AN ANALYSIS OF NONCONFORMING VIRTUAL ELEMENT METHODS ON POLYTOPAL MESHES WITH SMALL FACES

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Abstract. In this paper, we analyze nonconforming virtual element methods on polytopal meshes with small faces for the second-order elliptic problem. We propose new stability forms for 2D and 3D nonconforming virtual element methods. For the 2D case, the stability form is defined by the sum of an inner product of approximate tangential derivatives and a weighed L^2-inner product of certain projections on the mesh element boundaries. For the 3D case, the stability form is defined by a weighted L^2-inner product on the mesh element boundaries. We prove the optimal convergence of the nonconforming virtual element methods equipped with such stability forms. Finally, several numerical experiments are presented to verify our analysis and compare the performance of the proposed stability forms with the standard stability form [5].

1. Introduction

Recently, several numerical schemes for solving partial differential equations on polytopal meshes have been proposed, for example, mimetic finite difference (MFD) methods [16, 17, 30], hybrid high-order (HHO) methods [37, 43, 44], weak Galerkin (WG) methods [55, 60, 63], hybridizable discontinuous Galerkin (HDG) methods [38, 39], and so on. Among them, the virtual element method (VEM) was introduced in [7] as a generalization of the finite element method (FEM) to general polytopal meshes. The shape functions in the VEM are defined implicitly as the solution of a specific local boundary value problem. Although its explicit evaluation cannot be obtained in general, the virtual element function can be characterized by the degrees of freedom, and the VEM can be implemented by using the degrees of freedom only. This is the reason why the word "virtual" is used. The VEM has been successfully applied to a wide range of problems: elasticity problems [10, 54, 66], Stokes problems [19, 31, 56], Maxwell problems [8, 9, 14], etc. We also refer to [2, 11–13, 20, 52] and the references therein for more comprehensive survey.

The nonconforming FEM has been studied and developed by many researchers since its first introduction by Crouzeix and Raviart [41]. See, for instance, [6, 25, 35, 40, 48, 49, 53] and the references therein. There are several advantages of the nonconforming FEM. First, the low-order elements can be used to construct stable elements for the Stokes problem [41] and locking-free elements for the elasticity problem [27, 48, 51]. Second, one can implement efficient parallel algorithms [45, 46], since the basis functions of the nonconforming elements are supported on at most

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1991 *Mathematics Subject Classification.* 65N12, 65N15, 65N30, 35J15.

Key words and phrases. virtual element method, nonconforming method, polytopal mesh, stability analysis.

This work is partially supported by NRF, contract No. 2021R1A2C1003340.

two mesh elements. Third, a mixed formulation of the second-order elliptic problem can be cast into the nonconforming FEM [3,4,35,53]. These advantages carry over to the nonconforming VEM (see, e.g., [5,31,52,54,66]). Moreover, for the three-dimensional problem, the nonconforming VEM is easier than the conforming VEM, since the conforming element requires a recursive construction from the mesh faces to the mesh elements while the nonconforming element does not.

In the VEM literature, the underlying mesh is usually required to satisfy the following conditions: (i) each mesh element is star-shaped with respect to a ball whose radius is comparable to the diameter of the element, and (ii) all the edges or faces are not too small. However, several meshes may violate some of these conditions, such as anisotropic meshes, interface-fitted meshes, crack-fitted meshes, and Voronoi meshes. Although it has been observed that the VEMs perform robustly on such meshes in many numerical experiments (see, e.g., [15,21,22,34,57]), it is also important to analyze the performance of the VEM on such meshes rigorously.

For the case that the underlying mesh violates both conditions, we refer to [32, 33], where the authors considered the lowest-order conforming and nonconforming VEMs on anisotropic meshes. In this paper, we focus on the case that the mesh satisfies (i) but possibly violates (ii). In [18,28], the convergence of the conforming VEM on such meshes was analyzed. The authors of [18] observed that designing a suitable stability bilinear form plays an important role in the analysis, which is typically regarded as a minor issue under the usual mesh assumption. For the 2D case, the authors of [18,28] considered a tangential derivative-type stability form, which was first proposed in [65]. They proved an optimal error estimate for the 2D conforming virtual element solution equipped with this stability form. They also proved the optimal convergence of the 2D conforming VEM with the standard stability form, but the convergence depends on the maximum number of edges of the mesh elements and the ratio of the longest and shortest edges. The authors of [28] proposed a weighted L^2-inner product-type stability form for the 3D conforming VEM and proved that the 3D conforming VEM with this stability form yields an optimal error estimate, where the constant in the estimate depends on the maximum number of faces of the mesh elements and the ratio of the longest and shortest edges.

Unfortunately, the stability forms in [18,28] cannot be used for the nonconforming VEMs, because the forms require the function values on the mesh edges, but such values of the nonconforming virtual element functions are not known. To remedy this, the authors of [24] used the duality technique [23] to design stability forms for the 2D nonconforming VEMs. They first constructed several bilinear forms on the so-called dual space, and then the stability forms are defined by the reflexive generalized inverse of the constructed bilinear forms. However, the presented stability forms require that the maximum number of edges of the mesh elements be uniformly bounded, while the tangential derivative-type stability form for the 2D conforming VEM introduced in [18,28] does not. On the other hand, to the best of our knowledge, there are no results considering the 3D nonconforming VEM of arbitrary order on the meshes with small faces yet.

In this paper, we propose new stability forms for the nonconforming VEM in both 2D and 3D cases on polytopal meshes with small faces. In the 2D case, our proposed stability form is defined by the sum of an inner product of approximate tangential derivatives and a weighed L^2-inner product of certain projections on the mesh element boundaries. We then prove an optimal error estimate for the 2D nonconforming VEM with this stability form, without the assumption that the maximum number of edges of the mesh elements is uniformly bounded. In the 3D case, our proposed stability form is defined by a weighted L^2-inner product on the mesh element boundaries. We prove the optimal convergence of the 3D nonconforming VEM with this stability form, where the convergence depends on the maximum number of faces of the mesh elements, but not on the ratio of the longest and shortest edges.

The rest of the paper is organized as follows. In Section 2, we summarize some basic notions and the model problem. In Section 3, we describe the nonconforming VEM, and present a modified stability condition that the stability form must satisfy. In Section 4, we provide an error analysis for the nonconforming VEM equipped with the stability form satisfying the modified condition. In Section 5, we present new stability forms in both 2D and 3D cases, and prove that they indeed satisfy the modified stability condition under the mesh assumption weaker than the usual one. In Section 6, we offer several numerical tests verifying our theoretical results and comparing the performance of our proposed stability forms with the standard one. Finally, the conclusion is given in Section 7.

2. PRELIMINARIES

We first summarize basic definitions and notations. Let ω be a bounded open set in $\mathbb{R}^n$ with $n = 1, 2, 3$. We denote by h_ω its diameter and by $|\omega|$ its n-dimensional Lebesgue measure.

Throughout this paper, we follow the usual notation of Sobolev spaces (see, e.g., [26, 36, 50]). For $s \geq 0$, let $H^s(\omega)$ be the standard Sobolev space of order s, and let $|\cdot|_{s,\omega}$ and $\|\cdot\|_{s,\omega}$ be the corresponding seminorm and norm, respectively. In particular, $H^0(\omega)$ coincides with $L^2(\omega)$. Let $(\cdot,\cdot)_{0,\omega}$ be the standard L^2-inner product on ω. Let $\|\cdot\|_{L^\infty(\omega)}$ denote the standard L^∞-norm on ω. For any $v \in L^2(\omega)$, let $(v)_\omega$ denote the average of v on ω, that is, $(v)_\omega := \frac{1}{|\omega|}\int_\omega v$.

Let $H^{-1/2}(\omega)$ be the dual space of $H^{1/2}(\omega)$, and let $|\cdot|_{-1/2,\omega}$ be a seminorm on $H^{-1/2}(\omega)$ defined as follows [24]:

$$|u|_{-1/2,\omega} := \sup_{v \in H^{1/2}(\omega)/\mathbb{R}} \frac{\langle u, v\rangle_\omega}{|v|_{1/2,\omega}},$$

where $H^{1/2}(\omega)/\mathbb{R} := \{v \in H^{1/2}(\omega) : (v)_\omega = 0\}$, and $\langle\cdot,\cdot\rangle_\omega$ is the duality pairing.

For an integer $k \geq 0$, Let $\mathbb{P}_k(\omega)$ be the space of all polynomials of degree less than or equal to k on ω, and let $\mathbb{M}_k(\omega)$ be the set of all scaled monomials on ω. Let $\Pi_k^{0,\omega}$ be the L^2-projection operator onto $\mathbb{P}_k(\omega)$.

We next briefly describe the model problem. Let Ω be a bounded polytopal domain in $\mathbb{R}^d$ $(d = 2, 3)$. We consider the second-order elliptic problem: Given $f \in L^2(\Omega)$, find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}\boldsymbol{x} = \int_\Omega f v \,\mathrm{d}\boldsymbol{x} \quad \forall v \in H_0^1(\Omega). \tag{2.1}$$

It is well-known that (2.1) has a unique solution.

3. NONCONFORMING VIRTUAL ELEMENT METHOD

In this section, we describe the nonconforming VEM for (2.1).

3.1. **Mesh assumption.** Let $\mathcal{P}_h$ be a decomposition of Ω into finitely many non-overlapping polytopal elements, with $h = \max_{K\in\mathcal{P}_h} h_K$. Let $\mathcal{F}_h$ be the set of all $(d-1)$-dimensional mesh faces. Let $\mathcal{F}_h^i$ and $\mathcal{F}_h^b$ be the set of all interior and boundary faces, respectively. For each element K, let $\boldsymbol{n}_K$ be the unit normal vector on ∂K in the outward direction with respect to K. For each interior face F, $\boldsymbol{n}_F$ is defined by the unit normal vector on F whose direction is fixed once and for all. For each boundary face F, $\boldsymbol{n}_F$ is defined by the unit normal vector on F in the outward direction with respect to Ω.

Let v be a scalar function defined on Ω. For each $F \in \mathcal{F}_h^i$, we define the jump of v across F by

$$[\![v]\!]_F := (v|_{K^+})\boldsymbol{n}_{K^+} + (v|_{K^-})\boldsymbol{n}_{K^-},$$

where K^+ and K^- is the elements in $\mathcal{P}_h$ such that $F \subset \partial K^+ \cap \partial K^-$. For each $F \in \mathcal{F}_h^b$, we let $[\![v]\!]_F := v\boldsymbol{n}_F$.

Throughout this paper, we will assume that the following holds [18, 24, 28].

Assumption 3.1. There exists a positive constant ρ independent of h such that

- every mesh element $K \in \mathcal{P}_h$ is star-shaped with respect to a d-dimensional ball B_K with radius ρh_K;
- every mesh face $F \in \mathcal{F}_h$ is star-shaped with respect to a $(d-1)$-dimensional ball B_F with radius ρh_F.

The assumption above is weaker than the mesh assumption usually required in the VEM literature (see, e.g., [5, 7, 29]).

3.2. **Nonconforming virtual element spaces.** Let $k \geq 1$ be an integer, and let $K \in \mathcal{P}_h$. We first introduce an auxiliary space

$$N_h^\ell(\partial K) := \left\{g \in L^2(\partial K) : g|_F \in \mathbb{P}_\ell(F)\ \forall F \subset \partial K\right\}, \quad \ell \geq 0.$$

Then the nonconforming virtual element space on the element K is defined as follows:

$$V_h^k(K) := \left\{v \in H^1(K) : \Delta v \in \mathbb{P}_{k-2}(K),\ \partial v/\partial \boldsymbol{n} \in N_h^{k-1}(\partial K)\right\},$$

with the convention that $\mathbb{P}_{-1} = \{0\}$. Then the following degrees of freedom are unisolvent for $V_h^k(K)$ (see [5]): given $v \in V_h^k(K)$,

- the moments of order up to $k-1$ on each face $F \subset \partial K$:

$$\frac{1}{|F|}\int_F vq\,\mathrm{d}s, \quad q \in \mathbb{M}_{k-1}(F); \tag{3.1}$$

- the moments of order up to $k-2$ on K:

$$\frac{1}{|K|}\int_K vq\,\mathrm{d}\boldsymbol{x}, \quad q \in \mathbb{M}_{k-2}(K). \tag{3.2}$$

Next, the global nonconforming virtual element space $V_h^k(\Omega)$ is given by

$$V_h^k(\Omega) := \Bigg\{v_h \in L^2(\Omega) : v_h|_K \in V_h^k(K)\ \forall K \in \mathcal{P}_h,$$
$$\int_F [\![v_h]\!]_F \cdot \boldsymbol{n}_F q\,\mathrm{d}s = 0\ \forall q \in \mathbb{P}_{k-1}(F),\ \forall F \in \mathcal{F}_h\Bigg\}.$$

Then the following moments can be taken as degrees of freedom for $V_h^k(\Omega)$: The moments (3.1) of order up to $k-1$ on each interior mesh face F, and the moments (3.2) of order up to $k-2$ on each mesh element K.

Let $H_h(\Omega) := H_0^1(\Omega) + V_h^k(\Omega)$, and let $|\cdot|_{1,h}$ be the broken H^1-seminorm on $H_h(\Omega)$ given by

$$|v_h|_{1,h}^2 := \sum_{K\in\mathcal{P}_h} |v_h|_{1,K}^2, \quad \forall v_h \in H_h(\Omega).$$

For each $K \in \mathcal{P}_h$, let $I_h^K : H^1(K) \to V_h^k(K)$ be the canonical local interpolation operator based on the local degrees of freedom (3.1)-(3.2). Also, let $I_h : H_h(\Omega) \to V_h^k(\Omega)$ be the canonical global interpolation operator based on the global degrees of freedom of $V_h^k(\Omega)$.

3.3. **Discrete problem.** For each $K \in \mathcal{P}_h$, the elliptic projection operator $\Pi_k^{\nabla,K} : H^1(K) \to \mathbb{P}_k(K)$ is defined as follows:

$$\begin{aligned}\int_K \nabla\Pi_k^{\nabla,K} v \cdot \nabla q \,\mathrm{d}\boldsymbol{x} &= \int_K \nabla v \cdot \nabla q \,\mathrm{d}\boldsymbol{x} \quad \forall q \in \mathbb{P}_k(K),\\ \int_{\partial K} \Pi_k^{\nabla,K} v \,\mathrm{d}s &= \int_{\partial K} v \,\mathrm{d}s \quad \text{if } k = 1,\\ \int_K \Pi_k^{\nabla,K} v \,\mathrm{d}s &= \int_K v \,\mathrm{d}s \quad \text{if } k > 1.\end{aligned}$$

It is easy to verify that $\Pi_k^{\nabla,K} v$ is computable for any $v \in V_h^k(K)$ using only the degrees of freedom (3.1)-(3.2). For $v_h \in H_h(\Omega)$, we define $\Pi_k^{\nabla} v_h$ by the piecewise polynomial such that $(\Pi_k^{\nabla} v)|_K = \Pi_k^{\nabla,K}(v_h|_K)$ for any $K \in \mathcal{P}_h$.

We next define the global discrete bilinear form $a_h(\cdot,\cdot)$ on $H_h(\Omega)$ as

$$a_h(u_h, v_h) := \sum_{K\in\mathcal{P}_h} a_h^K(u_h, v_h),$$

and let $|\!|\!|\cdot|\!|\!| = \sqrt{a_h(\cdot,\cdot)}$ be the discrete energy norm on $H_h(\Omega)$. Here, for each $K \in \mathcal{P}_h$, $a_h(\cdot,\cdot)$ is the local discrete bilinear form on $H^1(K)$ given by

$$a_h^K(u, v) := \left(\nabla\Pi_k^{\nabla,K} u, \nabla\Pi_k^{\nabla,K} v\right)_{0,K} + S_h^K\left((I - \Pi_k^{\nabla,K})u, (I - \Pi_k^{\nabla,K})v\right),$$

where $S_h^K(\cdot,\cdot)$ is a symmetric positive semidefinite bilinear form on $H^1(K)$, called the *stability bilinear form.*

It is easy to check that $a_h^K(\cdot,\cdot)$ satisfies the property called *consistency*:

$$a_h^K(p, v) = (\nabla p, \nabla v)_{0,K} \quad \forall p \in \mathbb{P}_k(K),\ v \in H^1(K). \tag{3.3}$$

In the classical VEMs [5,29], the stability form $S_h^K(\cdot,\cdot)$ is constructed so that it can be computed using the degrees of freedom only and satisfies the *stability condition*: there exist positive constants c_* and c^* independent of h such that, for any $K \in \mathcal{P}_h$,

$$c_*\|\nabla v\|_{0,K}^2 \le a_h^K(v, v) \le c^*\|\nabla v\|_{0,K}^2 \quad \forall v \in V_h^k(K). \tag{3.4}$$

We will replace (3.4) with the following properties: there exists positive constants C_1 and C_2 such that

$$|v_h|_{1,h} \le C_1 |\!|\!|v_h|\!|\!| \quad \forall v_h \in V_h^k(\Omega), \tag{3.5}$$

$$|\!|\!|u - \Pi_k^{\nabla} u|\!|\!| \le C_2 h^\ell |u|_{\ell+1,\Omega} \quad \forall u \in H^{\ell+1}(\Omega),\ 1 \le \ell \le k. \tag{3.6}$$

In addition, we also assume that $S_h^K(\cdot,\cdot)$ satisfies the following property:

$$S_h^K(v, w) = S_h^K(I_h^K v, w) \quad \forall v, w \in H^1(K), \tag{3.7}$$

which also holds for the classical stability form in the nonconforming VEM [5]. In Section 5 we will present suitable stability bilinear forms $S_h^K(\cdot,\cdot)$ satisfying (3.5), (3.6) and (3.7), where C_1 and C_2 depend only on ρ and k in two dimensions and also N in three dimensions. Note that (3.4) together with the projection error estimates (4.3) implies (3.5)-(3.6), but the converse may not hold in general. Nevertheless, we can still obtain optimal error estimates, as shown in the next section.

Let $\langle f, \cdot \rangle_h$ be the discrete loading term on $H_h(\Omega)$ given by $\langle f, v_h \rangle_h := \big(f, \Pi_h v_h\big)_{0,\Omega}$, where $\Pi_h v_h$ is the projection such that, for any $K \in \mathcal{P}_h$,

$$(\Pi_h v_h)|_K = \Pi_k^{\nabla,K}(v_h|_K) \quad \text{if } k \le 2, \qquad (\Pi_h v_h)|_K = \Pi_{k-2}^{0,K} v_h \quad \text{if } k > 2.$$

With the above preparations, we state the nonconforming VEM for (2.1) as follows: Find $u_h \in V_h^k(\Omega)$ such that

$$a_h(u_h, v_h) = \langle f, v_h \rangle_h \quad \forall v_h \in V_h^k(\Omega). \tag{3.8}$$

4. Error analysis

In this section, we prove the optimal convergence of the nonconforming VEMs (3.8) with any stability form $S_h^K(\cdot,\cdot)$ satisfying (3.5), (3.6) and (3.7).

We first recall some standard estimates. Assume that $D = K$ for some $K \in \mathcal{P}_h$ or $D = F$ for some $F \in \mathcal{F}_h$. Then we have the trace inequality (see, e.g., [1, Theorem 3.2] and (2.18) of [28])

$$\|v\|_{0,\partial D}^2 \le C \left(h_D^{-1} \|v\|_{0,D}^2 + h_D |v|_{1,D}^2 \right) \quad \forall v \in H^1(D), \tag{4.1}$$

and the Poincaré-Friedrichs inequality (cf. [28, 64])

$$\|v - \overline{v}\|_{0,D}^2 \le C h_D^2 |v|_{1,D}^2 \quad \forall v \in H^1(D), \tag{4.2}$$

where $\overline{v} = (v)_{\partial D}$ or $\overline{v} = (v)_D$. Here C in (4.1) and (4.2) denotes a positive constant depending only on ρ.

We present the error estimates for the projection operators $\Pi_k^{\nabla,K}$ and $\Pi_k^{0,K}$ for each $K \in \mathcal{P}_h$ (cf. [26, 28]).

Lemma 4.1 (projection error estimates)**.** *Let $K \in \mathcal{P}_h$. Then there exists a constant $C > 0$ depending only on ρ and k such that, for any $v \in H^{\ell+1}(K)$ with $1 \le \ell \le k$ and any integer $0 \le m \le 2$,*

$$\left| v - \Pi_k^{\nabla,K} v \right|_{m,K} + \left| v - \Pi_k^{0,K} v \right|_{m,K} \le C h_K^{\ell+1-m} |v|_{\ell+1,K}. \tag{4.3}$$

We also present the interpolation error estimates. Note that their proofs can be done by following the arguments in [59, Proposition 3.1] and [58, Section 6.2].

Lemma 4.2 (interpolation error estimates)**.** *Let $K \in \mathcal{P}_h$. Then there exists a constant $C > 0$ depending only on ρ and k such that*

$$\left\| v - I_h^K v \right\|_{0,K} + h_K \left| v - I_h^K v \right|_{1,K} \le C h_K^{k+1} |v|_{k+1,K} \quad \forall v \in H^{k+1}(K). \tag{4.4}$$

In order to derive error estimates in $|\cdot|_{1,h}$ and $|\!|\!|\cdot|\!|\!|$, we first compute the consistency error as follows.

Lemma 4.3 (consistency error)**.** *Let $u \in H_0^1(\Omega) \cap H^{3/2+\varepsilon}(\Omega)$ be the solution of* (2.1)*, for any positive, arbitrary small $\varepsilon > 0$. Then, for any $v_h \in H_h(\Omega)$,*

$$\begin{aligned} & a_h(u, v_h) - \langle f, v_h \rangle_h \\ & \quad = \sum_{K \in \mathcal{P}_h} S_h^K \left(u - \Pi_k^\nabla u, v_h - \Pi_k^\nabla v_h\right) + \int_\Omega f(v_h - \Pi_h v_h)\,\mathrm{d}\boldsymbol{x} \\ & \qquad + \int_\Omega \nabla\left(\Pi_k^\nabla u - u\right) \cdot \nabla v_h \,\mathrm{d}\boldsymbol{x} + \sum_{F \in \mathcal{F}_h} \int_F \nabla u \cdot [\![v_h]\!]_F \,\mathrm{d}s. \end{aligned} \tag{4.5}$$

Proof. Let $v_h \in H_h(\Omega)$. Note that

$$\begin{aligned} & a_h(u, v_h) - \langle f, v_h \rangle_h \\ & \quad = \sum_{K \in \mathcal{P}_h} S_h^K \left(u - \Pi_k^\nabla u, v_h - \Pi_k^\nabla v_h\right) + \int_\Omega \nabla \Pi_k^\nabla u \cdot \nabla \Pi_k^\nabla v_h \,\mathrm{d}\boldsymbol{x} - \langle f, v_h \rangle_h \\ & \quad = \sum_{K \in \mathcal{P}_h} S_h^K \left(u - \Pi_k^\nabla u, v_h - \Pi_k^\nabla v_h\right) + \int_\Omega \nabla\left(\Pi_k^\nabla u - u\right) \cdot \nabla v_h \,\mathrm{d}\boldsymbol{x} \\ & \qquad + \int_\Omega \nabla u \cdot \nabla v_h \,\mathrm{d}\boldsymbol{x} - \int_\Omega f \Pi_h v_h \,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{4.6}$$

Since $u \in H^2(\Omega)$ and u is the solution of (2.1), integrating by parts yields

$$\begin{aligned} \int_\Omega \nabla u \cdot \nabla v_h \,\mathrm{d}\boldsymbol{x} &= \int_\Omega f v_h \,\mathrm{d}\boldsymbol{x} + \sum_{K \in \mathcal{P}_h} \int_{\partial K} \frac{\partial u}{\partial \boldsymbol{n}} v_h \,\mathrm{d}s \\ &= \int_\Omega f v_h \,\mathrm{d}\boldsymbol{x} + \sum_{F \in \mathcal{F}_h} \int_F \nabla u \cdot [\![v_h]\!]_F \,\mathrm{d}s. \end{aligned} \tag{4.7}$$

Now inserting (4.7) into (4.6), we get (4.5). □

The following lemma establishes the estimate of the second term on the right-hand side of (4.5). We skip the proof since it is essentially the same as the proof of [5, Lemma 3.4].

Lemma 4.4. *There exists a constant $C > 0$ depending only on ρ and k such that*

$$\left| \int_\Omega f(v_h - \Pi_h v_h)\,\mathrm{d}\boldsymbol{x} \right| \le C h^k |f|_{k-1,\Omega} |v_h|_{1,h} \quad \forall v_h \in H_h(\Omega). \tag{4.8}$$

We next consider the fourth term on the right-hand side of (4.5).

Lemma 4.5. *Suppose that $u \in H_0^1(\Omega) \cap H^{k+1}(\Omega)$ is the solution of* (2.1)*. Then there exists a constant $C > 0$ depending only on ρ and k such that*

$$\left| \sum_{F \in \mathcal{F}_h} \int_F \nabla u \cdot [\![v_h]\!]_F \,\mathrm{d}s \right| \le C h^k |u|_{k+1,\Omega} |v_h|_{1,h} \quad \forall v_h \in H_h(\Omega). \tag{4.9}$$

Proof. The proof is slightly different from that of Lemma 4.1 in [5], due to the presence of small edges/faces.

For an integer $\ell \geq 0$, $K \in \mathcal{P}_h$, and $v \in L^2(\partial K)$, define $\Pi_\ell^{0,\partial K} v$ by $(\Pi_\ell^{0,\partial K} v)|_F = \Pi_\ell^{0,F}(v|_F)$ for any $F \subset \partial K$. Then we have

$$\begin{aligned}
&\left| \sum_{F\in\mathcal{F}_h} \int_F \nabla u \cdot [\![v_h]\!]_F \,\mathrm{d}s \right| = \left| \sum_{F\in\mathcal{F}_h} \int_F \left(\nabla u - \Pi_{k-1}^{0,F}(\nabla u) \right) \cdot [\![v_h]\!]_F \,\mathrm{d}s \right| \\
&\quad = \left| \sum_{F\in\mathcal{F}_h} \int_F \left(\nabla u - \Pi_{k-1}^{0,F}(\nabla u) \right) \cdot \left([\![v_h]\!]_F - \Pi_0^{0,F}\left([\![v_h]\!]_F \right) \right) \mathrm{d}s \right| \\
&\quad \leq \sum_{F\in\mathcal{F}_h} \left\| \nabla u - \Pi_{k-1}^{0,F}(\nabla u) \right\|_{0,F} \left\| [\![v_h]\!]_F - \Pi_0^{0,F}\left([\![v_h]\!]_F \right) \right\|_{0,F} \\
&\quad \leq \left(\sum_{F\in\mathcal{F}_h} \left\| \nabla u - \Pi_{k-1}^{0,F}(\nabla u) \right\|_{0,F}^2 \right)^{\frac{1}{2}} \left(\sum_{F\in\mathcal{F}_h} \left\| [\![v_h]\!]_F - \Pi_0^{0,F}\left([\![v_h]\!]_F \right) \right\|_{0,F}^2 \right)^{\frac{1}{2}} \\
&\quad \leq C \left(\sum_{K\in\mathcal{P}_h} \left\| \nabla u - \Pi_{k-1}^{0,\partial K}(\nabla u) \right\|_{0,\partial K}^2 \right)^{\frac{1}{2}} \left(\sum_{K\in\mathcal{P}_h} \left\| v_h - \Pi_0^{0,\partial K} v_h \right\|_{0,\partial K}^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Let $K \in \mathcal{P}_h$ and $u_\pi := \Pi_k^{\nabla,K} u$. Then, by (4.1) and (4.3),

$$\begin{aligned}
&\left\| \nabla u - \Pi_{k-1}^{\partial K}(\nabla u) \right\|_{0,\partial K}^2 \leq 2 \left\| \nabla u - \nabla u_\pi \right\|_{0,\partial K}^2 + 2 \left\| \Pi_{k-1}^{\partial K}(\nabla u_\pi - \nabla u) \right\|_{0,\partial K}^2 \\
&\quad \leq 4 \left\| \nabla u - \nabla u_\pi \right\|_{0,\partial K}^2 \leq C h_K^{-1} |u - u_\pi|_{1,K}^2 + C h_K |u - u_\pi|_{2,K}^2 \leq C h_K^{2k-1} |u|_{k+1,K}^2.
\end{aligned}$$

Similarly, using (4.1) and (4.2), we obtain

$$\begin{aligned}
&\left\| v_h - \Pi_0^{\partial K} v_h \right\|_{0,\partial K}^2 \leq 2 \left\| v_h - (v_h)_K \right\|_{0,\partial K}^2 + 2 \left\| \Pi_0^{\partial K}((v_h)_K - v_h) \right\|_{0,\partial K}^2 \\
&\quad \leq 4 \left\| v_h - (v_h)_K \right\|_{0,\partial K}^2 \leq C h_K^{-1} \left\| v_h - (v_h)_K \right\|_{0,K}^2 + C h_K |v_h|_{1,K}^2 \leq C h_K |v_h|_{1,K}^2.
\end{aligned}$$

Combining the above estimates, we finally obtain (4.9). □

Now we derive the error estimates in the discrete energy norm $|\!|\!| \cdot |\!|\!|$ and the broken H^1-seminorm $|\cdot|_{1,h}$.

Theorem 4.6. *Suppose that* $u \in H_0^1(\Omega) \cap H^{k+1}(\Omega)$ *is the solution of* (2.1) *with* $f \in H^{k-1}(\Omega)$. *Let* $u_h \in V_h^k(\Omega)$ *be the solution of* (3.8). *Then there is a constant* $C > 0$ *depending only on* ρ, k, C_1 *and* C_2 *such that*

$$|u - u_h|_{1,h} + |\!|\!| u - u_h |\!|\!| \leq C h^k \left(|u|_{k+1,\Omega} + |f|_{k-1,\Omega} \right). \tag{4.10}$$

Proof. Using (4.4) and (3.5),

$$|u - u_h|_{1,h} \leq |u - I_h u|_{1,h} + |I_h u - u_h|_{1,h} \leq C h^k |u|_{k+1,\Omega} + |\!|\!| I_h u - u_h |\!|\!|.$$

Note that $|\!|\!| I_h u - u_h |\!|\!| = |\!|\!| u - u_h |\!|\!|$ by (3.7). Thus it is enough to estimate $|\!|\!| u - u_h |\!|\!|$. Let $\delta_h = u - u_h$. By (4.5),

$$\begin{aligned}
|\!|\!| u - u_h |\!|\!|^2 &= a_h(u, \delta_h) - \langle f, \delta_h \rangle \\
&= \sum_{K\in\mathcal{P}_h} S_h^K \left(u - \Pi_k^\nabla u, \delta_h - \Pi_k^\nabla \delta_h \right) + \int_\Omega f(\delta_h - \Pi_h \delta_h) \,\mathrm{d}\boldsymbol{x} \\
&\quad + \int_\Omega \nabla \left(\Pi_k^\nabla u - u \right) \cdot \nabla \delta_h \,\mathrm{d}\boldsymbol{x} + \sum_{F\in\mathcal{F}_h} \int_F \nabla u \cdot [\![\delta_h]\!]_F \,\mathrm{d}s \\
&=: T_1 + T_2 + T_3 + T_4. \qquad (4.11)
\end{aligned}$$

For T_1 and T_2, it follows from (3.6), (4.8) and (3.5) that

$$|T_1| + |T_2| \leq Ch^k\big(|u|_{k+1,\Omega} + |f|_{k-1,\Omega}\big)|||\delta_h|||.$$

For T_3 and T_4, using (4.3), (4.9) and (3.5), we obtain

$$|T_3| + |T_4| \leq Ch^k|u|_{k+1,\Omega}|\delta_h|_{1,h} \leq Ch^k|u|_{k+1,\Omega}|||\delta_h|||.$$

The conclusion follows by plugging the estimates for $T_1, \cdots, T_4$ into (4.11). □

Remark 4.7. Following the proof of Theorem 4.5 in [5], one can derive the optimal error estimate in the L^2-norm when Ω is convex: If $u \in H_0^1(\Omega) \cap H^{k+1}(\Omega)$ is the solution of (2.1) with $f \in H^{k-1}(\Omega)$ and if $u_h \in V_h^k(\Omega)$ is the solution of (3.8), then there exists a constant $C > 0$ depending only on Ω, ρ, k, C_1 and C_2 such that

$$\|u - u_h\|_{0,\Omega} \leq Ch^{k+1}\big(|u|_{k+1,\Omega} + |f|_{k-1,\Omega}\big).$$

5. STABILITY ANALYSIS

In this section, we present stability forms satisfying (3.5), (3.6) and (3.7) for the 2D and 3D cases. Let $K \in \mathcal{P}_h$.

5.1. **Some technical results.**

Before beginning, we present some results that are useful in the sequel. The following results hold in the presence of small edges/faces.

Lemma 5.1. *If $v \in V_h^k(K)$, then there exists $q \in \mathbb{P}_k(K)$ such that*

$$\Delta q = \Delta v, \quad |q|_{1,K} \leq C|v|_{1,K},$$

where C is a positive constant depending only on ρ and k.

Proof. The conclusion immediately follows via a similar argument in the proofs of Lemma 3.5 and Lemma 6.3 in [18]. □

By proceeding as in the proof of Lemma 6.2 in [18], together with the inverse trace theorem (see Subsection 2.7 in [28]), one can prove the following lemma.

Lemma 5.2. *Suppose that $\boldsymbol{w} \in [L^2(K)]^d$ satisfies $\operatorname{div} \boldsymbol{w} = 0$. Then there exists a constant $C > 0$ depending only on ρ such that*

$$|\boldsymbol{w} \cdot \boldsymbol{n}_K|_{-1/2,\partial K} \leq C\|\boldsymbol{w}\|_{0,K}.$$

We also state some useful estimates for $|\cdot|_{1/2,\partial K}$ as follows (see (2.16) and (2.17) of [28]).

Lemma 5.3. *The following estimates hold:*

$$|v|_{1/2,\partial K} \leq Ch_K^{1/2}|v|_{1,\partial K} \quad \forall v \in H^1(\partial K), \tag{5.1}$$

$$|v|_{1/2,\partial K} \leq C|v|_{1,K} \quad \forall v \in H^1(K), \tag{5.2}$$

where C is a positive constant depending only on ρ.

The following lemma is a modification of Lemma 4.6 in [24].

Lemma 5.4. *Let $g \in N_h^{k-1}(\partial K)$ satisfy $(g)_{\partial K} = 0$. Then*

$$h_K^{-1} \sum_{F \subset \partial K} h_F^2 \|g\|_{0,F}^2 \leq C|g|_{-1/2,\partial K}^2,$$

where C is a positive constant depending only on ρ and k.

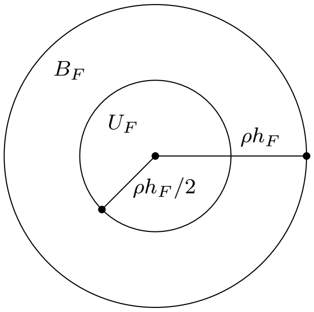

FIGURE 1. The balls B_F and U_F.

Proof. For each $F \subset \partial K$, let U_F is the ball concentric with B_F and has radius $\rho h_F/2$ (see Figure 1), and let η_F be a smooth (cut-off) function such that

(a) $\eta_F = 1$ on U_F, $\eta_F = 0$ outside B_F, and $0 \le \eta_F \le 1$;
(b) $\|\eta_F\|_{L^\infty(F)} \le C$ and $\|\nabla_F \eta_F\|_{L^\infty(F)} \le C h_F^{-1}$, where ∇_F denotes the $(d-1)$-dimensional gradient operator on F.

Let $p_F := g|_F$ for each $F \subset \partial K$, and define $\widetilde{g} \in L^2(\partial K)$ by

$$\widetilde{g}|_F = h_K^{-1} h_F^2 p_F \eta_F \quad \forall F \subset \partial K.$$

Then $\widetilde{g} \in C^1(\partial K)$, since p_F is a polynomial and η_F is a smooth function supported on B_F, for each $F \subset \partial K$. Then

$$\begin{aligned} \int_{\partial K} g\widetilde{g}\,\mathrm{d}s &= h_K^{-1} \sum_{F\subset\partial K} h_F^2 \int_F \eta_F |p_F|^2\,\mathrm{d}s \ge \sum_{F\subset\partial K} h_K^{-1} h_F^2 \|p_F\|_{0,U_F}^2 \\ &\ge C \sum_{F\subset\partial K} h_K^{-1} h_F^2 \|p_F\|_{0,F}^2 = C \sum_{F\subset\partial K} h_K^{-1} h_F^2 \|g\|_{0,F}^2. \end{aligned} \tag{5.3}$$

By $(g)_{\partial K} = 0$ and the definition of the norm $|\cdot|_{-1/2,\partial K}$, we have

$$\int_{\partial K} g\widetilde{g}\,\mathrm{d}s = \int_{\partial K} g\left(\widetilde{g} - (\widetilde{g})_{\partial K}\right)\mathrm{d}s \le |g|_{-1/2,\partial K} |\widetilde{g}|_{1/2,\partial K}. \tag{5.4}$$

From the inverse estimates for polynomials, we obtain, for any $F \subset \partial K$,

$$\begin{aligned} |\widetilde{g}|_{1,F} &\le h_K^{-1} h_F^2 \|\eta_F \nabla_F p_F\|_{0,F} + h_K^{-1} h_F^2 \|p_F \nabla_F \eta_F\|_{0,F} \\ &\le C(h_K^{-1} h_F^2 \|\nabla_F p_F\|_{0,F} + h_K^{-1} h_F \|p_F\|_{0,F}) \le C h_K^{-1} h_F \|p_F\|_{0,F}. \end{aligned} \tag{5.5}$$

Combining (5.1) with (5.5), and by the definition of p_F, we have

$$|\widetilde{g}|_{1/2,\partial K}^2 \le C h_K |\widetilde{g}|_{1,\partial K}^2 \le C \sum_{F\subset\partial K} h_K^{-1} h_F^2 \|g\|_{0,F}^2. \tag{5.6}$$

Now the assertion of the lemma follows from (5.3), (5.4), and (5.6). □

Lemma 5.5. *Let $v \in V_h^k(K)$ satisfy $\Pi_k^{\nabla,K} v = 0$, and let $q \in \mathbb{P}_k(K)$ be a polynomial satisfying $\Delta q = \Delta v$. Then we have*

$$\int_K |\nabla v|^2\,\mathrm{d}\boldsymbol{x} = \sum_{F\subset\partial K} \int_F (\nabla(v-q)\cdot \boldsymbol{n}_K)\left(\Pi_{k-1}^{0,F} v - (v)_{\partial K}\right)\mathrm{d}s. \tag{5.7}$$

Proof. Using integration by parts, we have

$$\begin{aligned}\int_K |\nabla v|^2 \,\mathrm{d}\boldsymbol{x} &= \int_K \nabla v \cdot \nabla (v - (v)_{\partial K}) \,\mathrm{d}\boldsymbol{x} \\ &= \int_K (-\Delta q)(v - (v)_{\partial K}) \,\mathrm{d}\boldsymbol{x} + \int_{\partial K} \frac{\partial v}{\partial \boldsymbol{n}} (v - (v)_{\partial K}) \,\mathrm{d}s \\ &= \int_K \nabla q \cdot \nabla v \,\mathrm{d}\boldsymbol{x} + \int_{\partial K} (\nabla (v - q)) \cdot \boldsymbol{n}_K (v - (v)_{\partial K}) \,\mathrm{d}s \\ &= \int_K \nabla q \cdot \nabla \Pi_k^{\nabla, K} v \,\mathrm{d}\boldsymbol{x} + \int_{\partial K} (\nabla (v - q)) \cdot \boldsymbol{n}_K (v - (v)_{\partial K}) \,\mathrm{d}s \\ &= \int_{\partial K} (\nabla (v - q) \cdot \boldsymbol{n}_K)(v - (v)_{\partial K}) \,\mathrm{d}s \\ &= \sum_{F \subset \partial K} \int_F (\nabla (v - q) \cdot \boldsymbol{n}_K) \big(\Pi_{k-1}^{0,F} v - (v)_{\partial K}\big) \,\mathrm{d}s.\end{aligned}$$

This completes the proof. ☐

5.2. **Stabilization in two dimensions.** In this subsection, we consider the two-dimensional case (that is, $d = 2$).

Let $B(\partial K)$ be the boundary space defined by

$$B(\partial K) := \left\{ \phi \in C^0(\partial K) : \phi|_F \in \mathbb{P}_2(F) \ \forall F \subset \partial K \right\}.$$

For $v \in H^1(K)$, let ϕ_v be the function in $B(\partial K)$ satisfying the following properties:

- v and ϕ_v have the same lowest-order face moments, that is,

$$\int_F v \,\mathrm{d}s = \int_F \phi_v \,\mathrm{d}s \quad \forall F \subset \partial K.$$

- If $\boldsymbol{a}$ is a common end point of two faces F_+ and F_- in ∂K, then

$$\phi_v(\boldsymbol{a}) = \frac{h_{F_-}}{h_{F_+} + h_{F_-}} (v)_{F_+} + \frac{h_{F_+}}{h_{F_+} + h_{F_-}} (v)_{F_-}.$$

Let us consider the stability bilinear form $S_h^K(\cdot, \cdot)$ given by

$$\begin{aligned} S_h^K(u, v) &= h_K \int_{\partial K} \partial_s \phi_u \partial_s \phi_v \,\mathrm{d}s \\ &\quad + h_K \sum_{F \subset \partial K} h_F^{-2} \int_F \big(\Pi_{k-1}^{0,F} u - (u)_F\big)\big(\Pi_{k-1}^{0,F} v - (v)_F\big) \,\mathrm{d}s, \end{aligned} \tag{5.8}$$

where $\partial_s \phi$ denotes the tangential derivative of ϕ along ∂K, for $\phi \in B(\partial K)$.

Note that the stability form given by (5.8) satisfies (3.7), since the face moments (3.1) of u and $I_h u$ are identical.

We will prove that the stability form given by (5.8) satisfies (3.5).

Lemma 5.6. *There exists a constant $C > 0$ depending only on ρ and k such that*

$$|v|_{1,K}^2 \le C S_h^K(v, v) \quad \forall v \in V_h^k(K) \text{ with } \Pi_k^{\nabla, K} v = 0.$$

Proof. Let $v \in V_h^k(K)$ satisfy $\Pi_k^{\nabla, K} v = 0$. By Lemma 5.1, there exists $q \in \mathbb{P}_k(K)$ such that $\Delta q = \Delta v$ and $|q|_{1,K} \le C |v|_{1,K}$. Let $\widetilde{v} := v - q$. Then we have

$$|\widetilde{v}|_{1,K} \le C |v|_{1,K}. \tag{5.9}$$

Using (5.7), we have

$$\begin{aligned}\int_K |\nabla v|^2 \,\mathrm{d}\boldsymbol{x} &= \sum_{F\subset\partial K}\int_F \frac{\partial\widetilde{v}}{\partial\boldsymbol{n}}\big(\Pi^{0,F}_{k-1}v-(v)_{\partial K}\big)\,\mathrm{d}s\\ &= \sum_{F\subset\partial K}\int_F \frac{\partial\widetilde{v}}{\partial\boldsymbol{n}}(\phi_v-(v)_{\partial K})\,\mathrm{d}s + \sum_{F\subset\partial K}\int_F \frac{\partial\widetilde{v}}{\partial\boldsymbol{n}}\big((v)_F-\phi_v\big)\,\mathrm{d}s\\ &\quad+\sum_{F\subset\partial K}\int_F \frac{\partial\widetilde{v}}{\partial\boldsymbol{n}}\big(\Pi^{0,F}_{k-1}v-(v)_F\big)\,\mathrm{d}s\\ &=: T_1+T_2+T_3. \end{aligned}\tag{5.10}$$

Since $\Delta\widetilde{v}=0$ on K, we have

$$0=\int_K \nabla\widetilde{v}\cdot\nabla 1\,\mathrm{d}\boldsymbol{x}=\int_{\partial K}\frac{\partial\widetilde{v}}{\partial\boldsymbol{n}}\,\mathrm{d}s.\tag{5.11}$$

For T_1, using the definition of $|\cdot|_{-1/2,\partial K}$, (5.1), Lemma 5.2, and (5.9), we have

$$\begin{aligned}T_1 &= \int_{\partial K}\frac{\partial\widetilde{v}}{\partial\boldsymbol{n}}(\phi_v-(v)_{\partial K})\,\mathrm{d}s \le |\nabla\widetilde{v}\cdot\boldsymbol{n}_K|_{-1/2,\partial K}|\phi_v|_{1/2,\partial K}\\ &\le Ch_K^{1/2}|\widetilde{v}|_{1,K}|\phi_v|_{1,\partial K}\le C|v|_{1,K}\left(S_h^K(v,v)\right)^{1/2}.\end{aligned}$$

Next, for T_2, since $(\phi_v)_F=(v)_F$, it follows from (5.11), Lemma 5.4, Lemma 5.2, (5.9) and (4.2) that

$$\begin{aligned}T_2 &\le \left(\sum_{F\subset\partial K}h_K^{-1}h_F^2\|\nabla\widetilde{v}\cdot\boldsymbol{n}_K\|_{0,F}^2\right)^{1/2}\left(\sum_{F\subset\partial K}h_Kh_F^{-2}\big\|(v)_F-\phi_v\big\|_{0,F}^2\right)^{1/2}\\ &\le Ch_K^{1/2}|\nabla\widetilde{v}\cdot\boldsymbol{n}_K|_{-1/2,\partial K}|\phi_v|_{1,\partial K}\le Ch_K^{1/2}|\widetilde{v}|_{1,K}|\phi_v|_{1,\partial K}\\ &\le C|v|_{1,K}\left(S_h^K(v,v)\right)^{1/2}.\end{aligned}$$

For T_3, using (5.11), Lemma 5.4, Lemma 5.2 and (5.9), we obtain

$$\begin{aligned}T_3 &\le \left(\sum_{F\subset\partial K}h_K^{-1}h_F^2\|\nabla\widetilde{v}\cdot\boldsymbol{n}_K\|_{0,F}^2\right)^{1/2}\left(\sum_{F\subset\partial K}h_Kh_F^{-2}\big\|\Pi^{0,F}_{k-1}v-(v)_F\big\|_{0,F}^2\right)^{1/2}\\ &\le C|\nabla\widetilde{v}\cdot\boldsymbol{n}_K|_{-1/2,\partial K}\left(S_h^K(v,v)\right)^{1/2}\le C|v|_{1,K}\left(S_h^K(v,v)\right)^{1/2}.\end{aligned}$$

Plugging the estimates for T_1, T_2 and T_3 into (5.10), we conclude the proof. □

The coercivity of $a_h^K(\cdot,\cdot)$ where $S_h^K(\cdot,\cdot)$ is given by (5.8) follows from the previous lemma. Its proof can be done by proceeding as in Lemma 3.3 in [5].

Theorem 5.7. *Suppose that the stability form $S_h^K(\cdot,\cdot)$ is given by (5.8). There exists a constant $C>0$ depending only on ρ and k such that*

$$|v_h|_{1,h}\le C|\!|\!|v_h|\!|\!| \quad \forall v_h\in V_h^k(\Omega).$$

We will next prove that the stability form given by (5.8) satisfies (3.6).

Lemma 5.8. *Let $F\subset\partial K$, and let $\boldsymbol{a}_+$ and $\boldsymbol{a}_-$ be the end points of F. For $s=+,-$, let $F_s\subset\partial K$ be the face having $\boldsymbol{a}_s$ as a common end point with F. Then there exists a constant $C>0$ depending only on ρ such that*

$$\|v-\phi_v\|_{0,F}^2\le Ch_F^2\big(|v|_{1,F}^2+|v|_{1,F_+}^2+|v|_{1,F_-}^2\big)\quad\forall v\in H^{3/2}(K).$$

Proof. Let $v \in H^{3/2}(K)$. Then $v|_F \in H^1(F)$ for each face F in ∂K by the trace theorem, and v is continuous on ∂K by the Sobolev embedding theorem (see, e.g., [26, 36]). Let $F \subset \partial K$. Let $\psi_0, \psi_+, \psi_- \in \mathbb{P}_2(F)$ be the polynomials such that

(a) $\psi_0(\boldsymbol{a}_+) = 0$, $\psi_0(\boldsymbol{a}_-) = 0$, and $(\psi_0)_F = 1$;
(b) $\psi_+(\boldsymbol{a}_+) = 1$, $\psi_+(\boldsymbol{a}_-) = 0$, and $(\psi_+)_F = 0$;
(c) $\psi_-(\boldsymbol{a}_+) = 0$, $\psi_-(\boldsymbol{a}_-) = 1$, and $(\psi_-)_F = 0$.

Then these polynomials form a basis of $\mathbb{P}_2(F)$. Moreover, we have

$$\psi_- + \psi_0 + \psi_+ = 1, \tag{5.12}$$

$$\|\psi_s\|_{L^\infty(F)} \le C \quad \forall s = +, 0, -, \tag{5.13}$$

$$\phi_v = \phi_v(\boldsymbol{a}_+)\psi_+ + (v)_F\psi_0 + \phi_v(\boldsymbol{a}_-)\psi_-. \tag{5.14}$$

Note that, from the fundamental theorem of calculus and Hölder's inequality,

$$|v(\boldsymbol{x}) - v(\boldsymbol{a}_s)| \le \int_F |\partial_s v| \,\mathrm{d}s \le h_F^{1/2}|v|_{1,F} \quad \forall \boldsymbol{x} \in F,\ s = +, -,$$

Then we have

$$\|v - v(\boldsymbol{a}_s)\|_{0,F}^2 = \int_F |v(\boldsymbol{x}) - v(\boldsymbol{a}_s)|^2 \,\mathrm{d}s \le h_F^2|v|_{1,F}^2, \quad s = +, -. \tag{5.15}$$

Similarly, since $\boldsymbol{a}_s$ is also an end point of F_s, we also have

$$\|v - v(\boldsymbol{a}_s)\|_{0,F_s}^2 \le h_{F_s}^2|v|_{1,F_s}^2, \quad s = +, -. \tag{5.16}$$

Let $\psi_v = v(\boldsymbol{a}_+)\psi_+ + (v)_F\psi_0 + v(\boldsymbol{a}_-)\psi_-$. Using (5.12), (5.14), (5.13), (5.15) and (4.2), we have

$$\begin{aligned}
\|v - \psi_v\|_{0,F}^2 &= \big\|(v - v(\boldsymbol{a}_+))\psi_+ + (v - (v)_F)\psi_0 + (v - v(\boldsymbol{a}_-))\psi_-\big\|_{0,F}^2 \\
&\le C\left(\|v - v(\boldsymbol{a}_+)\|_{0,F}^2 + \|v - (v)_F\|_{0,F}^2 + \|v - v(\boldsymbol{a}_-)\|_{0,F}^2\right) \\
&\le Ch_F^2|v|_{1,F}^2.
\end{aligned} \tag{5.17}$$

Thus it suffices to estimate $\|\phi_v - \psi_v\|_{0,F}^2$. By (5.13),

$$\|\phi_v - \psi_v\|_{0,F}^2 \le Ch_F\big(\,|\phi_v(\boldsymbol{a}_+) - v(\boldsymbol{a}_+)|^2 + |\phi_v(\boldsymbol{a}_-) - v(\boldsymbol{a}_-)|^2\,\big). \tag{5.18}$$

Combining (5.15)-(5.16) with (4.2), we have

$$\begin{aligned}
|(v)_F - v(\boldsymbol{a}_s)|^2 &= h_F^{-1}\|(v)_F - v(\boldsymbol{a}_s)\|_{0,F}^2 \le Ch_F|v|_{1,F}^2, \\
|(v)_{F_s} - v(\boldsymbol{a}_s)|^2 &= h_{F_s}^{-1}\|(v)_{F_s} - v(\boldsymbol{a}_s)\|_{0,F_s}^2 \le Ch_{F_s}|v|_{1,F_s}^2, \quad s = +, -.
\end{aligned}$$

Using the above estimates, for $s = +, -$,

$$\begin{aligned}
|\phi_v(\boldsymbol{a}_s) - v(\boldsymbol{a}_s)|^2 &= \left|\frac{h_{F_s}}{h_F + h_{F_s}}\big((v)_F - v(\boldsymbol{a}_s)\big) + \frac{h_F}{h_F + h_{F_s}}\big((v)_{F_s} - v(\boldsymbol{a}_s)\big)\right|^2 \\
&\le \frac{2h_{F_s}^2}{(h_F + h_{F_s})^2}|(v)_F - v(\boldsymbol{a}_s)|^2 + \frac{2h_F^2}{(h_F + h_{F_s})^2}|(v)_{F_s} - v(\boldsymbol{a}_s)|^2 \\
&\le \frac{2h_{F_s}^2 h_F}{(h_F + h_{F_s})^2}|v|_{1,F}^2 + \frac{2h_F^2 h_{F_s}}{(h_F + h_{F_s})^2}|v|_{1,F_s}^2
\end{aligned}$$

Plugging the above inequality into (5.18), we finally obtain

$$\begin{aligned}
\|\phi_v - \psi_v\|_{0,F}^2 &\le C h_F \big(|\phi_v(\boldsymbol{a}_+) - v(\boldsymbol{a}_+)|^2 + |\phi_v(\boldsymbol{a}_-) - v(\boldsymbol{a}_-)|^2 \big) \\
&\le C \sum_{s=+,-} \left(\frac{h_{F_s}^2 h_F^2}{(h_F + h_{F_s})^2} |v|_{1,F}^2 + \frac{h_F^3 h_{F_+}}{(h_F + h_{F_+})^2} |v|_{1,F_+}^2 + \frac{h_F^3 h_{F_-}}{(h_F + h_{F_-})^2} |v|_{1,F_-}^2 \right) \\
&\le C h_F^2 \big(|v|_{1,F}^2 + |v|_{1,F_+}^2 + |v|_{1,F_-}^2 \big).
\end{aligned}$$

Now the conclusion follows from the estimate above and (5.17). ☐

Lemma 5.9. *Let $1 \le \ell \le k$. Then there exists a constant $C > 0$ depending only on ρ and k such that*

$$S_h^K\big(v - \Pi_k^{\nabla,K} v, v - \Pi_k^{\nabla,K} v\big) \le C h_K^{2\ell} |v|_{\ell+1,K}^2 \quad \forall v \in H^{\ell+1}(K).$$

Proof. Let $v \in H^{\ell+1}(K)$ and $\xi = v - \Pi_k^{\nabla,K} v$. Then

$$S_h^K(\xi,\xi) = h_K |\phi_\xi|_{1,\partial K}^2 + h_K \sum_{F \subset \partial K} h_F^{-2} \big\| \Pi_{k-1}^{0,F} \xi - (\xi)_F \big\|_{0,F}^2.$$

Using an inverse estimate for polynomials, Lemma 5.8 and (4.2), we have

$$\begin{aligned}
h_K |\phi_\xi|_{1,\partial K}^2 &= h_K \sum_{F \subset \partial K} |\phi_\xi - (\xi)_F|_{1,F}^2 \le C h_K \sum_{F \subset \partial K} h_F^{-2} \|\phi_\xi - (\xi)_F\|_{0,F}^2 \\
&\le C h_K \sum_{F \subset \partial K} h_F^{-2} \left(\|\phi_\xi - \xi\|_{0,F}^2 + \|\xi - (\xi)_F\|_{0,F}^2 \right) \le C h_K \sum_{F \subset \partial K} |\xi|_{1,F}^2.
\end{aligned}$$

Using (4.2) again, we have

$$\begin{aligned}
h_K \sum_{F \subset \partial K} h_F^{-2} \big\| \Pi_{k-1}^{0,F} \xi - (\xi)_F \big\|_{0,F}^2 &= h_K \sum_{F \subset \partial K} h_F^{-2} \big\| \Pi_{k-1}^{0,F} \xi - \Pi_{k-1}^{0,F} (\xi)_F \big\|_{0,F}^2 \\
&\le h_K \sum_{F \subset \partial K} h_F^{-2} \|\xi - (\xi)_F\|_{0,F}^2 \le C h_K \sum_{F \subset \partial K} |\xi|_{1,F}^2.
\end{aligned}$$

Thus, it follows from (4.1) (applied to $\nabla \xi$) and the estimate (4.3) that

$$\begin{aligned}
S_h^K(\xi,\xi) &\le C h_K \sum_{F \subset \partial K} |\xi|_{1,F}^2 \le C h_K \|\nabla \xi\|_{0,\partial K}^2 \le C \left(|\xi|_{1,K}^2 + h_K^2 |\xi|_{2,K}^2 \right) \\
&\le C h_K^{2\ell} |v|_{\ell+1,K}^2.
\end{aligned}$$

This completes the proof of the lemma. ☐

Now we immediately obtain (3.6) from Lemma 5.9, as follows.

Theorem 5.10. *Suppose that $S_h^K(\cdot,\cdot)$ is given by (5.8). Let $1 \le \ell \le k$. Then there exists a constant $C > 0$ depending only on ρ and k such that*

$$|||u - \Pi_k^{\nabla} u||| \le C h^\ell |u|_{\ell+1,\Omega} \quad \forall u \in H^{\ell+1}(\Omega).$$

5.3. **Stabilization in three dimensions.** For the three-dimensional case, we consider the stability bilinear form $S_h^K(\cdot,\cdot)$ given by

$$S_h^K(u,v) = h_K \sum_{F \subset \partial K} h_F^{-2} \left(\Pi_{k-1}^{0,F} u, \Pi_{k-1}^{0,F} v \right)_{0,F}. \tag{5.19}$$

We will show that the above bilinear form satisfies the properties (3.5), (3.6) and (3.7).

Note that the stability form given by (5.19) satisfies (3.7), since the face moments (3.1) of u and $I_h u$ are identical.

The following assumption will be used to prove (3.6).

Assumption 5.11. There exists a positive integer N independent of h such that every element in $\mathcal{P}_h$ has at most N faces.

Lemma 5.12. *There exists a constant $C > 0$ depending only on ρ and k such that*

$$|v|_{1,K}^2 \le C S_h^K(v,v) \quad \forall v \in V_h^k(K) \text{ with } \Pi_k^{\nabla,K} v = 0.$$

Proof. Let $v \in V_h^k(K)$ satisfy $\Pi_k^{\nabla,K} v = 0$. By Lemma 5.1, there exists $q \in \mathbb{P}_k(K)$ such that $\Delta q = \Delta v$ and $|q|_{1,K} \le C|v|_{1,K}$. Using (5.7), we get

$$\begin{aligned} |v|_{1,K}^2 &= \sum_{F\subset\partial K} \int_F (\nabla(v-q)\cdot \boldsymbol{n}_K)\big(\Pi_{k-1}^{0,F} v - (v)_{\partial K}\big)\,\mathrm{d}s \\ &= \sum_{F\subset\partial K} \int_F (\nabla(v-q)\cdot \boldsymbol{n}_K)\big(\Pi_{k-1}^{0,F} v\big)\,\mathrm{d}s \end{aligned} \tag{5.20}$$

where the last equality follows from

$$0 = \int_K \nabla(v-q)\cdot\nabla 1\,\mathrm{d}\boldsymbol{x} = \int_{\partial K} (\nabla(v-q)\cdot\boldsymbol{n}_K)\,\mathrm{d}s. \tag{5.21}$$

Combining (5.21), Lemma 5.4 and Lemma 5.2, together with the inequality $|q|_{1,K} \le C|v|_{1,K}$, we obtain

$$\begin{aligned} &\sum_{F\subset\partial K} \int_F (\nabla(v-q)\cdot \boldsymbol{n}_K)\big(\Pi_{k-1}^{0,F} v\big)\,\mathrm{d}s \\ &\quad\le \left(\sum_{F\subset\partial K} \frac{h_F^2}{h_K}\|\nabla(v-q)\cdot\boldsymbol{n}_K\|_{0,F}^2\right)^{\frac12} \left(\sum_{F\subset\partial K}\frac{h_K}{h_F^2}\|\Pi_{k-1}^{0,F} v\|_{0,F}^2\right)^{\frac12} \\ &\quad\le C|\nabla(v-q)\cdot\boldsymbol{n}_K|_{-1/2,\partial K}\left(S_h^K(v,v)\right)^{\frac12} \le C|v|_{1,K}\left(S_h^K(v,v)\right)^{\frac12}. \end{aligned} \tag{5.22}$$

Now the assertion of the lemma follows by plugging (5.22) into (5.20). ☐

Now the coercivity of $a_h^K(\cdot,\cdot)$ with (5.19) follows from the previous lemma. Its proof can be done by proceeding as in Lemma 3.3 in [5].

Theorem 5.13. *Suppose that the stability form $S_h^K(\cdot,\cdot)$ is given by (5.19). There exists a constant $C > 0$ depending only on ρ and k such that*

$$|v_h|_{1,h} \le C|\!|\!|v_h|\!|\!| \quad \forall v_h \in V_h^k(\Omega).$$

We next show that the stability form (5.19) satisfies (3.6).

Lemma 5.14. *Suppose that Assumption 5.11 holds. Let $1 \le \ell \le k$. Then there exists a constant $C > 0$ depending only on ρ, k, and N such that*

$$S_h^K\big(v - \Pi_k^{\nabla,K} v, v - \Pi_k^{\nabla,K} v\big) \le C h_K^{2\ell}|v|_{\ell+1,K}^2 \quad \forall v \in H^{\ell+1}(K).$$

Proof. Let $v \in H^{\ell+1}(K)$ and let $\xi = v - \Pi_k^{\nabla,K} v$. It follows from the standard Sobolev embedding theorem that $\xi \in C^0(\overline{K})$. By Assumption 5.11,

$$\begin{aligned} S_h^K(\xi,\xi) &= h_K \sum_{F\subset\partial K} h_F^{-2}\|\Pi_{k-1}^{0,F}\xi\|_{0,F}^2 \le h_K \sum_{F\subset\partial K}\|\xi\|_{L^\infty(F)}^2 \\ &\le C h_K\|\xi\|_{L^\infty(\partial K)}^2 \le C h_K \|\xi\|_{L^\infty(K)}^2. \end{aligned} \tag{5.23}$$

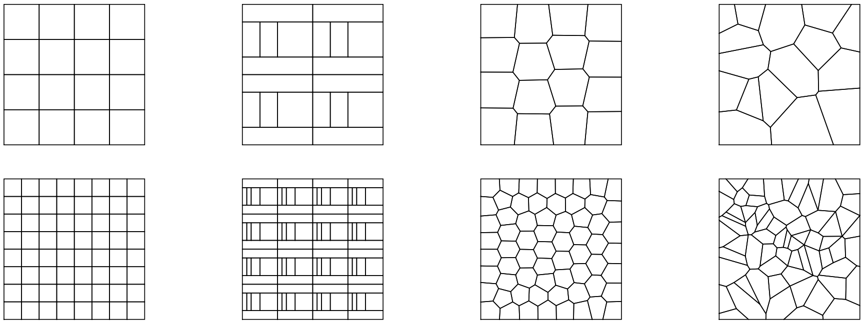

FIGURE 2. The meshes M1-2d, M2-2d, M3-2d and M4-2d.

Since $\xi \in H^2(K)$, it follows from the Sobolev inequality (see, e.g., [26,28]) and (4.3) that

$$h_K \|\xi\|^2_{L^\infty(K)} \le C h_K^{-2} \|\xi\|^2_{0,K} + |\xi|^2_{1,K} + h_K^2 |\xi|^2_{2,K} \le C h_K^{2\ell} |v|^2_{\ell+1,K}.$$

Now the inequality above together with (5.23) concludes the proof. □

The above lemma directly implies (3.6), as follows.

Theorem 5.15. *Suppose that Assumption 5.11 holds. Let $1 \le \ell \le k$. Consider the stability form $S_h^K(\cdot,\cdot)$ given by (5.19). Then there exists a constant $C > 0$ depending only on ρ, k and N such that*

$$\left|\!\left|\!\left| u - \Pi_k^\nabla u \right|\!\right|\!\right| \le C h_K^\ell |u|_{\ell+1,\Omega} \quad \forall u \in H^{\ell+1}(\Omega).$$

6. NUMERICAL EXPERIMENTS

In this section, we present some numerical experiments to confirm our theoretical analysis and compare the performance of the stability forms: (i) the standard one introduced in [5], and (ii) the new one given in (5.8) (2D case) or (5.19) (3D case). Note that the standard stability form is defined by

$$S_h^K(v_h, w_h) := h_K^{d-2} \sum_{i=1}^{N_K} \chi_i(v_h)\chi_i(w_h), \quad v_h, w_h \in V_h^k(K), \tag{6.1}$$

where N_K is the number of local degrees of freedom of $V_h^k(K)$, and χ_i is the operator associated with the i-th local degree of freedom of $V_h^k(K)$.

6.1. Test case 1: two-dimensional case.

In this test, we solve the problem (2.1) on $\Omega = (0,1)^2$ where the exact solution is given by

$$u(x,y) = x^5 + y^5 + (x-y)\exp(x+y), \quad (x,y) \in \Omega.$$

We consider four different types of meshes $\mathcal{P}_h$ with $h \approx 1/2^2, 1/2^3, \cdots, 1/2^6$:

- M1-2d: uniform rectangular meshes;
- M2-2d: Jenga meshes (see [61]);
- M3-2d: centroidal Voronoi tessellation meshes obtained by PolyMesher [62];
- M4-2d: Voronoi tessellation meshes associated with randomly distributed points inside Ω.

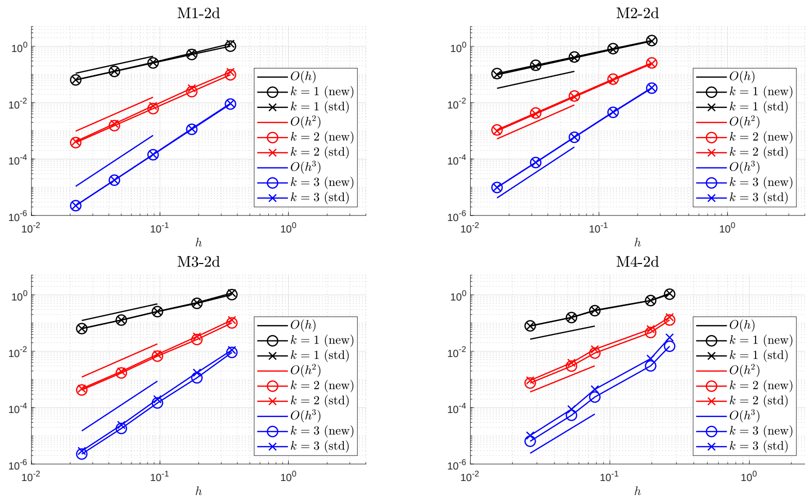

FIGURE 3. Test case 1: error curves.

Some examples of the meshes are shown in Figure 2.

We compute the discrete solution u_h^s and u_h^n of the nonconforming VEM for $k = 1, 2, 3$, where the stability form is chosen as (6.1) and (5.8), respectively. We then measure and report in Figure 3 the errors in the broken H^1-seminorm

$$e_{s,k} = \left|u - \Pi_k^\nabla u_h^{s,k}\right|_{1,h}, \quad e_{n,k} = \left|u - \Pi_k^\nabla u_h^{n,k}\right|_{1,h}, \tag{6.2}$$

labeled “std” and “new”, respectively. We observe that both the errors $e_{s,k}$ and $e_{n,k}$ behave similarly and converge to zero with rate $O(h^k)$. In particular, the behavior of the error curve $e_{n,k}$ is consistent with our theoretical analysis given in Sections 4 to 5.

6.2. **Test case 2: three-dimensional case.** In this test, we solve the problem (2.1) on $\Omega = (0,1)^3$ where the exact solution is given by

$$u(x, y, z) = xyz \sin(\pi x) \sin(\pi y) \sin(\pi z) - 10 \log(1 + x + y + z), \quad (x, y, z) \in \Omega.$$

We consider four different sequences of meshes $\mathcal{P}_h$ with $h \approx 1/4,\ 1/6,\ \cdots,\ 1/12$:

- M1-3d: uniform cubic meshes;
- M2-3d: uniform hexahedral meshes with small faces;
- M3-3d: centroidal Voronoi tessellation meshes generated by the Lloyd algorithm [47];
- M4-3d: Voronoi tessellation meshes associated with randomly distributed points inside Ω.

Some examples of the meshes are shown in Figure 4. In M2-3d, the ratio between the maximum and minimum diameters of the faces is chosen as $1/h$, which blows up as h goes to zero.

We compute the discrete solution $u_h^{s,k}$ and $u_h^{n,k}$ of the nonconforming VEM for $k = 1, 2, 3$, where the stability form is chosen as (6.1) and (5.19), respectively. We then measure and report in Figure 5 the errors in the broken H^1-seminorm (6.2), labeled “std” and “new” respectively. We also report in Table 1 the convergence rates for $h = 1/12$. We observe that the new stability form (5.19) exhibits the

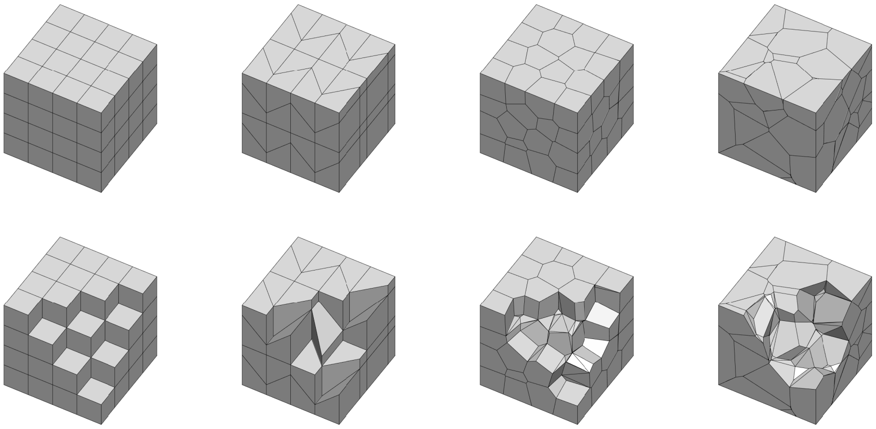

FIGURE 4. The meshes M1-3d, M2-3d, M3-3d and M4-3d.

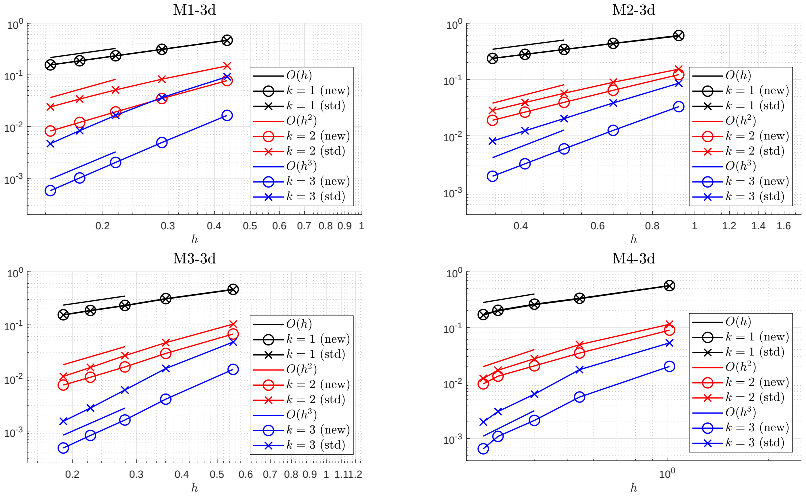

FIGURE 5. Test case 2: error curves.

optimal convergence rate for all the cases, while the standard stability form (5.19) does not for the case when $k = 3$ and the mesh is M2-3d. Furthermore, as shown in Figure 5 and Table 2, for $k = 2, 3$, the errors of the VEM with the new stability form is smaller than those of the standard stability form, and the ratio $e_{n,k}/e_{s,k}$ for $k = 3$ is smaller than the one for $k = 2$. This result shows that the standard stability form may not perform well for high-order schemes in the three-dimensional case, as in the conforming VEM [15, 42]. However, our new stability form seems robust for all the cases. Further investigation is needed, but it is beyond the scope of this paper. We leave it for a future work.

Table 1. Test case 2: convergence rates.

	new				std			
	M1-3d	M2-3d	M3-3d	M4-3d	M1-3d	M2-3d	M3-3d	M4-3d
$k = 1$	1.00	0.98	1.09	1.75	1.00	0.96	1.09	1.74
$k = 2$	2.10	1.95	1.97	3.15	1.94	1.90	2.26	3.39
$k = 3$	3.10	2.98	3.17	5.06	3.17	2.46	3.35	4.26

Table 2. Test case 2: the ratio $e_{n,k}/e_{s,k}$ for $k = 2, 3$.

h	$k = 2$				$k = 3$			
	M1-3d	M2-3d	M3-3d	M4-3d	M1-3d	M2-3d	M3-3d	M4-3d
1/4	0.52	0.80	0.65	0.79	0.18	0.38	0.31	0.38
1/6	0.42	0.72	0.63	0.70	0.13	0.33	0.26	0.32
1/8	0.38	0.69	0.61	0.74	0.12	0.29	0.27	0.34
1/10	0.35	0.68	0.65	0.78	0.12	0.26	0.30	0.36
1/12	0.34	0.67	0.69	0.80	0.12	0.24	0.31	0.33

7. Conclusion

We proposed new stability forms for 2D and 3D nonconforming VEMs where the underlying mesh may have arbitrarily small edges or faces. For the 2D case, the stability form is defined by the sum of an inner product of approximate tangential derivatives and a weighed L^2-inner product of certain projections on the mesh element boundaries. For the 3D case, the stability form is defined by a weighted L^2-inner product on the mesh element boundaries. We proved the optimal convergence of the nonconforming VEMs equipped with such stability forms under the mesh assumptions weaker than the usual one. We finally provided some numerical experiments that confirm our analysis and compare the performance of the proposed stability forms with the standard stability form. In the experiments, we observed that our proposed stability form performs as expected in the theoretical analysis.

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