha}_{11} = (0, \hat{x}^{k+1}, 0, 0) )</td>
<td>( \hat{\beta}_1 = (0, \hat{x}^{k+1}, 0, 0) )</td>
</tr>
<tr>
<td>( \hat{\alpha}_{12} = (0, \hat{x}^{k+1}, 0, 0) )</td>
<td>( \hat{\beta}_2 = (0, \hat{x}^{k+1}, 0, 0) )</td>
</tr>
<tr>
<td>( \hat{\alpha}_{21} = (0, \hat{x}^{k+1}, 0, 0) )</td>
<td>( \hat{\beta}_3 = (0, \hat{x}^{k+1}, 0, 0) )</td>
</tr>
<tr>
<td>( \hat{\alpha}_{22} = (0, \hat{x}^{k+1}, 0, 0) )</td>
<td>( \hat{\gamma}_1 = (0, \hat{x}^{k+1}, 0, 0) )</td>
</tr>
<tr>
<td>( \hat{\alpha}_{31} = (0, \hat{x}^{k+1}, 0, 0) )</td>
<td>( \hat{\gamma}_2 = (0, \hat{x}^{k+1}, 0, 0) )</td>
</tr>
<tr>
<td>( \hat{\alpha}_{32} = (0, \hat{x}^{k+1}, 0, 0) )</td>
<td>( \hat{\gamma}_3 = (0, \hat{x}^{k+1}, 0, 0) )</td>
</tr>
</tbody>
</table>

Here \( Q_{1,1,1} + Q_{0,2,2} \setminus Q_{0,1,1} \) (the first component of \( \hat{V}(\hat{K}) \)) is interpreted as

\[
\text{Span} \{1, \hat{x}, \hat{y}, \hat{z}, \hat{x}\hat{y}, \hat{x}\hat{z}, \hat{y}\hat{z}, \hat{x}^2, \hat{y}^2, \hat{z}^2, \hat{x}\hat{y}^2, \hat{x}^2\hat{z}, \hat{y}^2\hat{z}, \hat{x}\hat{y}\hat{z}^2 \}.
\]

Similar interpretation applies to the second and third components.

Now we shall define the degrees of freedom. For this purpose, we need two auxiliary spaces. First, we define \( \Phi_k^{\text{curl}}(\hat{x}, \hat{y}) \) to be the subspace of \( Q_{k-1,k}(\hat{x}, \hat{y}) \times Q_{k-1,k}(\hat{x}, \hat{y}) \) where the two elements \((\hat{x}^{k-1}\hat{y}, 0)\) and \((0, \hat{x}^{k-1}\hat{y})\) are replaced by the single element \((\hat{x}^{k-1}\hat{y}, \hat{x}^{k-1}\hat{y})\). To define the second space, we use a replacement rule similar to the Definition 3.1. Consider the vectors defined by the Table 2:

\[
\{\hat{\phi}_{i,j}\}_{j=1,2}, \hat{\psi}_j, \hat{\xi}_j (i = 1, 2, 3), \quad \text{and} \quad \hat{\zeta}.
\]

We define \( \hat{\Phi}_k^{\text{curl}}(\hat{K}) \) to be the subspace of \( Q_{k,k-1,k-1}(\hat{K}) \times Q_{k-1,k-1,k-1}(\hat{K}) \times Q_{k-1,k-1,k-1}(\hat{K}) \) where for \( i = 1, 2, 3 \) the elements \( \{\hat{\phi}_{ij}\}_{j=1,2} \) are replaced by the elements \( \hat{\psi}_i \) and the three elements \( \hat{\xi}_i \) are replaced by the single element \( \hat{\zeta} \). For \( k = 1 \), an element \( \hat{q} \in \hat{\Phi}_1^{\text{curl}}(\hat{K}) \) has the following form

\[
\hat{q} = \begin{pmatrix}
  a + d\hat{x} \\
  b + d\hat{y} \\
  c + d\hat{z}
\end{pmatrix}.
\]

For any \( \hat{u} = (\hat{u}_1, \hat{u}_2, \hat{u}_3) \in \hat{V}(\hat{K}) \), we consider the following degrees of freedom.

\[
\int_{\hat{\gamma}} \hat{u} \cdot \hat{q} d\hat{s}, \quad \hat{q} \in P_k(\hat{e}), \quad \text{for each edge} \ \hat{e} \ \text{with tangential vector} \ \hat{t}.
\]

\[
\int_{\hat{f}} (\hat{u} \times \hat{u}) \cdot \hat{q} d\hat{A}, \quad \hat{q} = (\hat{q}_1, \hat{q}_2) \in \hat{\Phi}_k^{\text{curl}}(\hat{f}), \quad \text{for each face} \ \hat{f}.
\]
Theorem 1 The finite element $\hat{V}(\hat{K})$ defined above is unisolvent and conforming in $H(\text{curl})$.

Proof Since the number of conditions equals the dimension of $\hat{V}(\hat{K})$, it suffices to show that if all the conditions are zero, then $\hat{u} = 0$. First, we consider the face $\hat{\zeta} = 0$. Then the tangential component of $\hat{u}$ on this face is $(\hat{u}_1, \hat{u}_2) = Q_{k+1}(\hat{x}, \hat{y})$.

where two dimensional vector $\hat{q}$ in (7) is understood as imbedded in $\mathbb{R}^3$ and we consider $\Phi_k^{\text{curl}}(\hat{f}) = \Phi_k^{\text{curl}}(\hat{x}, \hat{y})$ for the face $\hat{f}$ in $\hat{x}\hat{y}$-plane etc. Then the number of conditions is $12(k + 1) + 6[2k(k + 1) - 1] + 3k^2(k + 1) - (k - 1) - 2$ which is also the dimension of $\hat{V}(\hat{K})$.

Table 2 Vectors $\{\hat{\phi}_{i,j}\}_{j=1,2}$ are replaced by $\hat{\psi}_i$, and $\hat{\xi}_i$ are replaced by $\hat{\zeta}$.

<table>
<thead>
<tr>
<th>$\hat{\phi}_{11}$</th>
<th>$\hat{\phi}_{12}$</th>
<th>$\hat{\phi}_{21}$</th>
<th>$\hat{\phi}_{22}$</th>
<th>$\hat{\phi}_{31}$</th>
<th>$\hat{\phi}_{32}$</th>
<th>$\hat{\xi}_1$</th>
<th>$\hat{\xi}_2$</th>
<th>$\hat{\xi}_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${(\hat{x}^k \hat{x}^{k-1} \hat{x}^\ell, 0, 0)}_{\ell=0}$</td>
<td>${(0, \hat{x}^{k-1} \hat{y} \hat{x}^\ell, 0)}_{\ell=0}$</td>
<td>${(0, \hat{x} \hat{x}^k \hat{y} \hat{x}^{k-1}, 0)}_{\ell=0}$</td>
<td>${(0, 0, \hat{x}^{k-1} \hat{y} \hat{x}^{k-1})}_{\ell=0}$</td>
<td>${(\hat{x}^k \hat{x} \hat{x}^{k-1} \hat{z}^k, 0, 0)}_{\ell=0}$</td>
<td>${(0, 0, \hat{x} \hat{x}^{k-1} \hat{y} \hat{x}^{k-1})}_{\ell=0}$</td>
<td>${(\hat{x}^k \hat{x} \hat{x}^{k-1} \hat{z}^k, 0, 0)}_{\ell=0}$</td>
<td>${(0, 0, \hat{x} \hat{x}^{k-1} \hat{y} \hat{x}^{k-1})}_{\ell=0}$</td>
<td>${(0, 0, \hat{x} \hat{x}^{k-1} \hat{y} \hat{x}^{k-1})}_{\ell=0}$</td>
</tr>
</tbody>
</table>

where $\int_{\hat{K}} \hat{u} \cdot \hat{q} \, d\hat{x}$, $\hat{q} = (\hat{q}_1, \hat{q}_2, \hat{q}_3) \in \hat{V}_k^{\text{curl}}(\hat{K})$, (8)

where two dimensional vector $\hat{q}$ in (7) is understood as imbedded in $\mathbb{R}^3$ and we consider $\Phi_k^{\text{curl}}(\hat{f}) = \Phi_k^{\text{curl}}(\hat{x}, \hat{y})$ for the face $\hat{f}$ in $\hat{x}\hat{y}$-plane etc. Then the number of conditions is $12(k + 1) + 6[2k(k + 1) - 1] + 3k^2(k + 1) - (k - 1) - 2$ which is also the dimension of $\hat{V}(\hat{K})$. 

The number of conditions equals the dimension of $\hat{V}(\hat{K})$, it suffices to show that if all the conditions are zero, then $\hat{u} = 0$. First, we consider the face $\hat{\zeta} = 0$. Then the tangential component of $\hat{u}$ on this face is $(\hat{u}_1, \hat{u}_2) = Q_{k+1}(\hat{x}, \hat{y})$. On each edge of this face, the tangential component is polynomial of degree $k$. From the degrees of freedom in (6), we see that $\hat{u} \cdot \hat{t} = 0$ on each edge. This implies that on this face, we have

$$\hat{u}_1 = \hat{y}(1 - \hat{y}) \hat{v}_1, \quad \hat{u}_2 = \hat{x}(1 - \hat{x}) \hat{v}_2,$$

where $(\hat{v}_1, \hat{v}_2) \in \Phi_k^{\text{curl}}(\hat{x}, \hat{y})$. Then by choosing $\hat{q}_1 = \hat{v}_2$ and $\hat{q}_2 = -\hat{v}_1$ in the degrees of freedom (7), we see $\hat{v}_1 = \hat{v}_2 = 0$. Hence $\hat{u} \times \hat{n} = 0$ on this face. By the same reason, we see that $\hat{u} \times \hat{n} = 0$ on all faces. This proves the conformity in $H(\text{curl})$ space. And we have
For simplicity, we only show the degrees of freedom for $k = 1$. Left one is Nedelec element: there are two tangential component degrees of freedom per edge, four per face and six interior degrees of freedom. Right one is new element: there are two tangential component degrees of freedom per edge, three per face and four interior degrees of freedom

\[
\hat{u}_1 = \hat{y}(1 - \hat{y})\hat{z}(1 - \hat{z})\hat{w}_1, \\
\hat{u}_2 = \hat{x}(1 - \hat{x})\hat{z}(1 - \hat{z})\hat{w}_2, \\
\hat{u}_3 = \hat{x}(1 - \hat{x})\hat{y}(1 - \hat{y})\hat{w}_3,
\]

where $(\hat{w}_1, \hat{w}_2, \hat{w}_3) \in \hat{\Psi}^{curl}_k(\hat{K})$. Choosing $\hat{q} = (\hat{w}_1, \hat{w}_2, \hat{w}_3)$ in degrees of freedom (8), we know that $\hat{u} = 0$. \hfill \square

For a generic element $K$, we define the finite element space on $K$ as

\[
V_h(K) = \{(B^T_K)^{-1}\hat{v} \circ F^{-1}_K \mid \hat{v} \in \hat{V}(\hat{K})\},
\]

and set

\[
V_h = \{v \in H(\text{curl}, \Omega) \mid v|_K \in V_h(K) \text{ for all } K \in \mathcal{T}_h\}. \tag{9}
\]

Now, we define a projection operator $\hat{R} : H^{k+1}(\hat{K}) \to \hat{V}(\hat{K})$ satisfying

\[
\int_{\hat{e}} (\hat{u} - \hat{R}\hat{u}) \cdot \hat{t} \hat{\eta} d\hat{s}, \quad \hat{\eta} \in P_k(\hat{e}), \text{ for each edge } \hat{e} \text{ with tangential vector } \hat{t},
\]

\[
\int_{\hat{f}} ((\hat{u} - \hat{R}\hat{u}) \times \hat{n}) \cdot \hat{q} d\hat{A}, \quad \hat{q} = (\hat{q}_1, \hat{q}_2) \in \hat{\Phi}^{curl}_k(\hat{f}), \text{ for each face } \hat{f},
\]

\[
\int_{\hat{K}} (\hat{u} - \hat{R}\hat{u}) \cdot \hat{q} d\hat{x}, \quad \hat{q} = (\hat{q}_1, \hat{q}_2, \hat{q}_3) \in \hat{\Psi}^{curl}_k(\hat{K}).
\]

Then we can define the corresponding projection $R_K : H^{k+1}(K) \to P_F\hat{V}(\hat{K})$ for an arbitrary element $K = F(\hat{K})$ via the Piola transform $P_F$ in (2). Finally a global projection operator $R_h : H^{k+1}(\Omega) \to V_h$ is defined piecewise: $(R_h v)|_K = R_K (v|_K)$ (Fig. 1).

Remark 1 Our new element in $H(\text{curl})$ space has $3k^3 + 15k^2 + 21k + 7$ degrees of freedom and $3k + 5$ fewer degrees of freedom than Nedelec finite elements on parallelepipeds. For the case of $k = 1$, see Fig. 1.
Remark 2 The stiffness matrix resulting from our new element has a similar data structure as those of standard Nedelec space. In particular, our matrix can be obtained from collapsing the columns corresponding to the vectors \( \{\hat{\alpha}_{i,j}\}_{j=1,2} \) into those corresponding to \( \hat{\beta}_i \), for \( i = 1, 2, 3 \) and the three columns corresponding to the vectors \( \hat{\gamma}_i \) are collapsed into the single column of \( \hat{\delta} \).

Similarly, the rows corresponding to the test functions \( \{\hat{\phi}_{i,j}\}_{j=1,2} \) are collapsed into those corresponding to \( \hat{\psi}_i \) and the three rows corresponding to \( \hat{\xi}_i \) are collapsed to the column of \( \hat{\zeta} \). Thus the resulting stiffness matrix is smaller than the original system and the elementary operations involved in the above procedure does not increase the condition numbers. Hence the system can be solved by more efficiently than the original one. Furthermore, one can easily implement this new element by modifying the existing code, if any.

4 Construction of new divergence conforming elements

In this section, we introduce new \( H(\text{div}) \) conforming finite element spaces which will be used to discretize the magnetic induction. In order to define these new element spaces for \( k \geq 1 \), we need the following auxiliary spaces and vectors: Let

\[
\hat{W}^i(\hat{\kappa}) = \left( Q_{k+1,k,k}\{\hat{\xi}\}\right) \times \left( Q_{k+1,k,k}\{\hat{\zeta}\}\right) \times \left( Q_{k,k+1,k}\{\hat{\ell}\}\right)
\]

and

\[
\{\hat{\alpha}_{i,j}\}_{j=1,2} \quad \text{and} \quad \hat{b}_i
\]

be defined by the Table 3.

Definition 2 We define \( \hat{W}(\hat{\kappa}) \) to be the subspace of \( \hat{W}^i(\hat{\kappa}) \) where for \( i = 1, 2, 3 \) the vectors \( \{\hat{a}_{ij}\}_{j=1,2} \) are replaced by \( \hat{b}_i \).

Table 3 Vectors \( \{\hat{a}_{ij}\}_{j=1,2} \) are replaced by \( \hat{b}_i \)

<table>
<thead>
<tr>
<th>Vectors in ( \hat{W}^i(\hat{\kappa}) )</th>
<th>Are replaced by</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{a}_{11} = {\hat{\xi}_1, \hat{\xi}<em>2, 0, 0}</em>{\ell=0}^{k-1} )</td>
<td>( \hat{b}_1 = {\hat{\xi}_1, \hat{\xi}<em>2, 0, 0}</em>{\ell=0}^{k-1} )</td>
</tr>
<tr>
<td>( \hat{a}_{12} = {0, \hat{\xi}_1, \hat{\xi}<em>2, 0}</em>{\ell=0}^{k-1} )</td>
<td>( \hat{b}_2 = {0, \hat{\xi}_1, \hat{\xi}<em>2, 0}</em>{\ell=0}^{k-1} )</td>
</tr>
<tr>
<td>( \hat{a}_{21} = {0, \hat{\xi}_1, \hat{\xi}<em>2, 0}</em>{\ell=0}^{k-1} )</td>
<td>( \hat{b}_3 = {0, \hat{\xi}_1, \hat{\xi}<em>2, 0}</em>{\ell=0}^{k-1} )</td>
</tr>
<tr>
<td>( \hat{a}_{22} = {0, \hat{\xi}_1, \hat{\xi}<em>2, 0}</em>{\ell=0}^{k-1} )</td>
<td></td>
</tr>
<tr>
<td>( \hat{a}_{31} = {0, \hat{\xi}_1, \hat{\xi}<em>2, 0}</em>{\ell=0}^{k-1} )</td>
<td></td>
</tr>
<tr>
<td>( \hat{a}_{32} = {0, \hat{\xi}_1, \hat{\xi}<em>2, 0}</em>{\ell=0}^{k-1} )</td>
<td></td>
</tr>
</tbody>
</table>
With abuse of notation, we may write

\[ \hat{a}_{11} - \hat{a}_{12} \Rightarrow \hat{b}_1, \quad \hat{a}_{21} - \hat{a}_{22} \Rightarrow \hat{b}_2, \quad \hat{a}_{31} - \hat{a}_{32} \Rightarrow \hat{b}_3. \]

Then we see the dimension of \( \hat{W}(\hat{K}) \) is \( 3(k + 1)^2(k + 2) - 3(k + 2) - 3k = 3k^3 + 12k^2 + 9k \). For example, if \( k = 1 \) we see

\[
\hat{W}(\hat{K}) = \begin{pmatrix}
P_1(\hat{x}, \hat{y}, \hat{z}) + \hat{x} \hat{P}_1(\hat{x}, \hat{y}, \hat{z}) \\
P_1(\hat{x}, \hat{y}, \hat{z}) + \hat{y} \hat{P}_1(\hat{x}, \hat{y}, \hat{z}) \\
P_1(\hat{x}, \hat{y}, \hat{z}) + \hat{z} \hat{P}_1(\hat{x}, \hat{y}, \hat{z})
\end{pmatrix} \oplus \begin{pmatrix}
\hat{x}^2 \hat{y} \\
\hat{x} \hat{y}^2 \\
0
\end{pmatrix} \oplus \begin{pmatrix}
0 \\
\hat{y}^2 \hat{z} \\
-\hat{x} \hat{z}^2
\end{pmatrix} \oplus \begin{pmatrix}
-\hat{x}^2 \hat{z} \\
\hat{y}^2 \hat{z} \\
0
\end{pmatrix},
\]

where \( \hat{P}_1(\hat{x}, \hat{y}, \hat{z}) \) is the homogenous polynomial of total degree 1.

Now we need to define the degrees of freedom. For this purpose, we first consider the following spaces and vectors: Let

\[
\hat{U}_k(\hat{K}) = \left( Q_{k-1,k,k} \right\{ \hat{x}^k \hat{z}^k P_{k-1}(\hat{x}) \} \right) \times \left( Q_{k,k-1,k} \right\{ \hat{x}^k \hat{z}^k P_{k-1}(\hat{y}) \} \right)
\]

and consider the vectors

\[
\{ \hat{\phi}_{i,j} \}_{j=1,2}, \quad \text{and} \quad \hat{\psi}_i(i = 1, 2, 3)
\]

defined by the Table 4.

We define the new space \( \hat{U}_k^{div}(\hat{K}) \) to be the subspace of \( \hat{U}_k(\hat{K}) \) where for \( i = 1, 2, 3 \) vectors \( \{ \hat{\phi}_{ij} \}_{j=1,2} \) are replaced by \( \hat{\psi}_j \). Note that the definition of \( \hat{U}_k^{div}(\hat{K}) \) is similar to that of \( \hat{W}(\hat{K}) \) except that the highest exponent \( k + 1 \) is replaced by \( k - 1 \). When \( k = 1 \), \( \hat{U}_1^{div}(\hat{K}) \) consists of functions of the form:

\[
\hat{q} = \begin{pmatrix}
P_0 \\
0
\end{pmatrix} \oplus \begin{pmatrix}
\hat{y} \\
-\hat{x}
\end{pmatrix} \oplus \begin{pmatrix}
0 \\
\hat{z}
\end{pmatrix} \oplus \begin{pmatrix}
-\hat{x}
\end{pmatrix}.
\]
For any \( \hat{u} = (\hat{u}_1, \hat{u}_2, \hat{u}_3) \in \hat{W}(\hat{K}) \), we consider the following degrees of freedom.

\[
\int_{\hat{f}} \hat{u} \cdot \hat{n} \hat{q} \, d\hat{A}, \quad \hat{q} \in Q'_{k,k}(\hat{f}), \quad \text{for each face } \hat{f}, \quad (14)
\]

\[
\int_{\hat{K}} \hat{u} \cdot \hat{q} \, d\hat{x}, \quad \hat{q} \in \hat{\Psi}_k^{div}(\hat{K}), \quad (15)
\]

where \( \hat{n} \) is a unit outward normal vector to \( \hat{f} \) and we define \( Q'_{k,k}(\hat{f}) = Q_{k,k}(\hat{x}, \hat{y}) = Q_{k,k}(\hat{x}, \hat{y}) \setminus \{\hat{x}^k \hat{y}^k\} \) for the face \( \hat{f} \) in \( \hat{x} \hat{y} \)-plane etc. Then the number of conditions is \( 6((k + 1)^2 - 1) + 3(k(k + 1)^2 - k) - 3k \) which is also the dimension of \( \hat{W}(\hat{K}) \).

**Theorem 2** The finite element \( \hat{W}(\hat{K}) \) defined above is unisolvent and conforming in \( H(\text{div}) \).

**Proof** Since the number of conditions equals the dimension of \( \hat{W}(\hat{K}) \), it suffices to show that if the above conditions are zero, then \( \hat{u} = 0 \). Since \( \hat{u} \cdot \hat{n} \in Q'_{k,k}(\hat{f}) \), it is clear that (14) implies \( \hat{u} \cdot \hat{n} = 0 \) on each face and this proves the conformity in \( H(\text{div}) \) space. Also, \( \hat{u} = (\hat{u}_1, \hat{u}_2, \hat{u}_3) \) satisfies \( \hat{u}_1 = \hat{x}(1 - \hat{x})\hat{v}_1, \hat{u}_2 = \hat{y}(1 - \hat{y})\hat{v}_2, \) and \( \hat{u}_3 = \hat{z}(1 - \hat{z})\hat{v}_3 \), where \( \hat{v} = (\hat{v}_1, \hat{v}_2, \hat{v}_3) \in \hat{\Psi}_k^{div} \). Thus if we take \( \hat{q} = \hat{v} \) in (15), we obtain \( \hat{u} = 0 \). \( \square \)

For a generic element \( K \), we define the finite element space on \( K \) as

\[
W_h(K) = \left\{ \frac{1}{J_K} B_K \hat{w} \circ F^{-1}_K \mid \hat{w} \in \hat{W}(\hat{K}) \right\},
\]

and set

\[
W_h = \{ w \in H(\text{div}, \Omega) \mid w|_K \in W_h(K) \text{ for all } K \in T_h \}. \quad (16)
\]

For the error estimates, we define an projection operator \( \hat{\Pi} : H^{k+1}(\hat{K}) \to \hat{W}(\hat{K}) \) satisfying

\[
\int_{\hat{f}} (\hat{u} - \hat{\Pi}\hat{u}) \cdot \hat{n} \hat{q} \, d\hat{A}, \quad \hat{q} \in Q'_{k,k}(\hat{f}), \quad \text{for each face } \hat{f}, \quad (17)
\]

\[
\int_{\hat{K}} (\hat{u} - \hat{\Pi}\hat{u}) \cdot \hat{q} \, d\hat{x}, \quad \hat{q} \in \hat{\Psi}_k^{div}(\hat{K}). \quad (18)
\]

Then we can define the corresponding projection \( \Pi_K : H^{k+1}(K) \to P_F \hat{W}(\hat{K}) \) for an arbitrary element \( K = F(K) \) via the Piola transform \( P_F \) in (1). Finally a global projection operator \( \Pi_h : H^{k+1}(\Omega) \to W_h \) is defined piecewise: \( (\Pi_h w)|_K = \Pi_K (w|_K) \). The following theorem provides an error estimate for the projection operator whose proof is now standard.

**Theorem 3** If \( u \in H^{k+1}(\Omega) \) then there is a constant \( C \) independent of \( h \) and \( u \) such that \( \| u - \Pi_h u \|_0 \leq Ch^{k+1} \| u \|_{k+1} \).
Fig. 2. We show the degrees of freedom for \( k = 1 \). Left one is BDDF element: there are only three normal component degrees of freedom per face. Center figure is our new element: there are three normal component degrees of freedom per face and six interior degrees of freedom. Right one is RTN element: there are four normal component degrees of freedom per face and twelve interior degrees of freedom.

Remark 3 Our new element of index \( k \) lies between BDDF and RTN elements of index \( k \) (see Fig. 2). Although BDDF element has less degrees of freedom than our new element, there is no known finite element corresponding to \( H(\text{curl}) \) space in relation to de Rham diagram. Our new element has \( 6(k+1) \) fewer degrees of freedom than the RTN element, and there is a \( H(\text{curl}) \) conforming element (as introduced in Sect. 3) which satisfies de Rham diagram.

Also, implementation issues similar to Remark 2 applies here.

5 Error estimates of curl conforming finite element methods

For the error estimates of the projection operator, we first show the relation between the \( H(\text{curl}) \) space and \( H(\text{div}) \) space.

Lemma 3 Let \( S_1 \) be the subspace of \( \text{Nedl}_{[k]} \) where the three elements \( \hat{\gamma}_i, i = 1, 2, 3 \) are replaced by the single element \( \hat{\delta} \) as in Table 1. Let \( T_1 \) be the subspace of \( \text{RTNl}_{[k]} \) defined by

\[
T_1 = \left( Q_{k+1,k,k} \{ \hat{x}^{k+1} \hat{y}^k \hat{z}^k \} \right) \times \left( Q_{k,k+1,k} \{ \hat{x}^k \hat{y}^{k+1} \hat{z}^k \} \right) \times \left( Q_{k,k,k+1} \{ \hat{x}^k \hat{y}^k \hat{z}^{k+1} \} \right).
\]

Then \( \hat{\nabla} \times S_1 \subset T_1 \).

Proof Let \( \hat{u} \in S_1 \) be an element where the first component of \( \hat{\nabla} \times \hat{u} \) has a term of the form \( \hat{x}^{k+1} \hat{y}^k \hat{z}^k \). Then \( \hat{u} \) must be a linear combination of \( \hat{\gamma}_i, i = 1, 2, 3 \). But by definition of \( S_1 \), \( \hat{u} \) must be a constant multiple of \( \hat{\delta} \). Since \( \hat{\nabla} \times \hat{\delta} = 0 \), this is a contradiction. By the same reason, the second component of \( \hat{\nabla} \times \hat{u} \) cannot have a term of the form \( \hat{x}^k \hat{y}^{k+1} \hat{z}^k \) and the third component of \( \hat{\nabla} \times \hat{u} \) cannot have a term of the form \( \hat{x}^k \hat{y}^k \hat{z}^{k+1} \). This completes the proof. \( \square \)

Lemma 4 Let \( S_2 \) be the subspace of \( \text{Nedl}_{[k]} \) where for \( i = 1, 2, 3 \) and \( \ell = 0 \) the elements in \( \{ \hat{\alpha}_{i,j} \}_{j=1,2} \) are replaced by \( \hat{\beta}_i \) as in defined by Table 1. Let \( T_2 \) be the subspace of \( \text{RTNl}_{[k]} \) defined by

\[
T_2 = \left( Q_{k+1,k,k} \{ \hat{x}^k \hat{z}^k \} \right) \times \left( Q_{k,k+1,k} \{ \hat{x}^k \hat{z}^k \} \right) \times \left( Q_{k,k,k+1} \{ \hat{x}^k \hat{y}^k \} \right).
\]

Then \( \hat{\nabla} \times S_2 \subset T_2 \).
Let \( \hat{\mathbf{u}} \in S_2 \) be an element where the first component of \( \hat{\nabla} \times \hat{\mathbf{u}} \) has a term of the form \( \hat{x}^k \hat{z}^k \). By definition of \( S_2 \), \( \hat{\mathbf{u}} \) must be a linear combination of \( \hat{a}_{2j}, \ j = 1, 2 \) when \( \ell = 0 \). Since \( \hat{\nabla} \times \hat{\beta}_2 = 0 \), this is a contradiction. Similarly, the second component of \( \hat{\nabla} \times \hat{\mathbf{u}} \) cannot have a term of the form \( \hat{x}^k \hat{z}^k \) and the third component of \( \hat{\nabla} \times \hat{\mathbf{u}} \) has a term of the form \( \hat{x}^k \hat{z}^k \). Hence the proof is complete. \( \square \)

**Lemma 5** Let \( S_3 \) be the subspace of \( \text{Ned}_1[k] \) where for \( i = 1, 2, 3 \) and \( \ell = 1, \ldots, k \) the elements in \( \{\hat{a}_{i,j}\}_{j=1,2} \) are replaced by the elements in \( \hat{\beta}_1 \). Let \( T_3 \) be the subspace of \( \text{RTN}_1[k] \) defined by

\[
T_3 = \left( Q_{k+1,k,k} \setminus \{ \hat{x}^{\ell+1} \hat{y}^k \hat{z}^k \} \right) \times \left( Q_{k,k+1,k} \setminus \{ \hat{x}^{\ell} \hat{y}^{\ell+1} \hat{z}^k \} \right) \times \left( Q_{k,k,k+1} \setminus \{ \hat{x}^k \hat{y}^k \hat{z}^{\ell+1} \} \right),
\]

for \( \ell = 0, \ldots, k - 1 \). Then \( \hat{\nabla} \times S_3 \subset T_3 \).

**Proof** Since

\[
\hat{\nabla} \times \beta_1 = \left( -\ell \hat{x}^{k+1} \hat{y}^k \hat{z}^{\ell-1}, \ \ell \hat{x}^k \hat{y}^{k+1} \hat{z}^{\ell-1}, \ 0 \right), \ \ell = 1, \ldots, k \quad (19)
\]

the third component of \( \hat{\nabla} \times \hat{\beta}_1 \) cannot have the form \( \hat{x}^k \hat{y}^{k+1} \hat{z}^{\ell+1} \) for \( \ell = 0, \ldots, k - 1 \). Hence the first component of \( \hat{\nabla} \times \hat{\beta}_2 \) cannot have the form \( \hat{x}^{\ell+1} \hat{y}^k \hat{z}^k \) for \( \ell = 0, \ldots, k - 1 \) from

\[
\hat{\nabla} \times \beta_2 = (0, -\ell \hat{x}^k \hat{y}^k \hat{z}^k, \ (\ell + 1) \hat{x}^k \hat{y}^{k+1} \hat{z}^k), \ \ell = 0, \ldots, k - 1. \quad (20)
\]

Also, the second component of \( \hat{\nabla} \times \hat{\beta}_3 \)

\[
\hat{\nabla} \times \beta_3 = \left( (\ell + 1) \hat{x}^{k+1} \hat{y}^k \hat{z}^k, \ 0, \ -\ell \hat{x}^k \hat{y}^{k+1} \hat{z}^k \right), \ \ell = 0, \ldots, k - 1. \quad (21)
\]

cannot have the form \( \hat{x}^k \hat{y}^{\ell+1} \hat{z}^k \) for \( \ell = 0, \ldots, k - 1 \). Hence the proof is complete. \( \square \)

**Theorem 4** Suppose \( W_h \) is the divergence conforming finite element space in (16) and \( V_h \) is the curl conforming space given by (9). Then \( \nabla \times V_h \subset W_h \).

**Proof** It suffices to show it on the reference element. First, since \( \hat{\nabla}(\hat{\mathbf{K}}) = S_1 \cap S_2 \cap S_3 \) and \( \hat{W}(\hat{\mathbf{K}}) = T_1 \cap T_2 \cap T_3 \), we have \( \hat{\nabla} \times \hat{\nabla}(\hat{\mathbf{K}}) \subset \hat{W}(\hat{\mathbf{K}}) \) by Lemmas 3–5. Now we show \( \hat{\nabla} \times \hat{\nabla}(\hat{\mathbf{K}}) \) is indeed a subspace of \( \hat{W}(\hat{\mathbf{K}}) \). Among the elements of \( \hat{W}(\hat{\mathbf{K}}) \setminus \hat{W}(\hat{\mathbf{K}}) \) (actually those are elements of \( \cup_{i,j} \hat{a}_{ij} \)) that belongs to \( \hat{\nabla} \times \hat{\nabla}(\hat{\mathbf{K}}) \) must be in the form \( \hat{\nabla} \times \hat{\beta}_i, i = 1, 2, 3 \). Then from (19) to (21) of Lemma 5, we have \( \hat{\nabla} \times \hat{\nabla}(\hat{\mathbf{K}}) \subset \hat{W}(\hat{\mathbf{K}}) \). \( \square \)
Now we will prove that the following de Rham diagram commutes for suitable subspaces \( V \subset H(\text{curl} , \Omega) \) and \( W \subset H(\text{div} , \Omega) \):

\[
\begin{array}{ccc}
V & \xrightarrow{\nabla \times} & W \\
\downarrow{R_h} & & \downarrow{\Pi_h} \\
V_h & \xrightarrow{\nabla \times} & W_h
\end{array}
\]

For this purpose we need a lemma:

**Lemma 6** Let \( \hat{\Psi}_k^{\text{div}}(\hat{K}) \) be the space given in Sect. 4 and let \( \hat{\Psi}_k^{\text{curl}}(\hat{K}) \) be the space given in Sect. 3. If \( \hat{q} \in \hat{\Psi}_k^{\text{div}}(\hat{K}) \), then

\[
\hat{\nabla} \times \hat{q} \in \hat{\Psi}_k^{\text{curl}}(\hat{K}), \quad (22)
\]

\[
\hat{q} \in \hat{\Phi}_k^{\text{curl}}(\partial \hat{K}). \quad (23)
\]

**Proof** See appendix.

\[ \square \]

**Theorem 5** If \( u \) is smooth enough such that \( R_h u \) and \( \Pi_h(\nabla \times u) \) are defined, then

\[ \nabla \times R_h u = \Pi_h(\nabla \times u). \]

**Proof** To prove the commuting property, we map to the reference element and show that the degrees of freedom given in (14) and (15) vanish for \( \hat{\nabla} \times \hat{R}_h \hat{u} - \hat{\Pi}(\hat{\nabla} \times \hat{u}) \).

For the face degrees of freedom (14), we let \( \hat{f} \) be a face of \( \hat{K} \) with normal vector \( \hat{n} \) and \( \hat{q} \in Q'_k(\hat{f}) \). Let \( \hat{\nabla} \hat{f} \) be the surface gradient. Using the definition of projection operator \( \hat{\Pi} \), the fact that \( \hat{n} \cdot (\hat{\nabla} \times \hat{v})|_{\hat{f}} = -\hat{\nabla} \hat{f} \cdot (\hat{n} \times \hat{v}) \) and integration by parts (see Monk[14], p49), we have

\[
\int_{\hat{f}} (\hat{\nabla} \times \hat{R}_h \hat{u} - \hat{\Pi}(\hat{\nabla} \times \hat{u})) \cdot \hat{n} \hat{q} \, d\hat{A}
= \int_{\hat{f}} (\hat{\nabla} \times \hat{R}_h \hat{u} - \hat{\nabla} \times \hat{u}) \cdot \hat{n} \hat{q} \, d\hat{A}
= -\int_{\hat{f}} \hat{\nabla} \hat{f} \cdot (\hat{n} \times (\hat{R}_h \hat{u} - \hat{u})) \hat{q} \, d\hat{A}
= \int_{\hat{f}} \hat{n} \times (\hat{R}_h \hat{u} - \hat{u}) \cdot \hat{\nabla} \hat{f} \hat{q} \, d\hat{A} - \int_{\hat{f}} \hat{n} \times (\hat{R}_h \hat{u} - \hat{u}) \cdot \hat{n} \hat{f} \hat{q} \, d\hat{s},
= \int_{\hat{f}} \hat{n} \times (\hat{R}_h \hat{u} - \hat{u}) \cdot \hat{\nabla} \hat{f} \hat{q} \, d\hat{A} - \int_{\hat{f}} (\hat{R}_h \hat{u} - \hat{u}) \cdot (\hat{n} \hat{f} \times \hat{n}) \hat{q} \, d\hat{s},
\]

where \( \hat{n} \hat{f} \) is the unit normal vector to \( \hat{f} \) in the plane containing the face \( \hat{f} \). Since \( \hat{\nabla} \hat{f} \hat{q} \in \hat{\Phi}_k^{\text{curl}}(\hat{f}) \) and \( \hat{q} \in P_k(\hat{e}) \) for each edge \( \hat{e} \) of \( \hat{f} \), the right-hand side of above formula vanishes by the degrees of freedom (7) and (6).
For the volume degrees of freedom (15), we let \( \hat{q} \in \hat{\Psi}_1^{\text{div}}(\hat{K}) \). Using the definition of projection operator \( \hat{\Pi} \) and Green’s theorem of the following form:

\[
\int_{\Omega} \nabla \times u \cdot q \, dx = \int_{\Omega} u \cdot \nabla \times q \, dx + \int_{\partial \Omega} n \times u \cdot q \, dA,
\]

we have

\[
\int_{\hat{K}} (\hat{\nabla} \times \hat{R} \hat{u} - \hat{\Pi}(\hat{\nabla} \times \hat{u})) \cdot \hat{q} \, d\hat{x}
\]

\[
= \int_{\hat{K}} (\hat{\nabla} \times \hat{R} \hat{u} - \hat{\nabla} \times \hat{u}) \cdot \hat{q} \, d\hat{x}
\]

\[
= \int_{\hat{K}} (\hat{R} \hat{u} - \hat{u}) \cdot \hat{\nabla} \times \hat{q} \, d\hat{x} + \int_{\partial \hat{K}} (\hat{n} \times (\hat{R} \hat{u} - \hat{u})) \cdot \hat{q} \, d\hat{A}.
\]

Since \( \hat{\nabla} \times \hat{q} \in \hat{\Psi}_1^{\text{curl}}(\hat{K}) \) and \( \hat{q} \in \hat{\Phi}_1^{\text{curl}}(\partial \hat{K}) \) by Lemma 6, the right-hand side of above formula vanishes by the degrees of freedom (8) and (7).

**Theorem 6** If \( u \in H^{k+1}(\Omega) \) and \( \nabla \times u \in H^{k+1}(\Omega) \) then there is a constant \( C \) independent of \( h \) and \( u \) such that

\[
\| u - Rh u \|_0 \leq C h^{k+1} (\| u \|_{k+1} + \| \nabla \times u \|_{k+1}),
\]

\[
\| \nabla \times (u - Rh u) \|_0 \leq C h^{k+1} \| \nabla \times u \|_{k+1}.
\]

**Proof** By the definition of the norm and interpolation, we have

\[
\| u - Rh u \|^2_0 = \sum_{K \in \mathcal{T}_h} \| u - R_K u \|^2_0.
\]

From Lemma 2, we have

\[
\| u - R_K u \|_0 \leq \frac{1}{J_K} \left| J_K \right| \frac{1}{2} B_K^{-1} \| \hat{u} - \hat{R}_K u \|_0
\]

\[
\leq C h^{\frac{1}{2}} \| \hat{u} - \hat{R}_K u \|_0.
\]

Since \( \hat{R}_K u = R_K \hat{u} \) and \( (I - R_K) \hat{\phi} = 0 \) for all \( \hat{\phi} \in P_k \),

\[
\| \hat{u} - \hat{R}_K u \|_0 = \| \hat{u} - R_K \hat{u} \|_0 = \| (I - R_K)(\hat{u} + \hat{\phi}) \|_0.
\]

Using the Deny-Lions Theorem, we obtain

\[
\inf_{\hat{\phi} \in P_k} \| (I - R_K)(\hat{u} + \hat{\phi}) \|_0 \leq C (|\hat{u}|_{k+1} + |\hat{\nabla} \times \hat{u}|_{k+1}).
\]
Mapping back to the reference element, we obtain

\[ | \hat{\mathbf{u}}_{k+1} | \leq Ch^{k+\frac{1}{2}} | \mathbf{u}_{k+1} | \]

\[ | \hat{\nabla} \times \hat{\mathbf{u}}_{k+1} | \leq Ch^{k+\frac{3}{2}} | \nabla \times \mathbf{u}_{k+1} | . \]

Then

\[ \| \mathbf{u} - R_{K} \mathbf{u} \|_{0} \leq Ch^{k+1} (| \mathbf{u}_{k+1} | + | \nabla \times \mathbf{u}_{k+1} |). \]

Squaring and adding this estimate over all \( K \), we have the desired first estimate. From Theorems 5 and 3, we have the following second estimate

\[ \| \nabla \times (\mathbf{u} - R_{h} \mathbf{u}) \|_{0} \leq \| (I - \Pi_{h}) \nabla \times \mathbf{u} \|_{0} \leq Ch^{k+1} \| \nabla \times \mathbf{u} \|_{k+1} . \]

\[ \square \]

6 Appendix

In this section, we provide the proof of Lemma 6. For example, when \( k = 1 \), we see that \( \hat{\nabla} \times \hat{\Psi}_{1}^{\text{div}}(\hat{K}) \subset \hat{\Psi}_{1}^{\text{curl}}(\hat{K}) \) by direct computation (see 13 and 5).

In general, we first denote by \( (\hat{\Psi}_{k}^{\text{curl}})'(\hat{K}) \) the RTN space of order \( k - 1 \):

\[ (\hat{\Psi}_{k}^{\text{curl}})'(\hat{K}) = Q_{k-1,k-1}(\hat{K}) \times Q_{k-1,k-1}(\hat{K}) \times Q_{k-1,k-1}(\hat{K}), \]

and let \( (\hat{\Psi}_{k}^{\text{curl}})''(\hat{K}) \) be the subspace of \( (\hat{\Psi}_{k}^{\text{curl}})'(\hat{K}) \) where the three elements \( \hat{\xi}_{i}, \ i = 1, 2, 3 \) are replaced by the single element \( \hat{\xi} \) as in the Table 2.

Finally, let \( (\hat{\Psi}_{k}^{\text{curl}})'''(\hat{K}) \) be the subspace of \( (\hat{\Psi}_{k}^{\text{curl}})'(\hat{K}) \) where for \( i = 1, 2, 3 \), the vectors \( \{ \hat{\phi}_{ij} \}_{j=1,2} \) are replaced by \( \hat{\psi}_{i} \) as in the Table 1. Note that

\[ \hat{\Psi}_{k}^{\text{curl}}(\hat{K}) = (\hat{\Psi}_{k}^{\text{curl}})'(\hat{K}) \cap (\hat{\Psi}_{k}^{\text{curl}})''(\hat{K}) \cap (\hat{\Psi}_{k}^{\text{curl}})'''(\hat{K}). \quad (24) \]

Lemma 7 We have

\[ \hat{\nabla} \times \hat{\Psi}_{k}^{\text{div}}(\hat{K}) \subset (\hat{\Psi}_{k}^{\text{curl}})'(\hat{K}). \]

Proof Since \( \hat{\Psi}_{k}^{\text{div}}(\hat{K}) \) is a subspace of Nedelec space of order \( k - 1 \), it is clear that

\[ \hat{\nabla} \times \hat{\Psi}_{k}^{\text{div}}(\hat{K}) \subset \hat{\nabla} \times \text{Ned}_{[k-1]} \subset RTN_{[k-1]} \equiv (\hat{\Psi}_{k}^{\text{curl}})'(\hat{K}). \]

\[ \square \]

Lemma 8 We have

\[ \hat{\nabla} \times \hat{\Psi}_{k}^{\text{div}}(\hat{K}) \subset (\hat{\Psi}_{k}^{\text{curl}})''(\hat{K}). \]
**Proof** Let \( \hat{\mathbf{q}} = (\hat{q}_1, \hat{q}_2, \hat{q}_3) \in \hat{\mathbf{q}}^{\text{div}}(\hat{K}) \) be an element where the first component of \( \hat{\nabla} \times \hat{\mathbf{q}} \) has a term of the form \( \hat{x}^k \hat{y}^{k-1} \hat{z}^{k-1} \). Since the first component of \( \hat{\nabla} \times \hat{\mathbf{q}} \) is \( \frac{\partial q_0}{\partial y} - \frac{\partial q_0}{\partial z} \), \( \hat{q}_3 \) must have a term of the form \( \hat{x}^k \hat{y}^k \hat{z}^{k-1} \) or \( \hat{q}_2 \) has a term of the form \( \hat{x}^k \hat{y}^{k-1} \hat{z}^{k-1} \). From the definition of \( \hat{\mathbf{q}}^{\text{div}}(\hat{K}) \), this is contradiction. By the same reason, the second component of \( \hat{\nabla} \times \hat{\mathbf{q}} \) cannot have a term of the form \( \hat{x}^{k-1} \hat{y}^k \hat{z}^{k-1} \) and the third component of \( \hat{\nabla} \times \hat{\mathbf{q}} \) cannot have a term of the form \( \hat{x}^{k-1} \hat{y}^{k-1} \hat{z}^k \). This completes the proof. \( \square \)

**Lemma 9** We have

\[ \hat{\nabla} \times \hat{\mathbf{q}}^{\text{div}}(\hat{K}) \subset (\hat{\mathbf{q}}^{\text{curl}})^m(\hat{K}). \]

**Proof** It suffices to show \( \hat{\nabla} \times \hat{\mathbf{q}} = (\hat{\mathbf{q}}^{\text{curl}})^m(\hat{K}) \) for \( \hat{\mathbf{q}} \) defined by Table 4. First, we consider the case of \( \ell = 0 \). Let \( \hat{\mathbf{q}} = \hat{\psi}_1 \in \hat{\mathbf{q}}^{\text{div}}(\hat{K}) \). Then it holds clearly, since \( \hat{\nabla} \times \hat{\mathbf{q}} = (0, 0, -2k\hat{x}^{k-1} \hat{y}^{k-1}) \in (\hat{\mathbf{q}}^{\text{curl}})^m(\hat{K}) \). Similar conclusion holds when \( \hat{\mathbf{q}} = \hat{\psi}_2 \) or \( \hat{\mathbf{q}} = \hat{\psi}_3 \).

Now, we consider the case of \( \ell = 1, \ldots, k - 1 \). For \( \hat{\mathbf{q}} = \hat{\psi}_1 \), we can write \( \hat{\nabla} \times \hat{\psi}_1 \) as

\[ \hat{\nabla} \times \hat{\psi}_1 = ((\ell + 1)\hat{x}^k \hat{y}^{k-1} \hat{z}^{k-1}, (\ell + 1)\hat{x}^{k-1} \hat{y}^k \hat{z}^{k-1}, 0) + (0, 0, -2k\hat{x}^{k-1} \hat{y}^{k-1} \hat{z}^k) \equiv \hat{r}_1 + \hat{r}_2 \]

for \( \ell = 1, \ldots, k - 1 \). That is,

\[ \hat{r}_1 = ((\ell + 1)\hat{x}^k \hat{y}^{k-1} \hat{z}^{k-1}, (\ell + 1)\hat{x}^{k-1} \hat{y}^k \hat{z}^{k-1}, 0) \]

\[ \hat{r}_2 = (0, 0, -2k\hat{x}^{k-1} \hat{y}^{k-1} \hat{z}^k) \]

for \( \ell = 0, \ldots, k - 2 \). Then \( \hat{r}_1 \) is exactly a constant multiple of \( \hat{\psi}_1 \) defined by the Table 2 hence belongs to \( (\hat{\mathbf{q}}^{\text{curl}})^m(\hat{K}) \), while \( \hat{r}_2 \) belongs to \( (\hat{\mathbf{q}}^{\text{curl}})^m(\hat{K}) \) by definition. Hence \( \hat{\psi}_1 \in (\hat{\mathbf{q}}^{\text{curl}})^m(\hat{K}) \). By the same reason, \( \hat{\psi}_i \in (\hat{\mathbf{q}}^{\text{curl}})^m(\hat{K}) \) for \( i = 2, 3 \) and the proof is complete. \( \square \)

**Proof of Lemma 6** The relation (22) clearly holds from (24) and Lemmas 7–9.

Now we consider (23). Let \( \hat{f} \) be a face in \( \hat{x} \hat{y} \)-plane, for example. For \( \hat{\mathbf{q}} \in \hat{\mathbf{q}}^{\text{div}}(\hat{K}) \), we see that

\[ \hat{\mathbf{q}}|_{\hat{f}} \in \left( Q_{k-1,k} \{ \hat{x}^{k-1} P_{k-1}(\hat{x}) \} \right) \times \left( Q_{k,k-1} \{ \hat{x}^{k} P_{k-1}(\hat{y}) \} \right) \]

which clearly belongs to \( \hat{\mathbf{q}}^{\text{curl}}(\hat{f}) \) by definition.

**References**


