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Domain decomposition for model heterogeneous anisotropic problem

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SUMMARY

The main focus of this paper is to suggest a domain decomposition method for finite element approximations of elliptic problems with anisotropic coefficients in domains consisting of anisotropic shape rectangles. The theorems on traces of functions from Sobolev spaces play an important role in studying boundary value problems of partial differential equations. These theorems are commonly used for *a priori* estimates of the stability with respect to boundary conditions, and also play very important role in constructing and investigating effective domain decomposition methods. The trace theorem for anisotropic rectangles with anisotropic grids is the main tool in this paper to construct domain decomposition preconditioners. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: domain decomposition method; anisotropic problem; trace theorem

1. INTRODUCTION

The mathematical modelling of transient and diffusion of matter, energy or other substance is usually done by partial differential equations with appropriate initial and boundary conditions. Such problems arise in flow of fluids and gases, for example, for transport and diffusion pollutants in the air or ground water aquifers, etc. These problems are characterized in some practical problems by elliptic boundary value problems with anisotropic coefficients in anisotropic shape subdomains. It is of great practical importance to accurately compute the solution, especially in the layers and around the singular points. The numerical methods for such problems require techniques which are applicable for all scales of the parameters

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involved in the problem. In general, an efficient and accurate method will require new types of preconditioning for the resulting linear system and the use of efficient solution methods and algorithms based on domain decomposition, multilevel decomposition methods. The implementation of these methods with high aspect ratio in the coefficients is a widely open problem. In this paper a preconditioning technique based on the domain decomposition method with non-overlapping subdomains is suggested. Elliptic problems in the domains consisting of rectangles with anisotropic coefficients are considered. The trace theorem for anisotropic rectangles with anisotropic grids is the main tool in this paper to construct domain decomposition preconditioners. Some results were announced in References [1–4]. For the construction of preconditioners we use the so-called nested Chebyshev iterations. The general idea of this trick was suggested by Dyakonov [5] and in domain decomposition/multigrid methods this approach was used in References [6–8, 2]. Also the technique for low rank perturbed systems of linear algebraic equations is used [9]. The domain decomposition method for anisotropic elliptic problems was considered in Reference [10]. Each subproblem can be solved by multigrid method [11–13]. For anisotropic problems, one needs line smoothing. For convergence analysis of multigrid methods for anisotropic problems, we refer to References [14–16]. The remainder of this paper is organized as follows. First, we define an anisotropic problem together with some notations.

Let us consider the boundary value problem

$$\begin{aligned} -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} &= f(x) \quad \text{in } \Omega \\ u(x) &= 0 \quad \text{on } \Gamma = \partial\Omega \end{aligned} \tag{1}$$

Assume that the matrix $\{a_{ij}(x)\}$ is positive definite and the domain Ω is a union of n non-overlapping subdomains which are rectangles, i.e., $\bar{\Omega} = \bigcup_{i=1}^{n} \bar{\Omega}^{(i)}$, $\Omega^{(i)} \cap \Omega^{(j)} = \emptyset$, if $i \neq j$. Let $\Lambda = \bigcup_{i=1}^{k} \partial\Omega^{(i)} \backslash \Gamma$. Let $a(u,v)$ be the bilinear form corresponding to problem (1). Assume that there exist constants α_0 and α_1 and

$$\mathbf{p}(x) = \mathbf{p}^{(i)} = \begin{pmatrix} p_1^{(i)} & 0 \\ 0 & p_2^{(i)} \end{pmatrix}, \quad x \in \Omega^{(i)}$$

piecewise positive constant matrix such that

$$\alpha_0 a(v,v) \leqslant \int_{\Omega} (\mathbf{p}(x) \nabla v \cdot \nabla v)\, \mathrm{d}\Omega \leqslant \alpha_1 a(v,v), \quad \forall v \in H_0^1(\Omega)$$

The coefficient matrix $\mathbf{p}(x)$ can be anisotropic in each subdomain, that is, $p_1^{(i)} \gg p_2^{(i)}$ or $p_1^{(i)} \ll p_2^{(i)}$. Each subdomain where the problem is anisotropic can be transformed into a thin domain so that the problem becomes isotropic. For example, if $\Omega^{(i)} = (0, L_1) \times (0, L_2)$ and $p_1^{(i)} \gg p_2^{(i)}$, then the linear map $\mathbf{x} : R^2 \to R^2$ given by $\mathbf{x}(x_1, x_2) = ((1/\alpha)x_1, x_2)$, $\alpha = (p_1^{(i)}/p_2^{(i)})^{1/2}$, transforms the domain $\Omega^{(i)}$ onto $\tilde{\Omega}^{(i)} = (0, L_1/\alpha) \times (0, L_2)$ and problem (1) is changed to an isotropic problem in $\tilde{\Omega}_i$

$$-p^{(i)} \Delta u(x) = \tilde{f}(x)$$

for some $\tilde{f}(x)$ where $p^{(i)}=\min\{p_1^{(i)}, p_2^{(i)}\}$. In Section 2, we construct a trace semi-norm $|\cdot|_{\tilde{H}^{1/2}(\partial\tilde{\Omega}^{(k)})}$ for each thin domain $\tilde{\Omega}^{(k)}$ which satisfies

$$|\phi|^2_{\tilde{H}^{1/2}(\partial\tilde{\Omega}^{(k)})} \simeq \inf_{u\in H^1(\tilde{\Omega}^{(k)}),\, u|_{\partial\tilde{\Omega}^{(k)}}=\phi} |u|^2_{H^1(\tilde{\Omega}^{(k)})}, \quad \forall\phi\in H^{1/2}(\partial\tilde{\Omega}^{(k)})$$

Here $A\simeq B$ means that there exist c and C such that $cA\leqslant B\leqslant CA$. Here c and C are generic constants independent of $\mathbf{p}$ and the mesh size h to be specified later. For $\phi\in H^{1/2}(\Lambda)$, denote by $\phi^{(k)}$ its restriction on $\partial\Omega^{(k)}$ and by $\tilde{\phi}^{(k)}$ the corresponding function on $\tilde{\partial}\Omega^{(k)}$. Then we define a trace semi-norm $|\cdot|_{\tilde{H}^{1/2}(\Lambda)}$ on the whole interface Λ by

$$|\phi|^2_{\tilde{H}^{1/2}(\Lambda)} = \sum_{k=1}^{n} p^{(k)} |\tilde{\phi}^{(k)}|^2_{\tilde{H}^{1/2}(\partial\tilde{\Omega}^{(k)})}$$

which satisfies

$$|\phi|^2_{\tilde{H}^{1/2}(\Lambda)} \simeq \inf_{u\in H_0^1(\Omega),\, u|_\Lambda=\phi} a(u,u)$$

Let $\Omega_h=\bigcup_{i=1}^n \Omega_h^{(i)}$ be a quasi-uniform triangulation of rectangular grid of Ω of mesh size h and Λ_h be the triangulation of Λ induced by Ω_h. Denote by $H^h(\Omega_h)$ the space of real-valued continuous functions linear on each triangles of the triangulation Ω_h and by W the subspace of $H^h(\Omega_h)$ satisfying the Dirichlet boundary condition. Using the standard finite element method, we have a linear algebraic system

$$Au=f$$

The main purpose of this paper is the construction of the preconditioner B such that

$$c(Bv,v)\leqslant(Av,v)\leqslant C(Bv,v), \quad \forall v\in R^N$$

where N is the dimension of W. Denote

$$Au=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix} u_I \\ u_\Gamma\end{bmatrix}$$

where u_I and u_Γ are the vectors corresponding to interior nodes of each subdomains and nodes on Λ_h, respectively. If we let $S=A_{22}-A_{21}A_{11}^{-1}A_{12}$ be the Schur complement matrix of A then

$$(Su_\Gamma,u_\Gamma)=\inf_{u=u_I+u_\Gamma}(Au,u)$$

Here

$$(Au,u)=a(u,u)\simeq\sum_{k=1}^{n}\int_{\Omega^{(k)}}\mathbf{p}^{(k)}\nabla u\cdot\nabla u\,\mathrm{d}\Omega^{(k)}=\sum_{k=1}^{n}\int_{\tilde{\Omega}^{(k)}} p^{(k)}|\nabla u|^2\,\mathrm{d}\tilde{\Omega}^{(k)}$$

In Section 3, we construct a trace semi-norm $|\cdot|_{\tilde{H}_h^{1/2}(\partial\tilde{\Omega}^{(k)})}$ for each thin domain $\tilde{\Omega}^{(k)}$ which satisfies

$$|\phi|^2_{\tilde{H}_h^{1/2}(\partial\tilde{\Omega}^{(k)})} \simeq \inf_{u\in H^h(\tilde{\Omega}_h^{(k)}),\, u|_{\partial\tilde{\Omega}_h^{(k)}=\phi}} |u|^2_{H^1(\tilde{\Omega}^{(k)})}$$

Then we define a discrete trace semi-norm $|\cdot|_{\tilde{H}_h^{1/2}(\Lambda)}$ on the whole interface Λ by

$$|\phi|^2_{\tilde{H}_h^{1/2}(\Lambda)} = \sum_{k=1}^{n} p^{(k)} |\tilde{\phi}^{(k)}|^2_{\tilde{H}_h^{1/2}(\partial\tilde{\Omega}^{(k)})}$$

which satisfies

$$|\phi|^2_{\tilde{H}_h^{1/2}(\Lambda)} \simeq \inf_{u \in W, u|_{\Lambda_h} = \phi} a(u,u).$$

In Section 4, we construct a preconditioner B for A. Finally in Section 5, we provide numerical results.

2. CONSTRUCTION OF A TRACE NORM ON Λ

Let Ω be any open domain in R^2 and $\Gamma = \partial\Omega$. From trace theory in Sobolev spaces [17], we have the following two lemmas.

Lemma 2.1
There exist c and C such that for any $u \in H^1(\Omega)$, and $\phi \in H^{1/2}(\Gamma)$ with $u(x) = \phi(x)$ on Γ

$$\|\phi\|_{H^{1/2}(\Gamma)} \leqslant c \|u\|_{H^1(\Omega)} \tag{2}$$

holds, and for any given $\phi \in H^{1/2}(\Gamma)$, there exists $u \in H^1(\Omega)$ with $u(x) = \phi(x)$ on Γ such that

$$\|u\|_{H^1(\Omega)} \leqslant C \|\phi\|_{H^{1/2}(\Gamma)} \tag{3}$$

holds.

Moreover, we have similar results for semi-norm.

Lemma 2.2
There exist c and C such that for any $u \in H^1(\Omega)$, and $\phi \in H^{1/2}(\Gamma)$ with $u(x) = \phi(x)$ on Γ

$$|\phi|_{H^{1/2}(\Gamma)} \leqslant c |u|_{H^1(\Omega)} \tag{4}$$

holds, and for any given $\phi \in H^{1/2}(\Gamma)$, there exists $u \in H^1(\Omega)$ with $u(x) = \phi(x)$ on Γ such that

$$|u|_{H^1(\Omega)} \leqslant C |\phi|_{H^{1/2}(\Gamma)} \tag{5}$$

holds.

2.1. Trace theorem for domains with small diameter

We will consider several trace theorems for small domains. Let $0 < \varepsilon < 1$ be arbitrary real number. If we use the change of variables by $x = \varepsilon s$, and $y = \varepsilon t$, we can transform Ω, and Γ into Ω_ε, and Γ_ε, respectively. Let $u_\varepsilon(x, y) = u(x/\varepsilon, y/\varepsilon)$ and ϕ_ε be defined similarly.

We note

$$|\phi|^2_{H^{1/2}(\Gamma)} = \int_\Gamma \int_\Gamma \frac{(\phi(x) - \phi(y))^2}{|x-y|^2} \, \mathrm{d}x \, \mathrm{d}y$$

Lemma 2.3
For any $u \in H^1(\Omega)$, and $\phi \in H^{1/2}(\Gamma)$, we have

$$|\phi_\varepsilon|_{H^{1/2}(\Gamma_\varepsilon)} = |\phi|_{H^{1/2}(\Gamma)} \tag{6}$$

$$|u_\varepsilon|_{H^1(\Omega_\varepsilon)} = |u|_{H^1(\Omega)} \tag{7}$$

Lemma 2.3 implies that the semi-norms $|\cdot|_{H^{1/2}(\Gamma_\varepsilon)}$ and $|\cdot|_{H^1(\Omega_\varepsilon)}$ do not depend on ε but only on the shape of Ω. From Lemmas 2.2 and 2.3, we have the following result.

Lemma 2.4
For any $u_\varepsilon \in H^1(\Omega_\varepsilon)$, and $\phi_\varepsilon \in H^{1/2}(\Gamma_\varepsilon)$ with $u_\varepsilon(x) = \phi_\varepsilon(x)$ on Γ_ε,

$$|\phi_\varepsilon|_{H^{1/2}(\Gamma_\varepsilon)} \leqslant c|u_\varepsilon|_{H^1(\Omega_\varepsilon)} \tag{8}$$

holds, and for any given $\phi_\varepsilon \in H^{1/2}(\Gamma_\varepsilon)$, there exists $u_\varepsilon \in H^1(\Omega_\varepsilon)$ with $u_\varepsilon(x) = \phi_\varepsilon(x)$ on Γ_ε such that

$$|u_\varepsilon|_{H^1(\Omega_\varepsilon)} \leqslant C|\phi_\varepsilon|_{H^{1/2}(\Gamma_\varepsilon)} \tag{9}$$

holds.

It is clear that the above constants c and C are independent of ε. To obtain a trace theory for the full norm on Γ_ε which is independent of ε we first need to define the corresponding trace norm on Γ_ε.

Definition 1
For any $\phi_\varepsilon \in H^{1/2}(\Gamma_\varepsilon)$, define the norm of ϕ_ε by

$$\|\phi_\varepsilon\|^2_{H^{1/2,\varepsilon}(\Gamma_\varepsilon)} = \varepsilon\|\phi_\varepsilon\|^2_{L_2(\Gamma_\varepsilon)} + |\phi_\varepsilon|^2_{H^{1/2}(\Gamma_\varepsilon)}$$

Remark 2.1
Note that $\|\phi_\varepsilon\|_{H^{1/2,\varepsilon}(\Gamma_\varepsilon)}$ and the standard norm $\|\phi_\varepsilon\|_{H^{1/2}(\Gamma_\varepsilon)}$ are not equivalent with constant independent of ε.

Lemma 2.5
There exist c and C which are independent of ε such that for any $u_\varepsilon \in H^1(\Omega_\varepsilon)$, and $\phi_\varepsilon \in H^{1/2,\varepsilon}(\Gamma_\varepsilon)$ with $u_\varepsilon(x) = \phi_\varepsilon(x)$ on Γ_ε

$$\|\phi_\varepsilon\|_{H^{1/2,\varepsilon}(\Gamma_\varepsilon)} \leqslant c\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \tag{10}$$

holds, and for any given $\phi_\varepsilon \in H^{1/2}(\Gamma_\varepsilon)$, there exists $u_\varepsilon \in H^1(\Omega_\varepsilon)$ with $u_\varepsilon(x) = \phi_\varepsilon(x)$ on Γ_ε such that

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leqslant C\|\phi_\varepsilon\|_{H^{1/2,\varepsilon}(\Gamma_\varepsilon)} \tag{11}$$

holds.

Proof
Let $\phi_\varepsilon \in H^{1/2}(\Gamma_\varepsilon)$ be arbitrary. Let $u_\varepsilon \in H^1(\Omega_\varepsilon)$ with $u_\varepsilon(x) = \phi_\varepsilon(x)$ on Γ_ε. Then

$$\begin{aligned}\varepsilon\|\phi_\varepsilon\|^2_{L_2(\Gamma_\varepsilon)} &= \varepsilon \int_{\Gamma_\varepsilon} \phi_\varepsilon(x)^2 \, \mathrm{d}\Gamma_\varepsilon \\ &= \varepsilon^2 \int_\Gamma \phi(s)^2 \, \mathrm{d}\Gamma \\ &\leqslant c\varepsilon^2 \|u\|^2_{H^1(\Omega)}\end{aligned}$$

by Lemma 2.1. Since $0 < \varepsilon < 1$, we have

$$\begin{aligned}\varepsilon^2\|u\|^2_{H^1(\Omega)} &= \|u_\varepsilon\|^2_{L_2(\Omega_\varepsilon)} + \varepsilon^2|u_\varepsilon|^2_{H^1(\Omega_\varepsilon)} \\ &\leqslant \|u_\varepsilon\|^2_{H^1(\Omega_\varepsilon)}\end{aligned}$$

Hence by Lemma 2.4, we have (10). To prove (11), let $u_\varepsilon \in H^1(\Omega_\varepsilon)$ be the extension of ϕ_ε given by Lemma 2.4. The corresponding function u satisfies

$$\|u\|^2_{L_2(\Omega)} \leqslant c(\|\phi\|^2_{L_2(\Gamma)} + |u|^2_{H^1(\Omega)})$$

by Friedrich's inequality. Therefore

$$\begin{aligned}\|u_\varepsilon\|^2_{L_2(\Omega_\varepsilon)} &= \varepsilon^2\|u\|^2_{L_2(\Omega)} \\ &\leqslant c\varepsilon^2(\|\phi\|^2_{L_2(\Gamma)} + |u|^2_{H^1(\Omega)}) \\ &\leqslant c(\varepsilon\|\phi_\varepsilon\|^2_{L_2(\Gamma_\varepsilon)} + |u|^2_{H^1(\Omega_\varepsilon)})\end{aligned}$$

Hence

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leqslant C\|\phi_\varepsilon\|_{H^{1/2,\varepsilon}(\Gamma_\varepsilon)} \qquad \square$$

The following lemma is due to Sobolev [17].

Lemma 2.6
Let $\phi \in H^{1/2}(0,1)$ and let A be any subinterval of $(0,1)$. Then there exists a constant C depending only on the measure of A such that for any $\phi \in H^{1/2}(0,1)$,

$$\|\phi\|^2_{H^{1/2}(0,1)} \leqslant C\left(|\phi|^2_{H^{1/2}(0,1)} + \left(\int_A \phi \, \mathrm{d}x\right)^2\right)$$

From this lemma, we obtain the following corollary.

Corollary 2.1
Let $\phi \in H^{1/2}(0,\varepsilon)$ and let A_ε be any subinterval of $(0,\varepsilon)$ whose measure is of order ε. Assume

that $\int_{A_\varepsilon} \phi \, \mathrm{d}x = 0$. Then there exists a constant C independent of ε such that for any $\phi \in H^{1/2}(0,\varepsilon)$,

$$\frac{1}{\varepsilon}\|\phi\|^2_{L_2(0,\varepsilon)} \leqslant C|\phi|^2_{H^{1/2}(0,\varepsilon)}$$

Lemma 2.7
Let $\phi \in H^{1/2}(0,1)$ be arbitrary. Define $\psi(x)$ in the interval $(0,3)$ such that

$$\psi(x) = \begin{cases} \phi(x), & \text{if } x \in (0,1] \\ (2-x)\phi(2-x), & \text{if } x \in (1,2] \\ 0, & \text{if } x \in (2,3) \end{cases}$$

Then there exists a constant c independent of ϕ and ψ such that

$$\|\psi\|_{H^{1/2}(0,3)} \leqslant c\|\phi\|_{H^{1/2}(0,1)}$$

Moreover, if $\int_0^1 \phi(x)\,\mathrm{d}x = 0$, then there exists a constant C independent of ϕ and ψ such that

$$\|\psi\|^2_{L_2(0,3)} + |\psi|^2_{H^{1/2}(0,3)} \leqslant C|\phi|^2_{H^{1/2}(0,1)}$$

Remark 2.2
The first inequality in the statement of Lemma 2.7 can be proved by interpolation theory between L_2 and H^1 in order to define the $H^{1/2}$ norms. And by Lemma 2.6, we have the following:

$$\int_0^1 \phi^2 \,\mathrm{d}x \leqslant c\left(|\phi|^2_{H^{1/2}(0,1)} + \left(\int_0^1 \phi(x)\,\mathrm{d}x\right)^2\right)$$

Using this, the second inequality can be proved.

The following is obtained just by scaling.

Corollary 2.2
Let $\phi \in H^{1/2}(0,3\varepsilon)$ be arbitrary. Define $\psi(x)$ in the interval $(0,3\varepsilon)$ such that

$$\psi(x) = \begin{cases} \phi(x), & \text{if } x \in (0,\varepsilon] \\ (2\varepsilon-x)\phi(2\varepsilon-x), & \text{if } x \in (\varepsilon,2\varepsilon] \\ 0, & \text{if } x \in (2\varepsilon,3\varepsilon) \end{cases}$$

Then there exists a constant c independent of ϕ, ψ, and ε such that

$$\|\psi\|_{H^{1/2,\varepsilon}(0,3\varepsilon)} \leqslant c\|\phi\|_{H^{1/2,\varepsilon}(0,\varepsilon)} \tag{12}$$

Moreover, if $\int_0^\varepsilon \phi(x)\,\mathrm{d}x = 0$, then there exists a constant C independent of ϕ, ψ, and ε such that

$$\frac{1}{\varepsilon}\|\psi\|^2_{L_2(0,3\varepsilon)} + |\psi|^2_{H^{1/2}(0,3\varepsilon)} \leqslant C|\phi|^2_{H^{1/2}(0,\varepsilon)}$$

2.2. Trace theorem for thin domain problem for continuous case

Let Ω be a rectangular domain $\Omega=(0,H)\times(0,L)$ with a boundary Γ. Assume that this domain is thin, that is, $H\ll L$. Define $[x]$ as the largest integer which is less than or equal to x. Denote $k=[L/H]$ and $\tilde{H}=L/k$. Let $\sigma_0=\tau_0=(0,H)\times\{0\}$, $\sigma_{k+1}=\tau_{k+1}=(0,H)\times\{L\}$, and for each $i=1,\dots,k$, let

$$\sigma_i=\{0\}\times((i-1)\tilde{H},i\tilde{H}) \quad \text{and} \quad \tau_i=\{H\}\times((i-1)\tilde{H},i\tilde{H})$$

For each $i=0,1,\dots,k$, let l_i and τ_i be the connected open subset of Γ such that

$$\bar{l}_i=\bar{\sigma}_i\cup\bar{\sigma}_{i+1} \quad \text{and} \quad \bar{r}_i=\bar{\tau}_i\cup\bar{\tau}_{i+1}$$

Definition 2
Let A and B be any subset of Γ. For any $\phi\in H^{1/2}(\Gamma)$, define

$$I_{A,B}(\phi)=\int_A\int_B\frac{(\phi(x)-\phi(y))^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y$$

Definition 3
For any $\phi\in H^{1/2}(\Gamma)$, define

$$\|\phi\|^2_{\tilde{L}_2(\Gamma)}=\sum_{i=0}^{k}H(\|\phi\|^2_{L_2(l_i)}+\|\phi\|^2_{L_2(r_i)})$$

$$|\phi|^2_{\tilde{H}^{1/2}(\Gamma)}=\sum_{i=0}^{k}(I_{l_i,l_i}(\phi)+I_{r_i,r_i}(\phi)+I_{l_i,r_i}(\phi))$$

$$\|\phi\|^2_{\tilde{H}^{1/2}(\Gamma)}=\|\phi\|^2_{\tilde{L}_2(\Gamma)}+|\phi|^2_{\tilde{H}^{1/2}(\Gamma)}$$

Theorem 2.1
There exist constants c and C independent of H such that for any $u\in H^1(\Omega)$, with $u(x)=\phi(x)$, on Γ,

$$\|\phi\|^2_{\tilde{H}^{1/2}(\Gamma)}\leqslant c\|u\|^2_{H^1(\Omega)}$$

and for all $\phi\in H^{1/2}(\Gamma)$, there exists $u\in H^1(\Omega)$ with $u(x)=\phi(x)$, on Γ such that

$$\|u\|^2_{H^1(\Omega)}\leqslant C\|\phi\|^2_{\tilde{H}^{1/2}(\Gamma)}$$

Proof
Let $\tilde{\Omega}_i=(0,H)\times((i-1)\tilde{H},i\tilde{H})$, $i=1,2,\dots,k$, and $\Omega_0=\tilde{\Omega}_1,\Omega_k=\tilde{\Omega}_k$, and Ω_i be the overlapping subdomains of Ω such that

$$\bar{\Omega}_i=\bar{\tilde{\Omega}}_i\cup\bar{\tilde{\Omega}}_{i+1}, \quad i=1,2,\dots,k-1$$

For any given $u\in H^1(\Omega)$, we have

$$\|u\|^2_{H^1(\Omega)}\geqslant\frac{1}{2}\sum_{i=0}^{k}\|u\|^2_{H^1(\Omega_i)}$$

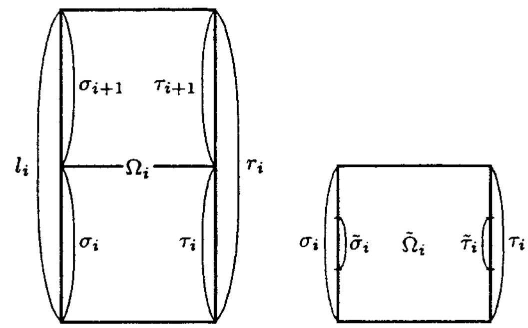

Figure 1. Subdivision of thin domains.

By Lemma 2.5, we have

$$\|u\|^2_{H^1(\Omega_i)} \geqslant c(H\|\phi\|^2_{L_2(\partial\Omega_i)} + I_{\partial\Omega_i,\partial\Omega_i}(\phi))$$

Hence

$$\begin{aligned}\|u\|^2_{H^1(\Omega_i)} &\geqslant \frac{1}{2}\sum_{i=0}^{k}\|u\|^2_{H^1(\Omega_i)} \\ &\geqslant c\sum_{i=0}^{k}(H\|\phi\|^2_{L_2(\partial\Omega_i)} + I_{\partial\Omega_i,\partial\Omega_i}(\phi)) \\ &\geqslant c\|\phi\|^2_{\tilde{H}^{1/2}(\Gamma)}\end{aligned}$$

This completes the first part. For $i=0,1,\ldots,k+1$, let $\tilde{\sigma}_i$ be the open interval contained in σ_i satisfying that the length of $\tilde{\sigma}_i$ is of order H and the distance to the end points of σ_i is also of order H. Similarly define $\tilde{\tau}_i$ (see Figure 1). For example, we set $\tilde{\sigma}_0=\tilde{\tau}_0=(\frac{1}{3}H,\frac{2}{3}H)\times\{0\}$, $\tilde{\sigma}_{k+1}=\tilde{\tau}_{k+1}=(\frac{1}{3}H,\frac{2}{3}H)\times\{L\}$, and for each $i=1,\ldots,k$,

$$\tilde{\sigma}_i=\{0\}\times((i-\tfrac{2}{3})\tilde{H},(i-\tfrac{1}{3})\tilde{H}) \quad\text{and}\quad \tilde{\tau}_i=\{H\}\times((i-\tfrac{2}{3})\tilde{H},(i-\tfrac{1}{3})\tilde{H})$$

Given $\phi\in H^{1/2}(\Gamma)$, let us define a piecewise linear function $\phi^H\in H^{1/2}(\Gamma)$ which has a constant value on $\tilde{\sigma}_i,\ \tilde{\tau}_i,\ i=0,1,\ldots,k+1$

$$\phi^H|_{\tilde{\sigma}_i}=\alpha_i=\frac{3}{\tilde{H}}\int_{\tilde{\sigma}_i}\phi(s)\,\mathrm{d}s$$

$$\phi^H|_{\tilde{\tau}_i}=\beta_i=\frac{3}{\tilde{H}}\int_{\tilde{\tau}_i}\phi(s)\,\mathrm{d}s$$

Between $\tilde{\sigma}_i$ and $\tilde{\sigma}_{i+1}$ ($\tilde{\tau}_i$ and $\tilde{\tau}_{i+1}$) we extend the function ϕ^H as a linear function. To define function ϕ^H on $\partial\tilde{\Omega}_i$, we define a linear extension between points $(0,i\tilde{H})$ and $(H,i\tilde{H})$. So we have a function ϕ^H on $\partial\tilde{\Omega}_i$, $i=1,\dots,k$. By Corollary 2.2, it can be shown that

$$\sum_{i=1}^{k}(H\|\phi^H\|^2_{L_2(\partial\tilde{\Omega}_i)} + I_{\partial\tilde{\Omega}_i,\partial\tilde{\Omega}_i}(\phi^H)) \leqslant c\|\phi\|^2_{\tilde{H}^{1/2}(\Gamma)} \tag{13}$$

By Lemma 2.5, we can extend ϕ^H to u^H on each $\tilde{\Omega}_i$, $i=1,\dots,k$, such that

$$u^H(x) = \phi^H(x), \quad x \in \partial\tilde{\Omega}_i$$

and

$$\|u^H\|^2_{H^1(\tilde{\Omega}_i)} \leqslant c(H\|\phi^H\|^2_{L_2(\partial\tilde{\Omega}_i)} + I_{\partial\tilde{\Omega}_i,\partial\tilde{\Omega}_i}(\phi^H))$$

Summing these,

$$\|u^H\|^2_{H^1(\Omega)} \leqslant c\|\phi\|^2_{\tilde{H}^{1/2}(\Gamma)}$$

Let $\psi(x) = \phi(x) - \phi^H(x)$. Then for $i = 0, 1, \dots, k-1$,

$$\int_{\tilde{\sigma}_i} \psi(s)\,\mathrm{d}s = \int_{\tilde{\tau}_i} \psi(s)\,\mathrm{d}s = 0$$

By Corollary 2.2, we can represent the function $\psi(x)$ in the following form:

$$\psi(x) = \sum_{i=1}^{k} \psi_{i,l}(x) + \psi_{i,r}(x)$$

$$\psi_{i,l}(x) = 0, \quad x \notin l_i$$

$$\psi_{i,r}(x) = 0, \quad x \notin r_i, \;\; i = 1, 2, \dots, k$$

and the following estimate is valid

$$\sum_{i=1}^{k} \|\psi_{i,l}\|^2_{\tilde{H}^{1/2}(\Gamma)} + \|\psi_{i,r}\|^2_{\tilde{H}^{1/2}(\Gamma)} \leqslant C\|\psi\|^2_{\tilde{H}^{1/2}(\Gamma)}$$

By Lemma 2.5, we can extend the functions $\psi_{i,l}, \psi_{i,r}$ inside Ω, i.e. there exist $u_{i,l}, u_{i,r} \in H^1(\Omega)$ such that

$$u_{i,l}(x) = \psi_{i,l}(x), \quad x \in \Gamma$$

$$u_{i,l}(x) = 0, \quad x \notin \Omega_i$$

and

$$\|u_{i,l}\|^2_{H^1(\Omega_i)} \leqslant C\|\psi_{i,l}\|^2_{\tilde{H}^{1/2}(\Gamma)}$$

The same inequality holds for $u_{i,r}$ with

$$u_{i,r}(x) = \psi_{i,l}(x), \quad x \in \Gamma$$
$$u_{i,r}(x) = 0, \quad x \notin \Omega_i$$

Setting

$$u = u^H + \sum_{i=1}^{k} u_{i,l} + u_{i,r}$$

and summing the estimates obtained, we prove the second part of theorem. □

With the same technique, we have the following theorem.

Theorem 2.2
There exist c and C independent of H such that for all $\phi \in H^{1/2}(\Gamma)$,

$$c(S\phi, \phi) \leqslant |\phi|^2_{\tilde{H}^{1/2}(\Gamma)} \leqslant C(S\phi, \phi)$$

where

$$(S\phi, \phi) = \inf_{u \in H^1(\Omega), u|_\Gamma = \phi} |u|^2_{H^1(\Omega)}$$

3. CONSTRUCTION OF A TRACE NORM ON Λ_h

3.1. Trace theorem for anisotropic grid in FEM case

Assume that Ω is any rectangle. Let $\Gamma = \partial\Omega$. Let I_1, I_2, I_3, and I_4 be the bottom, the right, the top, and the left side of Γ, respectively. Let Ω_h be the triangulation of Ω with grid sizes h_1 and h_2 in x and y directions, respectively. From here, the generic constants c and C are independent of h_1 and h_2. Let $H^{1,h}$ be the finite element space consisting of continuous piecewise linear functions and $H^{1/2,h}$ be the trace space of $H^{1,h}$. For any subset T of Γ, define

$$H^{1,h}_T = \{u \in H^{1,h} \mid u = 0 \text{ on } T\}$$
$$H^{1/2,h}_T = \{\phi \in H^{1/2,h} \mid \phi = 0 \text{ on } T\}$$

Let n_1 and n_2 be the number of nodes in x and y direction, respectively. In this subsection, we denote by A the generic matrix corresponding to the Laplacian with various boundary condition. Let us define a norm and a seminorm on trace.

Definition 4
For each $\phi \in H^{1/2,h}$, define

$$|\phi|^2_{H^{1,h}(\Gamma)} = h_1(|\phi|^2_{H^1(I_2)} + |\phi|^2_{H^1(I_4)}) + h_2(|\phi|^2_{H^1(I_1)} + |\phi|^2_{H^1(I_3)})$$
$$\|\phi\|^2_{H^{1/2,h}(\Gamma)} = \|\phi\|^2_{H^{1/2}(\Gamma)} + |\phi|^2_{H^{1,h}(\Gamma)}$$
$$|\phi|^2_{H^{1/2,h}(\Gamma)} = |\phi|^2_{H^{1/2}(\Gamma)} + |\phi|^2_{H^{1,h}(\Gamma)}$$

Remark 3.1
Note that $\|\phi\|_{H^{1/2,h}(\Gamma)}$ and the standard norm $\|\phi\|_{H^{1/2}(\Gamma)}$ are not equivalent with constant independent of h_1 and h_2.

Definition 5
Define S by

$$(S\phi,\phi)=\inf_{u\in H^{1,h},u|_\Gamma=\phi}\|u\|^2_{H^1(\Omega)}$$

Lemma 3.1
If $\phi=0$ on $I_2\cup I_3\cup I_4$, then there exist c and C such that

$$c(S\phi,\phi)\leqslant\|\phi\|^2_{H^{1/2}(\Gamma)}+h_2|\phi|^2_{H^1(I_1)}\leqslant C(S\phi,\phi)$$

Proof
Let $T=I_2\cup I_3\cup I_4$. For any $u\in H^{1,h}_T$, we have

$$|u|^2_{H^1(\Omega)}=(Au,u)$$

Let $\sigma=(h_2/h_1)^2$, and

$$A_0=\begin{bmatrix}2 & -1 & & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2\end{bmatrix}$$

Then

$$A=\sigma^{-1/2}\begin{bmatrix}\sigma A_0+2I & -I & & & \\ -I & \sigma A_0+2I & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & \sigma A_0+2I & -I \\ & & & -I & \frac{1}{2}\sigma A_0+I\end{bmatrix}$$

A can be also expressed as

$$Au=\sigma^{-1/2}\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

where u_2 is the vector corresponding to the nodes on I_1. Let $\Sigma=\sigma^{-1/2}(A_{22}-A_{21}A_{11}^{-1}A_{12})$. It is clear that A_0 is symmetric and positive definite. Let λ be an arbitrary eigenvalue of A_0 and ξ is the corresponding eigenvector. If there exist y_i's such that

$$A_{11}\begin{bmatrix}y_1\\ \vdots\\ y_{m-1}\\ y_m\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ 0\\ \xi\end{bmatrix}$$

then

$$A_{21}A_{11}^{-1}\begin{bmatrix}0\\ \vdots\\ 0\\ \xi\end{bmatrix}=-y_m$$

Assume that $y_i=\alpha_i\xi$, for $i=1,2,\ldots,m$. Then we obtain

$$A_{11}\begin{bmatrix}\alpha_1\xi\\ \vdots\\ \alpha_{m-1}\xi\\ \alpha_m\xi\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ 0\\ \xi\end{bmatrix}$$

that is,

$$\begin{bmatrix}(\sigma\lambda+2)I & -I & & & \\ -I & (\sigma\lambda+2)I & -I & & \\ & & \ddots & & \\ & & -I & (\sigma\lambda+2)I & -I\\ & & & -I & (\sigma\lambda+2)I\end{bmatrix}\begin{bmatrix}\alpha_1\xi\\ \alpha_2\xi\\ \vdots\\ \alpha_{m-1}\xi\\ \alpha_m\xi\end{bmatrix}=\begin{bmatrix}0\\ 0\\ \vdots\\ 0\\ \xi\end{bmatrix}$$

or

$$\begin{bmatrix}\sigma\lambda+2 & -1 & & & \\ -I & \sigma\lambda+2 & -1 & & \\ & & \ddots & & \\ & & -I & \sigma\lambda+2 & -1\\ & & & -1 & \sigma\lambda+2\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_{m-1}\\ \alpha_m\end{bmatrix}=\begin{bmatrix}0\\ 0\\ \vdots\\ 0\\ 1\end{bmatrix}$$

Let $\beta=1+\frac{1}{2}\sigma\lambda$. Since $A_{21}A_{11}^{-1}A_{12}\xi=y_m=\alpha_m\xi, \Sigma\xi=\sigma^{-1/2}(\beta-\alpha_m)\xi$. Now define a sequence $d_i,\ i=0,1,\ldots,m$ by

$$d_0=1$$
$$d_1=2\beta$$
$$\vdots$$
$$d_j=2\beta d_{j-1}-d_{j-2},\quad j=2,3,\ldots,m$$

then we see that $\alpha_i = d_{i-1}/d_m$, for $i=1,2,\dots,m$. It turns out $d_j = U_j(\beta)$, where U_j is the Chebyshev polynomial of the second kind of degree j. If we let $\mu(\Sigma)$ be the eigenvalue of Σ corresponding to the eigenvector ξ, then

$$\mu(\Sigma) = \sigma^{-1/2}\left(\beta - \frac{d_{m-1}}{d_m}\right) = \sigma^{-1/2}\left(\beta - \frac{U_{m-1}(\beta)}{U_m(\beta)}\right)$$

Let us note the formula $\beta U_m(\beta) - U_{m-1}(\beta) = T_{m+1}(\beta)$, where $T_i,\ i=1,2,\dots$ is the Chebyshev polynomial of the first kind of degree i. Then

$$\mu(\Sigma) = \sigma^{-1/2}\frac{T_{m+1}(\beta)}{U_m(\beta)} = \sigma^{-1/2}\sqrt{\beta^2-1}\times\frac{(\beta+\sqrt{\beta^2-1})^{m+1} + (\beta+\sqrt{\beta^2-1})^{-m-1}}{(\beta+\sqrt{\beta^2-1})^{m+1} - (\beta+\sqrt{\beta^2-1})^{-m-1}}$$

It is easy to show that

$$1 \leqslant \frac{(\beta+\sqrt{\beta^2-1})^{m+1} + (\beta+\sqrt{\beta^2-1})^{-m-1}}{(\beta+\sqrt{\beta^2-1})^{m+1} - (\beta+\sqrt{\beta^2-1})^{-m-1}} \leqslant C$$

Since

$$\frac{1}{2}\left(\sqrt{\sigma\lambda} + \frac{\sigma\lambda}{2}\right) \leqslant \sqrt{\sigma\lambda + \frac{\sigma^2\lambda^2}{4}} \leqslant \sqrt{\sigma\lambda} + \frac{\sigma\lambda}{2}$$

we have

$$c(\sqrt{\lambda} + \sqrt{\sigma}\lambda) \leqslant \mu(\Sigma) \leqslant C(\sqrt{\lambda} + \sqrt{\sigma}\lambda)$$

Since this holds for any eigenvalue of A_0, we obtain

$$c\left(\left(A_0^{1/2} + \frac{h_2}{h_1}A_0\right)\phi, \phi\right) \leqslant (\Sigma\phi, \phi) \leqslant C\left(\left(A_0^{1/2} + \frac{h_2}{h_1}A_0\right)\phi, \phi\right), \quad \forall \phi \in R^{n_i-2}$$

Since $(h_1 I\phi, \phi) \simeq \|\phi\|^2_{L_2(I_1)} = \|\phi\|^2_{L_2(\Gamma)}$ and $((1/h_1)A_0\phi, \phi) \simeq |\phi|^2_{H^1(I_1)} = |\phi|^2_{H^1(\Gamma)}$, by interpolation we have $(A_0^{1/2}\phi, \phi) \simeq |\phi|^2_{H^{1/2}(\Gamma)}$.

Observe that $|\phi|^2_{H^{1/2}(\Gamma)} \simeq |\phi|^2_{H^{1/2}(\Gamma)}$ when $\phi = 0$ on T. Hence

$$c(\|\phi\|^2_{H^{1/2}(\Gamma)} + h_2|\phi|^2_{H^1(I_1)}) \leqslant (\Sigma\phi, \phi) \leqslant C(\|\phi\|^2_{H^{1/2}(\Gamma)} + h_2|\phi|^2_{H^1(I_1)})$$

Observe that for any $w \in H_T^{1,h}$, $\|w\|_{H^1(\Omega)} \simeq |w|_{H^1(\Omega)}$. Hence we have the following estimation

$$c(S\phi, \phi) \leqslant \|\phi\|^2_{H^{1/2}(\Gamma)} + h_2|\phi|^2_{H^1(I_1)} \leqslant C(S\phi, \phi) \qquad \square$$

Lemma 3.2
Let u be the discrete harmonic function satisfying $\partial u/\partial n = 0$ on $I_2 \cup I_4$ and $u = 0$ on I_4 and $u = \phi$ on I_1. Then

$$c(S\phi, \phi) \leqslant \|\phi\|^2_{H^{1/2}(\Gamma)} + h_2|\phi|^2_{H^1(I_1)} \leqslant C(S\phi, \phi)$$

Proof
Let $\sigma=(h_2/h_1)^2$ and

$$A_0=\begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}$$

$$J=\begin{bmatrix} \frac{1}{2} & 0 & & & \\ 0 & 1 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & 1 & 0 \\ & & & 0 & \frac{1}{2} \end{bmatrix}$$

Then

$$A=\sigma^{-1/2}\begin{bmatrix} \sigma A_0+2J & -J & & & \\ -J & \sigma A_0+2J & -J & & \\ & \ddots & \ddots & \ddots & \\ & & -J & \sigma A_0+2J & -J \\ & & & -J & \frac{1}{2}\sigma A_0+J \end{bmatrix}$$

To estimate the Schur complement matrix of A, let us define a matrix B which is equivalent to A.

$$B=\sigma^{-1/2}\begin{bmatrix} \sigma A_0+2I & -I & & & \\ -I & \sigma A_0+2I & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & \sigma A_0+2I & -I \\ & & & -I & \frac{1}{2}\sigma A_0+I \end{bmatrix}$$

The proof of the equivalence of A and B is simple and omitted. Note that if A and B are equivalent then the corresponding Schur complement matrices are also equivalent. The rest of the proof is quite similar to that of Lemma 3.1 except that A_0 is only positive semi-definite, that is, it has 0 eigenvalue. If λ is non-zero eigenvalue of A_0 and ξ be the corresponding eigenvector then by using the same technique, we have

$$(\Sigma_B\xi,\xi)\simeq\left(A_0+\frac{h_2}{h_1}\xi,\xi\right)\simeq|\xi|^2_{H^{1/2}(\Gamma)}+h_2|\xi|^2_{H^1(I_1)} \tag{14}$$

Let $\lambda=0$. It is obvious that the corresponding eigenvector ξ is just a constant vector. If we let $\mu(\Sigma_B)$ be the eigenvalue corresponding to ξ then it is easy to obtain that $\mu(\Sigma_B)=$

$h_1/(h_2+1)\simeq h_1$, so

$$(\Sigma_B\xi,\xi)\simeq(h_1 I\xi,\xi)\simeq\|\xi\|^2_{L_2(\Gamma)} \tag{15}$$

Combining (14) and (15), we have

$$c(S\phi,\phi)\leqslant\|\phi\|^2_{H^{1/2}(\Gamma)}+h_2|\phi|^2_{H^1(I_1)}\leqslant C(S\phi,\phi) \qquad \square$$

Theorem 3.1

There exist c and C such that

$$c(S\phi,\phi)\leqslant\|\phi\|^2_{H^{1/2,h}(\Gamma)}\leqslant C(S\phi,\phi),\quad \forall\phi\in H^{1/2,h}(\Gamma)$$

Proof

Let $\phi_i=\phi|_{I_i}$ for $i=1,2,3,4$. Let u_1 and u_2 be the discrete harmonic functions satisfying $\partial u_1/\partial n=\partial u_2/\partial n=0$ on $I_2\cup I_4$ and $u_1=0$ on I_3 and $u_1=\phi_1$ on I_1 and $u_2=\phi_3$ on I_3 and $u_2=0$ on I_1. By Lemma 3.2, we have

$$\|u_1\|^2_{H^1(\Omega)}\simeq\|\phi_1\|^2_{H^{1/2}(\Gamma)}+h_2|\phi|^2_{H^1(I_1)}$$

$$\|u_2\|^2_{H^1(\Omega)}\simeq\|\phi_3\|^2_{H^{1/2}(\Gamma)}+h_2|\phi|^2_{H^1(I_3)}$$

For $j=2,4$, let $\psi_j=\phi_j-u_1|_{I_j}-u_2|_{I_j}$. Let u_3 be the discrete harmonic function satisfying $u_3=\psi_2$ on I_2 and vanishing elsewhere and u_4 be the discrete harmonic function satisfying $u_4=\psi_4$ on I_4 and vanishing elsewhere. Then by Lemma 3.1,

$$\begin{aligned}\|u_3\|^2_{H^1(\Omega)}&\simeq\|\psi_2\|^2_{H^{1/2}(\Gamma)}+h_1|\psi_2|^2_{H^1(I_2)}\\&\leqslant c(\|\phi_2\|^2_{H^{1/2}(\Gamma)}+\|u_1|_{I_2}\|^2_{H^{1/2}(\Gamma)}+\|u_2|_{I_2}\|^2_{H^{1/2}(\Gamma)})\\&\quad+c(h_1|\phi_2|^2_{H^1(I_2)}+h_1|u_1|_{I_2}|^2_{H^1(I_2)}+h_1|u_2|_{I_2}|^2_{H^1(I_2)})\end{aligned}$$

By Lemma 3.2, u_i for $i=1,2$, satisfies

$$\|u_i|_{I_j}\|^2_{H^{1/2}(\Gamma)}\leqslant c\|u_i\|^2_{H^1(\Omega)},\quad \text{for } j=2,4$$

And it is easy to show that

$$h_1|u_i|_{I_j}|^2_{H^1(I_j)}\leqslant c\|u_i\|^2_{H^1(\Omega)}$$

Hence by Lemma 3.2

$$\begin{aligned}\|u_3\|^2_{H^1(\Omega)}&\leqslant c(\|\phi_2\|^2_{H^{1/2}(\Gamma)}+\|\phi_1\|^2_{H^{1/2}(\Gamma)}+\|\phi_2\|^2_{H^{1/2}(\Gamma)})\\&\quad+c(h_1|\phi_2|^2_{H^1(I_2)}+h_2|\phi_1|^2_{H^1(I_1)}+h_2|\phi_3|^2_{H^1(I_3)})\end{aligned}$$

Similar estimation holds for u_4. Now let $u=\sum_{i=1}^{4} u_i$, then $u|_\Gamma=\phi$. It is clear that

$$\begin{aligned}\|u\|^2_{H^1(\Omega)} &\leqslant \sum_{i=1}^{4}\|u_i\|^2_{H^1(\Omega)}\\ &\leqslant C\|\phi\|^2_{H^{1/2,h}(\Gamma)}\end{aligned}$$

Now consider the opposite inequality. Let u be any function in $H^h(\Omega)$ satisfying $u|_\Gamma=\phi$. By trace theorem,

$$\|\phi\|_{H^{1/2}(\Gamma)}\leqslant C\|u\|_{H^1(\Omega)}$$

Let $z_i,\ i=0,1,\ldots,m$ be the nodes on I_2. Then

$$\begin{aligned}h_1|\phi|^2_{H^1(I_2)} &\leqslant C\sum_{i=0}^{m-1}\frac{h_1}{h_2}(\phi(z_{i+1})-\phi(z_i))^2\\ &= C\sum_{i=0}^{m-1}h_1h_2\left(\frac{u(z_{i+1})-u(z_i)}{h_2}\right)^2\\ &\leqslant C\|u\|_{H^1(\Omega)}\end{aligned}$$

Similar inequalities hold for I_1, I_3 and I_4. From above inequalities, we have

$$\|\phi\|_{H^{1/2,h}(\Gamma)}\leqslant C\inf_{w\in H^h(\Omega),w|_\Gamma=\phi}\|w\|_{H^1(\Omega)}$$

This completes the proof. □

Definition 6
Define $\tilde{S}$ by

$$(\tilde{S}\phi,\phi)=\inf_{u\in H^1(\Omega),u|_\Gamma=\phi}|u|^2_{H^1(\Omega)}$$

Theorem 3.2
There exist c and C such that

$$c(\tilde{S}\phi,\phi)\leqslant|\phi|^2_{H^{1/2,h}(\Gamma)}\leqslant C(\tilde{S}\phi,\phi)$$

Proof
Consider the second inequality. Let u be such that $u|_\Gamma=\phi$. Decompose $u=u_0+u_1$, where u_0 is constant and u_1 satisfies

$$\int_\Omega u_1\,\mathrm{d}\Omega=0$$

Let $\phi_0 = u_0|_\Gamma$ and $\phi_1 = u_1|_\Gamma$. Then $\phi = \phi_0 + \phi_1$ and

$$
\begin{aligned}
|u|^2_{H^1(\Omega)} = |u_1|^2_{H^1(\Omega)} &\geqslant c\|u_1\|^2_{H^1(\Omega)} \\
&\geqslant c\|\phi_1\|^2_{H^{1/2,h}(\Gamma)} \geqslant c|\phi_1|^2_{H^{1/2,h}(\Gamma)} = c|\phi|^2_{H^{1/2,h}(\Gamma)}
\end{aligned}
$$

By taking the infimum, we obtain the second inequality. Now consider the first inequality. Decompose $\phi = \phi_0 + \phi_1$, where ϕ_0 is constant and ϕ_1 satisfies

$$\int_\Gamma \phi_1 \, \mathrm{d}\Gamma = 0$$

Let u_0 be the trivial extension of ϕ_0. By Theorem 3.1 we can choose u_1 such that

$$\frac{1}{2}\|u_1\|^2_{H^1(\Omega)} \leqslant (S\phi_1, \phi_1) \leqslant c\|\phi_1\|^2_{H^{1/2,h}(\Gamma)} \tag{16}$$

Let $u = u_0 + u_1$. Then by the definition of $\tilde{S}$, (16) and Poincaré inequality, we see

$$
\begin{aligned}
(\tilde{S}\phi, \phi) &\leqslant C|u|^2_{H^1(\Omega)} = C|u_1|^2_{H^1(\Omega)} \\
&\leqslant C\|u_1\|^2_{H^1(\Omega)} \leqslant C\|\phi_1\|^2_{H^{1/2,h}(\Gamma)} \leqslant C|\phi_1|^2_{H^{1/2,h}(\Gamma)} = C|\phi|^2_{H^{1/2,h}(\Gamma)}
\end{aligned}
$$

This completes the first inequality. □

3.2. *Trace theorem for domains with small diameter in FEM case*

We will consider trace theory for a small domain in finite element case. Let $0 < \varepsilon < 1$ be arbitrary real number. By the change of variables, $x = \varepsilon s$, and $y = \varepsilon t$, we can transform Ω, Ω_h, Γ, and I_i into $\Omega_\varepsilon, \Omega_{h(\varepsilon)}, \Gamma_\varepsilon$, and $I_{i,\varepsilon}$, for $1 \leqslant i \leqslant 4$, respectively. Denote by $H^{1,h(\varepsilon)}$ the space of finite element functions on $\Omega_{h(\varepsilon)}$, and denote by $H^{1/2,h(\varepsilon)}$ the space of finite element functions on $\Gamma_{h(\varepsilon)}$. For $u \in H^{1,h}$ and $\phi \in H^{1/2,h}$, let $u_\varepsilon(x, y) = u(s, t)$ and $\phi_\varepsilon(x, y) = \phi(s, t)$.

Definition 7
For each $\phi_\varepsilon \in H^{1/2,h(\varepsilon)}$, define

$$
\begin{aligned}
|\phi_\varepsilon|^2_{H^{1,h(\varepsilon)}(\Gamma_\varepsilon)} &= \varepsilon h_1(|\phi_\varepsilon|^2_{H^1(I_{2,\varepsilon})} + |\phi_\varepsilon|^2_{H^1(I_{4,\varepsilon})}) + \varepsilon h_2(|\phi_\varepsilon|^2_{H^1(I_{1,\varepsilon})} + |\phi_\varepsilon|^2_{H^1(I_{3,\varepsilon})}) \\
|\phi_\varepsilon|^2_{H^{1/2,h(\varepsilon)}(\Gamma_\varepsilon)} &= |\phi_\varepsilon|^2_{H^{1/2}(\Gamma_\varepsilon)} + |\phi_\varepsilon|^2_{H^{1,h(\varepsilon)}(\Gamma_\varepsilon)} \\
\|\phi_\varepsilon\|^2_{H^{1/2,h(\varepsilon)}(\Gamma_\varepsilon)} &= \varepsilon\|\phi_\varepsilon\|^2_{L_2(\Gamma_\varepsilon)} + |\phi_\varepsilon|^2_{H^{1/2,h(\varepsilon)}(\Gamma_\varepsilon)}
\end{aligned}
$$

From the definition, we immediately obtain the following:

Lemma 3.3
For any $\phi \in H^{1/2,h}$, we have

$$|\phi_\varepsilon|^2_{H^{1/2,h(\varepsilon)}(\Gamma_\varepsilon)} = |\phi|^2_{H^{1/2,h}(\Gamma)}$$

The following two lemmas are direct consequence of Theorems 3.1, 3.2, Definition 7, and Lemma 3.3.

Lemma 3.4
For any $u_\varepsilon \in H^{1,h(\varepsilon)}$, and $\phi_\varepsilon \in H^{1/2,h(\varepsilon)}$ with $u_\varepsilon(x) = \phi_\varepsilon(x)$ on Γ_ε

$$|\phi_\varepsilon|_{H^{1/2,h(\varepsilon)}(\Gamma_\varepsilon)} \leqslant c|u_\varepsilon|_{H^1(\Omega_\varepsilon)} \tag{17}$$

holds, and for any given $\phi_\varepsilon \in H^{1/2,h(\varepsilon)}$, there exists $u_\varepsilon \in H^{1,h(\varepsilon)}$ with $u_\varepsilon(x) = \phi_\varepsilon(x)$ on Γ_ε such that

$$|u_\varepsilon|_{H^1(\Omega_\varepsilon)} \leqslant C|\phi_\varepsilon|_{H^{1/2,h(\varepsilon)}(\Gamma_\varepsilon)} \tag{18}$$

holds.

Lemma 3.5
There exist c and C which are independent of ε such that for any $u_\varepsilon \in H^{1,h(\varepsilon)}$, and $\phi_\varepsilon \in H^{1/2,h(\varepsilon)}$ with $u_\varepsilon(x) = \phi_\varepsilon(x)$ on Γ_ε

$$\|\phi_\varepsilon\|_{H^{1/2,h(\varepsilon)}(\Gamma_\varepsilon)} \leqslant c\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}$$

holds, and for any given $\phi_\varepsilon \in H^{1/2,h(\varepsilon)}$, there exists $u_\varepsilon \in H^{1,h(\varepsilon)}$ with $u_\varepsilon(x) = \phi_\varepsilon(x)$ on Γ_ε such that

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leqslant C\|\phi_\varepsilon\|_{H^{1/2,h(\varepsilon)}(\Gamma_\varepsilon)}$$

holds.

Lemma 3.6
Let I_h and J_h be the uniform triangulation of $(0,1)$ and $(0,3)$, respectively, with mesh size h. For $i = 0,\ldots,3/h$, let $x_i = ih$. Let H_I and H_J be the spaces of continuous and piecewise linear functions on I_h and J_h, respectively. Let $\phi \in H_I$ be arbitrary. Define $\psi(x) \in H_J$ in the interval $(0,3)$ such that

$$\psi(x_i) = \begin{cases} \phi(x_i), & \text{if } x_i \in (0,1] \\ (2-x_i)\phi(2-x_i), & \text{if } x_i \in (1,2] \\ 0, & \text{if } x_i \in (2,3) \end{cases}$$

Then there exists a constant c independent of ϕ and ψ such that

$$\|\psi\|^2_{H^{1/2}(0,3)} + h|\psi|^2_{H^1(0,3)} \leqslant c(\|\phi\|^2_{H^{1/2}(0,1)} + h|\phi|^2_{H^1(0,1)})$$

Moreover, if $\int_0^1 \phi(x)\,\mathrm{d}x = 0$, then there exists a constant C independent of ϕ and ψ such that

$$\psi|_{H^{1/2}(0,3)} + h|\psi|^2_{H^1(0,3)} \leqslant C(|\phi|_{H^{1/2}(0,1)} + h|\phi|^2_{H^1(0,1)})$$

Proof
Define the following discrete norms:

$$\|\phi\|^2_{\tilde{L}_2^h(I_h)} = h\sum_i \phi^2(x_i)$$

$$\|\phi\|^2_{\tilde{H}^{1/2,h}(I_h)} = h^2\sum_i\sum_j \frac{(\phi(x_i)-\phi(x_j))^2}{|x_i-x_j|^2}$$

$$\|\phi\|^2_{\tilde{H}^{1,h}(I_h)} = \frac{1}{h}\sum_i(\phi(x_{i+1})-\phi(x_i))^2$$

Then it can be shown that these discrete norms are equivalent to their corresponding continuous norms. Then the following inequalities can be shown by direct computation, from which the proof of the lemma is derived.

$$\|\psi\|^2_{\tilde{L}_2^h(J_h)} \leqslant c\|\phi\|^2_{\tilde{L}_2^h(I_h)}$$

$$|\psi|^2_{\tilde{H}^{1/2,h}(J_h)} \leqslant c(|\phi|^2_{\tilde{H}^{1/2,h}(I_h)} + \|\phi\|^2_{\tilde{L}_2^h(I_h)})$$

$$h|\psi|^2_{\tilde{H}^{1,h}(J_h)} \leqslant h|\phi|^2_{\tilde{H}^{1,h}(I_h)}$$

□

We can directly obtain the following just by scaling.

Corollary 3.1
Denote by H_I^ε and H_J^ε be the scaled finite element spaces with respect to H_I and H_J, respectively. Let $\tilde{x}_i = \varepsilon x_i$, for $i=0,\dots,3/h$. Let $\phi\in H_I^\varepsilon$ be arbitrary. Define $\psi(x)$ in the interval $(0,3\varepsilon)$ such that

$$\psi(\tilde{x}_i) = \begin{cases} \phi(\tilde{x}_i), & \text{if } \tilde{x}_i\in(0,\varepsilon] \\ (2\varepsilon-\tilde{x}_i)\phi(2\varepsilon-\tilde{x}_i), & \text{if } \tilde{x}_i\in(\varepsilon,2\varepsilon] \\ 0, & \text{if } \tilde{x}_i\in(2\varepsilon,3\varepsilon) \end{cases}$$

Then there exists a constant c independent of ϕ,ψ, and ε such that

$$\varepsilon\|\psi\|^2_{L_2(0,3\varepsilon)} + |\psi|^2_{H^{1/2}(0,3\varepsilon)} + \varepsilon h|\psi|^2_{H^1(0,3\varepsilon)} \leqslant c(\varepsilon\|\phi\|^2_{L_2(0,\varepsilon)} + |\phi|^2_{H^{1/2}(0,\varepsilon)} + \varepsilon h|\phi|^2_{H^1(0,\varepsilon)}) \tag{19}$$

Moreover, if $\int_0^\varepsilon \phi(x)\,\mathrm{d}x = 0$, then there exists a constant C independent of ϕ,ψ, and ε such that

$$\frac{1}{\varepsilon}\|\psi\|^2_{L_2(0,3\varepsilon)} + |\psi|^2_{H^{1/2}(0,3\varepsilon)} + \varepsilon h|\psi|^2_{H^1(0,3\varepsilon)} \leqslant C(|\phi|^2_{H^{1/2}(0,\varepsilon)} + \varepsilon h|\phi|^2_{H^1(0,\varepsilon)})$$

3.3. Trace theorem for anisotropic grid on thin domain in FEM case

This section is closely related to Section 2.1, but we deal with finite element case. Let $\Omega=(0,H)\times(0,L)$ and let $\Gamma=\partial\Omega$. Assume that $H\ll L$. Let $k=[L/H]$ and $\tilde{H}=L/k$. Let Ω^h

be a triangulation on Ω with uniform grid size h_1 and h_2 in x and y direction, respectively. Let $n_1 = H/h_1$ and $n_2 = L/h_2$. Assume that h_2 is sufficiently small so that there exists an integer $m \geqslant 3$ so that $m \cdot h_2 \leqslant H < (m+1) \cdot h_2$. We can choose a sequence $\{s_i\}_{i=0}^{k}$ of integers such that $0 = s_0 < s_1 < \cdots < s_k = n_2$ and $m_i = s_i - s_{i-1} \geqslant 3$ of order m, and $m_1 = m_k = m$. Decompose the interval $[0, L] = \bigcup_{i=1}^{k} \bar{I}_i$ where $I_i = (s_{i-1} \cdot h_2, s_i \cdot h_2)$. Let $\sigma_0 = \tau_0 = (0, H) \times \{0\}$, $\sigma_k = \tau_k = (0, H) \times \{L\}$, and $\sigma_i = \{0\} \times I_i$, $\tau_i = \{H\} \times I_i$. Define $l_i = \sigma_i \cup \sigma_{i+1}$ and $r_i = \tau_i \cup \tau_{i+1}$.

Definition 8
For any $\phi \in H^{1/2,h}$, define

$$
\begin{aligned}
|\phi|^2_{H^{1,h}(\Gamma)} &= h_1(|\phi_h|^2_{H^1(\Gamma_L)} + |\phi_h|^2_{H^1(\Gamma_R)}) + h_2(|\phi_h|^2_{H^1(\Gamma_B)} + |\phi_h|^2_{H^1(\Gamma_T)}) \\
\|\phi\|^2_{L_2(\Gamma)} &= \sum_{i=0}^{k} H(\|\phi\|^2_{L_2(l_i)} + \|\phi\|^2_{L_2(r_i)}) \\
|\phi|^2_{\tilde{H}^{1/2,h}(\Gamma)} &= \sum_{i=0}^{k} (I_{l_i,l_i}(\phi) + I_{r_i,r_i}(\phi) + I_{l_i,r_i}(\phi)) + |\phi|^2_{H^{1,h}(\Gamma)} \\
\|\phi\|^2_{\tilde{H}^{1/2,h}(\Gamma)} &= \|\phi\|^2_{\tilde{L}_2(\Gamma)} + |\phi|^2_{\tilde{H}^{1/2,h}(\Gamma)}
\end{aligned}
$$

Theorem 3.3
There exist c and C independent of H, h_1, and h_2 such that for all $\phi \in H^{1/2,h}$,

$$c(S\phi, \phi) \leqslant \|\phi\|^2_{\tilde{H}^{1/2,h}(\Gamma)} \leqslant C(S\phi, \phi)$$

where

$$(S\phi, \phi) = \inf_{u \in H^{1,h}, u|_\Gamma = \phi} \|u\|^2_{H^1(\Omega)}$$

Proof
Define $\tilde{\sigma}_i$ and $\tilde{\tau}_i$ similarly as in the proof of Theorem 2.1. Then we can define α_i and β_i to construct $\phi^H, \psi_0, \psi_{k+1}$, and ψ_i, for $i = 1, \ldots, k$ in the same way as in the proof of Theorem 2.1. To complete the proof, it is sufficient to show that

$$\sum_{i=1}^{k} |\phi^H|^2_{H^{1,h}(\partial\tilde{\Omega}_i)} \leqslant c\|\phi\|^2_{\tilde{H}^{1/2,h}(\Gamma)} \tag{20}$$

and

$$|\psi_{-1}|^2_{H^{1,h}(\partial\Omega_0)} + |\psi_k|^2_{H^{1,h}(\partial\Omega_k)} + \sum_{i=0}^{k-1} (|\psi_{i,l}|^2_{H^{1,h}(\partial\Omega_i)} + |\psi_{i,r}|^2_{H^{1,h}(\partial\Omega_i)}) \leqslant c\|\phi\|^2_{\tilde{H}^{1/2,h}(\Gamma)} \tag{21}$$

By direct calculation,

$$
\begin{aligned}
\sum_{i=1}^{k} |\phi^H|^2_{H^{1,h}(\partial\tilde{\Omega}_i)} &\leqslant c\sum_{i=0}^{k} ((\alpha_{i+1} - \alpha_i)^2 + (\beta_{i+1} - \beta_i)^2) \\
&\quad + c\sum_{i=0}^{k} ((\alpha_{i+1} - \beta_i)^2 + (\beta_{i+1} - \alpha_i)^2) + c\sum_{i=0}^{k+1} (\alpha_i - \beta_i)^2 \qquad (22) \\
&\leqslant c\|\phi\|_{\tilde{H}^{1/2,h}(\Gamma)} \qquad (23)
\end{aligned}
$$

where the last inequality is obtained as the proof of Theorem 2.1. This completes (20). Equation (21) follows from Corollary 3.1. □

For the semi norm, we have the following theorem.

Theorem 3.4
There exist c and C independent of H such that for all $\phi \in H^{1/2,h}$,

$$c(\tilde{S}\phi, \phi) \leqslant |\phi|^2_{\tilde{H}^{1/2,h}(\Gamma_h)} \leqslant C(\tilde{S}\phi, \phi)$$

where

$$(\tilde{S}\phi, \phi) = \inf_{u \in H^{1,h}, u|_\Gamma = \phi} |u|^2_{H^1(\Omega)}$$

The semi-norm, $|\phi|_{\tilde{H}^{1/2,h}(\Gamma)}$ is complicated to implement, but we can replace it by an equivalent norm $|\phi|_{\tilde{H}^{1/2,h}(\Gamma)}$, which is simpler.

Definition 9
For any $\phi \in H^{1/2,h}$, define

$$|\phi|^2_{\tilde{H}^{1/2,h}(\Gamma)} = \sum_{i=0}^{k} (I_{l_i,l_i}(\phi) + I_{r_i,r_i}(\phi)) + \sum_{i=1}^{k} I_{\sigma_i,\tau_i}(\phi) + |\phi|^2_{H^{1,h}(\Gamma)} \tag{24}$$

Theorem 3.5
There exist c and C independent of H such that for all $\phi \in H^{1/2,h}$,

$$c(\tilde{S}\phi, \phi) \leqslant |\phi|^2_{\tilde{H}^{1/2,h}(\Gamma)} \leqslant C(\tilde{S}\phi, \phi)$$

where

$$(\tilde{S}\phi, \phi) = \inf_{u \in H^{1,h}, u|_\Gamma = \phi} |u|^2_{H^1(\Omega)}$$

4. CONSTRUCTION OF PRECONDITIONER *B*

Let us decompose W into two subspaces W_0, W_1, and construct a preconditioner for each subspaces. The subspaces W_0 and W_1 are defined as follows. Let

$$W_0 = \{u \in W \mid u(x) = 0, \ x \in \Lambda_h\}$$

Set $W_{1/2}$ be the trace space of W on Λ_h. Let t be the extension operator from $W_{1/2}$ to W. In fact,

$$t = \begin{bmatrix} A_{11}^{-1} A_{12} \\ I \end{bmatrix}$$

and t maps a discrete function defined on S^h to the A-discrete harmonic function in Ω_h. Let t^* be the adjoint map of t. In matrix form,

$$t^* = [A_{21} A_{11}^{-1} \quad I]$$

Set $W_1 = tW_{1/2}$ and

$$W_0^{(j)} = \{u \in W_0 \mid u(x) = 0,\ x \notin \Omega_h^{(j)}\}, \quad j = 1, 2, \ldots, n$$

Let $B_0^{(j)} : W_0^{(j)} \to W_0^{(j)}$ such that there exist c and C satisfying

$$c(B_0^{(j)} v, v) \leqslant \int_{\Omega^{(j)}} \mathbf{p}^{(j)} \nabla u \cdot \nabla u \, \mathrm{d}\Omega \leqslant C(B_0^{(j)} v, v), \quad \forall v \in W_0^{(j)}$$

For any linear operator T, denote by T^+ the pseudo inverse of T. Set $B_0 = B_0^{(1)} + B_0^{(2)} + \cdots + B_0^{(n)}$ and $B_0^+ = (B_0^{(1)})^+ + (B_0^{(2)})^+ + \cdots + (B_0^{(n)})^+$. For the preconditioner for W_1, let $\phi^{(j)} = \phi|_{\partial\tilde{\Omega}^{(j)}}$, $\phi \in W_{1/2}$. Assume that we have $\Sigma^{(j)}$ which induces a semi norm on $\partial\tilde{\Omega}^{(j)}$ which is equivalent to the norm defined in Theorem 3.5. Let $p^{(j)} = \min\{p_1^{(j)}, p_2^{(j)}\}$, for $j = 1, \ldots, n$. Let

$$(\Sigma\phi, \psi) = \sum_{j=1}^{n} p^{(j)} (\Sigma^{(j)} \phi^{(j)}, \psi^{(j)})$$

Then $(\Sigma\phi, \phi)$ is equivalent to the trace norm on $W_{1/2}$. To give a preconditioner B_1 for Σ, we define $\Sigma^{(j)}$. Fix a subdomain $\Omega^{(j)}$. For convenience assume that $\Omega^{(j)} = (0, L_1) \times (0, L_2)$, where $L_1 = n_1 h$ and $L_2 = n_2 h$ for some positive integers n_1 and n_2. Assume that $p_i^{(j)} \gg p_2^{(j)}$. Let $\alpha^{(j)} = \sqrt{p_1^{(j)}/p_2^{(j)}}$. Assume that h is sufficiently small so that there exists an integer $m \geqslant 3$ so that $mh < L_2/\alpha^{(j)} < (m+1)h$. By a change of variable $(x_1, x_2) = (x_1/\alpha^{(j)}, x_2), \Omega^{(j)}$ is transformed onto $\tilde{\Omega}^{(j)} = (0, L_1/\alpha^{(j)}) \times (0, L_2)$. Let $H = L_1/\alpha^{(j)}$, $L = L_2$, $\tilde{H} = L_2/\alpha^{(j)}$, $h_1 = h/\alpha^{(j)}$, and $h_2 = h$. Note that $H \ll L$ and $h_1 \ll h_2$. Each $\tilde{\Omega}^{(j)}$ will be decomposed exactly as in Section 3.2, and we use the same notations $m_i, \sigma_i, \tau_i, l_i, r_i,$. By Theorem 3.5, we have that for all $\phi \in H^{1/2,h}(\tilde{\Gamma}^{(j)})$,

$$(S_i \phi, \phi) \simeq |\phi|^2_{\tilde{H}^{1/2,h}(\tilde{\Gamma}^{(j)})}$$

Let us construct $\Sigma^{(j)}$ which is equivalent to $|\cdot|^2_{\tilde{H}^{1/2,h}(\tilde{\Gamma}^{(j)})}$. For given $\phi \in H^{1/2,h}(\tilde{\Gamma}^{(j)})$, let ϕ_S be the restriction of ϕ on S, for any subset S of $\tilde{\Gamma}^{(j)}$. From now on, denote by $\mathscr{A}_{(n)}$ the generic matrix corresponding to the 1-dimensional laplacian with Neumann boundary condition where the size is equal the number of grid points n:

$$\mathscr{A}_{(n)} = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}$$

In fact, $(\mathscr{A}_{(n)}\phi, \phi) = h|\phi|^2_{H^1(I)}$ whenever ϕ is a finite element function on interval I which is discretized by n points with uniform mesh size h. Then for $i = 1, \ldots, k-1$,

$$I_{l_i, l_i}(\phi) \simeq (\mathscr{A}^{1/2}_{(m_i + m_{i+1} + 1)} \phi_{l_i}, \phi_{l_i})$$

Let $\Sigma_{l_i} = \mathscr{A}^{1/2}_{(m_i + m_{i+1} + 1)}$ and $T_{l_i} = \mathscr{A}_{(m_i + m_{i+1} + 1)}$.

For $I_{l_0,l_0}(\phi)$, the situation is quite different since l_0 is composed of two different meshes h_1 and h_2 (similarly for $I_{l_k,l_k}(\phi)$, $I_{r_0,r_0}(\phi)$, and $I_{r_k,r_k}(\phi)$). Let σ_{0,h_1} and σ_{1,h_2} be the triangulation on σ_0 and σ_1 with grid sizes h_1 and h_2, respectively. Let $l_{0,h}=\sigma_{0,h_1}\cup\sigma_{1,h_2}$. Let $\{y_i\}_{i=0}^{m}$ and $\{y_i\}_{i=m}^{m+n_1}$ be the grid points of σ_1 and σ_0, respectively. We will insert new nodes z_i and renumber the whole nodes on l_0 as follows. Since $h_2\gg h_1$, there exists a positive integer p such that $ph_1\leqslant h_2<(p+1)h_1$. Let $z_{ip+s}=[(p-s)y_i+sy_{i+1}]/p$, $(i=0,\dots,m-1$ and $s=0,1,\dots,p-1)$, and $z_{mp+i}=y_{m+i}$, $(i=0,\dots,n_1)$. Let $\ell=m\cdot p+n_1$. Observe that $\{z_i\}_{i=0}^{\ell}$ is a quasi-uniform grid on l_0 with grid size of order h_1. Let l_{0,h_1} be the triangulation induced by $\{z_i\}_{i=0}^{\ell}$. Then we have two different finite element spaces on $l_0, H^{1/2}_{l_{0,h}}$ and $H^{1/2}_{l_{0,h_1}}$, which are finite element spaces on $l_{0,h}$ and l_{0,h_1}, respectively. Let J_{l_0} be the natural extension of the function in $H^{1/2}_{l_{0,h}}$ to the same function in $H^{1/2}_{l_{0,h_1}}$. In fact, $J_{l_0}[\phi(x_i)]^{\mathrm{T}}=[\phi(z_j)]^{\mathrm{T}}$. For any $\phi\in H^{1/2}_{l_{0,h}}$, we have

$$(J_{l_0}^{\mathrm{T}}\mathscr{A}^{1/2}_{(\ell+1)}J_{l_0}\phi,\phi)=(\mathscr{A}^{1/2}_{(\ell+1)}J_{l_0}\phi,J_{l_0}\phi)\simeq|\phi|^2_{H^{1/2}(l_0)}=I_{l_0,l_0}(\phi)$$

Let $\Sigma_{l_0}=J_{l_0}^{\mathrm{T}}\mathscr{A}^{1/2}_{(\ell+1)}J_{l_0}$ and $T_{l_0}=J_{l_0}^{\mathrm{T}}\mathscr{A}_{(\ell+1)}J_{l_0}$. Note that T_{l_0} is a tridiagonal matrix. In fact,

$$T_{l_0}=\begin{bmatrix} \frac{1}{p} & -\frac{1}{p} & & & & & & & \\ & -\frac{1}{p} & 2\frac{1}{p} & -\frac{1}{p} & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & -\frac{1}{p} & 1+\frac{1}{p} & -1 & & & \\ & & & & -1 & 2 & -1 & & \\ & & & & & -1 & 2 & -1 & \\ & & & & & & \ddots & \ddots & \\ & & & & & & & -1 & 1 \end{bmatrix}$$

Similarly we can define $\Sigma_{r_0},\Sigma_{l_k},\Sigma_{r_k},T_{r_0},T_{l_k}$, and T_{r_k}. Now consider the term $I_{\sigma_i,\tau_i}(\phi)$. For $i=1,\dots,k$, denote $\{x_{\sigma_i,j}\}_{j=0}^{m_i}$ and $\{x_{\tau_i,j}\}_{j=0}^{m_i}$ be the mesh points on σ_i and τ_i, respectively. Then

$$\begin{aligned} I_{\sigma_i,\tau_i}(\phi)&=\int_{\sigma_i}\int_{\tau_i}\frac{(\phi(x)-\phi(y))^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y \\ &\simeq\frac{1}{H^2}\int_{\sigma_i}\int_{\tau_i}(\phi(x)-\phi(y))^2\,\mathrm{d}x\,\mathrm{d}y \\ &\simeq\frac{1}{(m_i+1)^2}\sum_{j,k}(\phi(x_{\sigma_i,j})-\phi(x_{\tau_i,k}))^2 \\ &=(\Xi_{\sigma_i,\tau_i}\phi,\phi) \end{aligned}$$

where

$$\Xi_{\sigma_i,\tau_i} = \frac{1}{(m_i+1)^2}\begin{bmatrix} & & & -1 & \cdots & -1 \\ & (m_i+1)I & & \vdots & \cdots & \vdots \\ & & & -1 & \cdots & -1 \\ -1 & \cdots & -1 & & & \\ \vdots & \cdots & \vdots & & (m_i+1)I & \\ -1 & \cdots & -1 & & & \end{bmatrix}$$

Denote by ϕ_L the restriction of ϕ on $\Gamma_L^{(j)}$. Then

$$h_1|\phi|^2_{H^1(\Gamma_L^{(j)})} \simeq (\Xi_L\phi_L, \phi_L)$$

where $\Xi_L = (h_1/h_2)\mathscr{A}_{(n_2+1)}$. Define Ξ_R, Ξ_B and Ξ_T similarly. Now define $\Sigma^{(j)}$ by

$$\begin{aligned}(\Sigma^{(j)}\phi,\psi) = {} & \sum_{i=1}^{k-1}((\Sigma_{l_i}\phi_{l_i},\psi_{l_i}) + (\Sigma_{r_i}\phi_{r_i},\psi_{r_i})) + \sum_{i=1}^{k}(\Xi_{\sigma_i,\tau_i}\phi_{\sigma_i\cup\tau_i},\psi_{\sigma_i\cup\tau_i}) \\ & + (\Sigma_{l_0}\phi_{l_0},\psi_{l_0}) + (\Sigma_{r_0}\phi_{r_0},\psi_{r_0}) + (\Sigma_{l_k}\phi_{l_k},\psi_{l_k}) + (\Sigma_{r_k}\phi_{r_k},\psi_{r_k}) \\ & + (\Xi_L\phi_L,\psi_L) + (\Xi_R\phi_R,\psi_R) + (\Xi_B\phi_B,\psi_B) + (\Xi_T\phi_T,\psi_T)\end{aligned}$$

and $T^{(j)}$ by

$$\begin{aligned}(T^{(j)}\phi,\phi) = {} & \sum_{i=1}^{k-1}((T_{l_i}\phi_{l_i},\psi_{l_i}) + (T_{r_i}\phi_{r_i},\psi_{r_i})) + \sum_{i=1}^{k}(\Xi_{\sigma_i,\tau_i}\phi_{\sigma_i\cup\tau_i},\psi_{\sigma_i,\cup\tau_i}) \\ & + (T_{l_0}\phi_{l_0},\psi_{l_0}) + (T_{r_0}\phi_{r_0},\psi_{r_0}) + (T_{l_k}\phi_{l_k},\psi_{l_k}) + (T_{r_k}\phi_{r_k},\psi_{r_k}) \\ & + (\Xi_L\phi_L,\psi_L) + (\Xi_R\phi_R,\psi_R) + (\Xi_B\phi_B,\psi_B) + (\Xi_T\phi_T,\psi_T)\end{aligned}$$

One can show that

$$\check{c}^{(j)}(T^{(j)}\phi,\phi) \leqslant (\Sigma^{(j)}\phi,\phi) \leqslant \hat{c}^{(j)}(T^{(j)}\phi,\phi) \tag{25}$$

where $\check{c}^{(j)} = \frac{1}{2}$ and $\hat{c}^{(j)} = 1/\sin(\pi/2(\ell+1))$. Note that ℓ is of order $1/h$. Denote by $\psi^{(k)}$ be the restriction of ψ on $\partial\tilde{\Omega}^{(k)}$ for all $\psi \in W_{1/2}$. Define bilinear forms Σ and T by

$$(\Sigma\phi,\psi) = \sum_{k=1}^{n} p^{(k)}(\Sigma^{(k)}\phi^{(k)},\psi^{(k)})$$

$$(T\phi,\psi) = \sum_{k=1}^{n} p^{(k)}(T^{(k)}\phi^{(k)},\psi^{(k)})$$

From (25), we can see that

$$\check{c}(T\phi,\phi)\leqslant(\Sigma\phi,\phi)\leqslant\hat{c}(T^{(j)}\phi,\phi) \tag{26}$$

where $\check{c}=\frac{1}{2}$ and $\hat{c}$ is the maximum of $\{\hat{c}^{(j)}\}_{j=1}^{n}$. Observe that each $\Sigma^{(j)}$ is a full matrix and hard to construct, hence it is cost-expensive to invert Σ. However the inversion of T is reasonably simple. Note that each Ξ_{σ_i,τ_i} is of the form

$$\Xi_{\sigma_i,\tau_i}=\begin{bmatrix} K_i^{11} & K_i^{12} \\ K_i^{21} & K_i^{22} \end{bmatrix}$$

where K_i^{12} and K_i^{21} are matrices of rank 1. Hence we decompose $T^{(j)}$ into

$$T^{(j)}=\tilde{T}^{(j)}+\sum_{i=1}^{2k}K_i$$

where $\tilde{T}^{(j)}$ is a tridiagonal matrix and K_i's are the matrices corresponding to the rank 1 matrices K_i^{12}'s and K_i^{21}'s. Hence T can be inverted by using rank reduction method [9]. The cost to invert T is of order $(1/h)\max_{i=1}^{n}\{\alpha^{(i)}\}$.

Now we are in position to solve the following complicated system.

$$\Sigma\phi=\psi$$

Since Σ is a full matrix, we shall use Chebyshev iteration as a preconditioner. Consider the following iteration:

$$\begin{aligned} \phi^{0}&=0 \\ \phi^{i+1}-\phi^{i}&=-t_i T^{-1}(\Sigma\phi^{i}-\psi) \end{aligned}$$

where t_i's are Chebyshev set of iteration parameter [4]. Set

$$B_{1/2}^{-1}=\left(I-\prod_{i=0}^{n(\varepsilon)}(I-t_iT^{-1}\Sigma)\right)\Sigma^{-1}$$

where

$$n(\varepsilon)\leqslant\frac{\ln(2/\varepsilon)}{\ln(1/q)},\quad q=\frac{\check{c}^{1/2}-\hat{c}^{1/2}}{\check{c}^{1/2}+\hat{c}^{1/2}}$$

Then, $\phi^{n(\varepsilon)}=B_{1/2}^{-1}\psi$ and if we choose $\varepsilon=\frac{1}{2}$, we see $n(\varepsilon)=\mathcal{O}(h^{1/2})$ and

$$\tfrac{1}{2}(B_{1/2}\phi,\phi)\leqslant(\Sigma\phi,\phi)\leqslant\tfrac{3}{2}(B_{1/2}\phi,\phi),\quad\forall\phi\in W_{1/2}$$

Set $B_1^{+}=tB_{1/2}^{-1}t^{*}$ and $B^{-1}=B_0^{+}+B_1^{+}$. Then the following theorem holds.

Theorem 4.1
There exist positive constants c and C independent of h and p such that

$$c(Bv, v) \leqslant (Av, v) \leqslant C(Bv, v) \quad \forall v \in W$$

5. NUMERICAL EXPERIMENT

In this section, we present some numerical results to verify the performance of our domain decomposition algorithm. The region Ω is the unit square $(0,1)\times(0,1)$ and divided into four squares as in Figure 2. We consider the example

$$-\nabla \cdot p(x)\nabla u(x) = 0 \quad \text{in } \Omega$$

with Dirichlet boundary condition, where $p(x)$ is a piecewise constant function whose value is p_i on each subdomain Ω_i.

The initial guess is $u(x, y) = x(1-x)y(1-y)$. We denote by N the number of iteration of preconditioned conjugate gradient method. On the first experiment, we fix $p(x)$ and vary h from $1/2^3$ to $1/2^8$. Tables I and II show that N is stable with respect to h. Next, for each $h = 1/2^3 \cdots 1/2^8$ we vary $p(x)$, $x \in \Omega_2 \cup \Omega_4$ from 1 to 10^5. Table III shows that N is stable with respect $p(x)$.

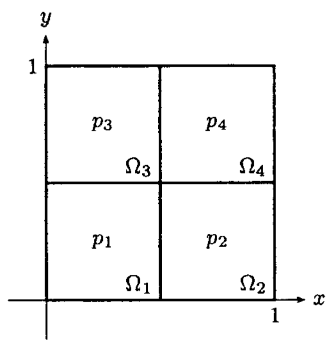

Figure 2. Diffusion coefficient $p(x)$.

Table I. $p_1 = p_4 = 1$, $p_2 = 100$ and $p_3 = 1000$.

$1/h$	N
2^3	5
2^4	6
2^5	7
2^6	7
2^7	9
2^8	8

Table II. $p_1 = 1$ and $p_2 = p_3 = p_4 = 1000$.

$1/h$	N
2^3	5
2^4	7
2^5	8
2^6	8
2^7	9
2^8	9

Table III. The number of iteration $N : p_1 = p_4 = 1$ and $p_2 = p_3 = p^*$.

$h\backslash p^*$	10^0	10^1	10^2	10^3	10^4	10^5
$1/2^3$	4	5	5	5	5	5
$1/2^4$	5	6	6	6	7	7
$1/2^5$	6	7	7	8	8	9
$1/2^6$	5	6	7	8	8	9
$1/2^7$	7	7	8	9	10	10
$1/2^8$	6	7	8	9	9	10

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