

Mixed virtual volume methods for elliptic problems

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ABSTRACT

We develop a class of mixed virtual volume methods for elliptic problems on polygonal/polyhedral grids. Unlike the mixed virtual element methods introduced in [22,13], our methods are reduced to symmetric, positive definite problems for the primary variable without using Lagrangian multipliers. We start from the usual way of changing the given equation into a mixed system using the Darcy's law, $\mathbf{u} = -\mathcal{K}\nabla p$. By integrating the system of equations with some judiciously chosen test spaces on each element, we define new mixed virtual volume methods of all orders. We show that these new schemes are equivalent to the nonconforming virtual element methods for the primal variable p .

Once the primary variable is computed solving the symmetric, positive definite system, all the degrees of freedom for the Darcy velocity are locally computed. Also, the L^2 -projection onto the polynomial space is easy to compute. Hence our work opens an easy way to compute Darcy velocity on the polygonal/polyhedral grids. For the lowest order case, we give a formula to compute a Raviart-Thomas space like representation which satisfies the conservation law.

An optimal error analysis is carried out and numerical results are presented which support the theory.

1. Introduction

The virtual element method (VEM), introduced by Beirão da Veiga, et al. [7], is a generalization of the conventional finite element method to general polygonal (or polyhedral) meshes, where thorough error analysis and numerical tests for more general cases for elliptic problems were developed in [7,1,9,10,33,25]. VEM is similar to the mimetic finite difference method (MFD) [20,17,21,23,6] in the sense of flexibility of mesh handling and using degrees of freedom only to construct the bilinear form. However, MFD does not use basis functions while VEM assumes basis functions as solutions of local partial differential equations. The word *virtual* comes from the fact that no explicit knowledge of the shape function is necessary. By designing suitable elliptic projection operators on the local approximation space, VEM can be implemented using only the degrees of freedom and the polynomial part of the approximation space, while the integration of source-term multiplied by virtual element test function on the right hand sides is carefully handled using certain L^2 -projection (see [1]).

The detailed guidelines for the implementation of VEM for elliptic problems including the construction of the projection operators can

be found in [9,40]. Also, nonconforming versions of VEM were studied in [33,25]. Recently, p - and hp -versions of VEM were analyzed in [11,38,14,26]. The developments and theories of VEM for elasticity problems and Stokes problems can be found in [8,34,16,4,5,41] and [3,24], respectively. Meanwhile, couplings of VEM with boundary element methods were studied in [35,31,32].

On the other hand, the idea of VEM was extended to the $H(\text{div})$ -conforming space on general polygons/polyhedral, called the mixed virtual element method (MVEM) in [22,12,13], where the approximation spaces for the vector variables have degrees of freedom similar to those of Brezzi–Douglas–Marini (BDM) [19] or Raviart-Thomas (RT) space [39].

The inner product term in the MVEM is defined through an L^2 -projection, thus the computations of the local integral are possible from the knowledge of degrees of freedom of elements, plus a stabilizing term which makes it compatible with ordinary inner product. The MVEM leads to a saddle point problem similar to that of the mixed finite element methods, which is a disadvantage of the mixed FEM. Thus, it is necessary to devise a fast solution method for the algebraic equations arising from the mixed formulation of VEM. For example, an Uzawa

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type of solver may be used, or a hybridization technique as in [2,19,30] can be employed. Still, the resulting system involving the Lagrange multipliers is nontrivial to solve; one has to invert the local matrix to find the Schur complement.

In this paper, we develop new mixed VEM formulations for two and three dimensional problems along the line of mixed finite volume method (MFVM) introduced in [28,36], where for the momentum equation, the gradient of test functions of a nonconforming space and some subspace of polynomials are applied on each element, while the mass equation is tested by a space of polynomials. One of the advantages of the MFVM proposed in [28,36] is that the formulation can be converted to the nonconforming finite element method for the primary variable with modified forcing term. Once the primary variable is obtained from solving the symmetric positive definite system, the velocity variable can be recovered locally. Another advantage of this scheme is that the conservation of the momentum as well as the mass hold.

We develop a similar mixed volume formulation using virtual elements on general polygonal/polyhedral meshes, by modifying the weak formulation introduced in [36]. The $H(\text{div})$ -conforming VEM space in [12] or [13] is used for the vector variable, and the nonconforming VEM (NCVEM) space developed in [33] is used for the primary variable.

Our method is more naturally related to the NCVEM than MFEM is to nonconforming FEM, in the sense that the treatment of the forcing term is exactly the same as NCVEM (i.e., one uses the L^2 projection on the right hand side.)

As is usual in VEMs, the variation form involves elliptic projection operators and stability terms for the primary variables, see (3.5), (4.2a). By eliminating the velocity field from the first equation, we obtain an equation for the NCVEM in the primary variable. Once the primary variable is obtained by solving the symmetric, positive definite (SPD) system, all the moments of the velocity variable can be recovered locally. Also, one can compute the L^2 -projection of velocity variable easily. Thus, the whole process can be implemented efficiently, avoiding the saddle point problems. We name our method a **mixed virtual volume method** (MVVM).

The proposed method is the first success in MVEM to compute the $H(\text{div})$ -conforming velocity variables by solving SPD problems in the primary variable. Optimal error estimates for the proposed schemes are provided for 2D case. Numerical results supporting our analysis are presented. One may raise questions regarding the relationship of the proposed scheme with the reconstruction of velocity variable as in [37]. The possibility is discussed in Section 4.2. In the lowest order case, we propose a way to reconstruct Raviart - Thomas type velocity similar to [37] in general polygonal/polyhedral mesh.

The rest of our paper is organized as follows. The governing equation and brief review of MVEM are given in Section 2. In Section 3, we review the nonconforming virtual element methods for the variable coefficient. In Section 4, we introduce an MVVM and show that it is equivalent to the NCVEM. The error analysis is given in Section 5. The numerical tests supporting our analysis are given in Section 6. The conclusion follows in Section 7.

2. Preliminaries

Let Ω be a bounded polygonal/polyhedral domain in $\mathbb{R}^d, d = 2, 3$ with the boundary $\partial\Omega$. We consider the second-order elliptic boundary value problem

$$\begin{cases} -\text{div } \mathcal{K}\nabla p = g & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.1)$$

where \mathcal{K} is a smooth, bounded, symmetric and uniformly positive definite tensor.

We introduce some notations here: For any domain D , let $H^k(D)$ (or $\mathbf{H}^k(D)$) be the scalar and vector Sobolev spaces of order $k \geq 0$. We use the standard notations $|\cdot|_{k,D}, \|\cdot\|_{k,D}$ for the (semi)-norms on $H^k(D)$,

$\|\cdot\|_{\partial D}$ for the $L^2(\partial D)$, and $(\cdot, \cdot)_D$ for the L^2 inner product. When $D = \Omega$, we drop the subscript Ω and write $|\cdot|_k, \|\cdot\|_k$ instead. In two dimensions, we let

$$\text{rot } \mathbf{v} = \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \text{ and } \text{rot } q = \left(\frac{\partial q}{\partial y}, -\frac{\partial q}{\partial x} \right),$$

for smooth enough vector and scalar functions \mathbf{v} and q . Let

$$\mathbf{H}(\text{div}; D) = \{ \mathbf{u} \in (L^2(D))^d, (d = 2, 3) \text{ with } \text{div } \mathbf{u} \in L^2(D) \},$$

$$\mathbf{H}(\text{rot}; D) = \{ \mathbf{u} \in (L^2(D))^2, \text{ with } \text{rot } \mathbf{u} \in L^2(D) \},$$

$$\mathbf{H}(\text{curl}; D) = \{ \mathbf{u} \in (L^2(D))^3, \text{ with } \text{curl } \mathbf{u} \in (L^2(D))^3 \}.$$

Let us introduce the vector variable $\mathbf{u} = -\mathcal{K}\nabla p$ and rewrite problem (2.1) in the mixed form

$$\begin{cases} \mathbf{u} = -\mathcal{K}\nabla p & \text{in } \Omega, \\ \text{div } \mathbf{u} = g & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.2)$$

Throughout this paper, we assume the following regularity hold: The solution (\mathbf{u}, p) of (2.2) satisfies $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega), p \in H^{k+2}(\Omega)$, and there exists some constant $C > 0$ such that

$$\|\mathbf{u}\|_{k+1} + \|p\|_{k+2} \leq C \|g\|_k. \quad (2.3)$$

Its weak form is: Find $\mathbf{u} \in \mathbf{H}(\text{div}; D)$ and $p \in L^2(\Omega)$ such that

$$(\mathcal{K}^{-1} \mathbf{u}, \mathbf{v}) - (p, \text{div } \mathbf{v}) = 0, \quad \mathbf{v} \in \mathbf{H}(\text{div}; \Omega), \quad (2.4)$$

$$(\text{div } \mathbf{u}, q) = (g, q), \quad q \in L^2(\Omega). \quad (2.5)$$

2.1. Mixed virtual element methods

We briefly review the mixed virtual element methods (MVEM) introduced in [22,12,13]. Let \mathcal{T}_h be a decomposition of Ω into regular polygons/polyhedra, and let \mathcal{E}_h^o be the set of all interior edges (faces), \mathcal{E}_h^d be the set of boundary edges (faces), and $\mathcal{E}_h = \mathcal{E}_h^o \cup \mathcal{E}_h^d$. Following [13,33], we mean by “regular” that, there exists some $\rho > 0$ such that

- $h_f \geq \rho h_p$ holds for every element $\mathcal{P} \in \mathcal{T}_h$ and for every edge (face) $f \subset \partial\mathcal{P}$,
- every element \mathcal{P} is star-shaped with respect to all points of a sphere of radius $\geq \rho h_p$,
- when $d = 3$, every face $f \in \mathcal{E}_h$ is star-shaped with respect to all points of a sphere of radius $\geq \rho h_f$,

where h_f (resp. h_p) is the diameter of edge (face) f (resp. \mathcal{P}). We denote the maximum diameter of elements $\mathcal{P} \in \mathcal{T}_h$ by h . Throughout the paper, the constants C, C_* and C^* will be independent of mesh size h , not necessarily the same for each occurrence.

For any integer $k \geq 0$, we denote by $P_k(D)$ the set of all polynomials of total degree less than or equal to k , and set $P_{-1}(D) = \{0\}$. Also, we let the scaled polynomials:

$$M_k(D) = \left\{ \left(\frac{\mathbf{x} - \mathbf{x}_D}{h_D} \right)^\alpha, |\alpha| \leq k \right\}, \quad (2.6)$$

where $\alpha = (\alpha_1, \dots, \alpha_d)$ ($d = 2, 3$) is the multi-index and \mathbf{x}_D is the center of mass.

Let

$$\mathcal{G}_k(\mathcal{P}) := \nabla P_{k+1}(\mathcal{P}),$$

$$\mathcal{G}_k(\mathcal{P})^\perp := \text{orthogonal complement of } \mathcal{G}_k(\mathcal{P}) \text{ in } (P_k(\mathcal{P}))^d,$$

$$\mathcal{R}_k(\mathcal{P}) := \text{curl}(P_{k+1}(\mathcal{P}))^3 \text{ if } d = 3 \text{ and } \text{rot}(P_{k+1}(\mathcal{P}))^2 \text{ if } d = 2.$$

If we let $\pi_{k,d}$ be the dimension of $P_k(\mathbb{R}^d)$, then we see

$$\dim \mathcal{G}_k(\mathcal{P}) = \pi_{k+1,d} - 1, \dim \mathcal{G}_k(\mathcal{P})^\perp = d\pi_{k,d} - \pi_{k+1,d} + 1. \tag{2.7}$$

Given $\mathcal{P} \in \mathcal{T}_h$, the local $\mathbf{H}(\text{div})$ -conforming virtual element space is defined as follows:

$$\mathbf{V}_h^k(\mathcal{P}) := \{ \mathbf{v} \in \mathbf{H}(\text{div}; \mathcal{P}) \cap \mathbf{H}(\text{curl}; \mathcal{P}) : \mathbf{v} \cdot \mathbf{n}|_f \in P_k(f), \forall \text{ edges (faces)} \\ f \subset \partial \mathcal{P}, \text{div } \mathbf{v} \in P_k(\mathcal{P}), \text{curl } \mathbf{v} \in \mathcal{R}_{k-1}(\mathcal{P}) \}, \tag{2.8}$$

where in two dimensional case, the 'curl' operator is replaced by the 'rot' operator and the space $\mathcal{R}_{k-1}(\mathcal{P})$ is replaced by $P_{k-1}(\mathcal{P})$.

The global space of order k is the space \mathbf{V}_h^k defined as

$$\mathbf{V}_h^k = \{ \mathbf{v} \in \mathbf{H}(\text{div}; \Omega) : \mathbf{v}|_{\mathcal{P}} \in \mathbf{V}_h^k(\mathcal{P}), \forall \mathcal{P} \in \mathcal{T}_h \}. \tag{2.9}$$

The degrees of freedom for \mathbf{V}_h^k are

$$\frac{1}{|f|} \int_f \mathbf{v} \cdot \mathbf{n} \xi_k ds, \quad \forall \xi_k \in M_k(f), \forall f \in \mathcal{E}_h, \tag{2.10}$$

$$\frac{1}{|\mathcal{P}|} \int_{\mathcal{P}} \mathbf{v} \cdot \mathbf{g}_{k-1} dx, \quad \forall \mathbf{g}_{k-1} \in \mathcal{G}_{k-1}(\mathcal{P}), \forall \mathcal{P} \in \mathcal{T}_h, \tag{2.11}$$

$$\frac{1}{|\mathcal{P}|} \int_{\mathcal{P}} \mathbf{v} \cdot \mathbf{g}_k^\perp dx, \quad \forall \mathbf{g}_k^\perp \in \mathcal{G}_k(\mathcal{P})^\perp, \forall \mathcal{P} \in \mathcal{T}_h. \tag{2.12}$$

Here, $|\cdot|$ for any geometrical object means its Lebesgue measure and $g_k, \mathbf{g}_{k-1}, \mathbf{g}_k^\perp$ are taken from the scaled monomials. Let $\Psi_h(\mathcal{P}) = \mathcal{G}_{k-1}(\mathcal{P}) \oplus \mathcal{G}_k^\perp(\mathcal{P})$. The conditions (2.11), (2.12) can be replaced by a single condition.

$$\frac{1}{|\mathcal{P}|} \int_{\mathcal{P}} \mathbf{v} \cdot \mathbf{g} dx, \quad \forall \mathbf{g} \in \Psi_h(\mathcal{P}), \forall \mathcal{P} \in \mathcal{T}_h.$$

The pressure space is

$$W_h^k := \{ q \in L^2(\Omega), q|_{\mathcal{P}} \in P_k(\mathcal{P}) \}.$$

Remark 2.1. Let $k \geq 1$. Replacing the condition $\text{div } \mathbf{v} \in P_k(\mathcal{P})$ by $\text{div } \mathbf{v} \in P_{k-1}(\mathcal{P})$ in (2.8) and replacing $k - 1$ by $k - 2$ in (2.11), we obtain a BDM like virtual element space defined in [12]. However, we get $\mathcal{O}(h^k)$ instead of $\mathcal{O}(h^{k+1})$ in $H(\text{div})$ -norm. See Remark 4.1 in Section 4.

2.2. Interpolations and L^2 -projections

The L^2 -projection operators $\Pi_k^0 : L^2(\mathcal{P}) \rightarrow P_k(\mathcal{P})$ and $\mathbf{\Pi}_k^0 : (L^2(\mathcal{P}))^d \rightarrow (P_k(\mathcal{P}))^d$ are defined as follows: On each \mathcal{P} , we define

$$\begin{cases} \int_{\mathcal{P}} (q - \Pi_k^0 q) \xi_k dx = 0, & \forall \xi_k \in P_k(\mathcal{P}), \\ \int_{\mathcal{P}} (\mathbf{v} - \mathbf{\Pi}_k^0 \mathbf{v}) \mathbf{q}_k dx = 0, & \forall \mathbf{q}_k \in (P_k(\mathcal{P}))^d. \end{cases} \tag{2.13}$$

When no confusion arises, we use the same notations Π_k^0 and $\mathbf{\Pi}_k^0$ to denote the L^2 -projections from some virtual element spaces of $L^2(\Omega)$ or $(L^2(\Omega))^d$, although the computations are sometimes nontrivial (see the definition of nonconforming virtual spaces in the next section).

As is shown in [12], we can compute the L^2 -projection $\mathbf{\Pi}_k^0 \mathbf{v}$ for $\mathbf{v} \in \mathbf{V}_h^k$ from the degrees of freedom of \mathbf{v} and the following properties hold:

$$\|q - \Pi_k^0 q\|_0 \leq Ch^{k+1} |q|_{k+1}, \quad \|\mathbf{v} - \mathbf{\Pi}_k^0 \mathbf{v}\|_0 \leq Ch^{k+1} |\mathbf{v}|_{k+1}.$$

The local interpolation operator $\mathbf{\Pi}_k^F : (H^1(\mathcal{P}))^d \rightarrow \mathbf{V}_h^k(\mathcal{P})$ is defined by

$$\int_f (\mathbf{v} - \mathbf{\Pi}_k^F \mathbf{v}) \cdot \mathbf{n} \xi_k d\sigma = 0, \quad \forall \xi_k \in M_k(f), \tag{2.14}$$

$$\int_{\mathcal{P}} (\mathbf{v} - \mathbf{\Pi}_k^F \mathbf{v}) \cdot \mathbf{g} dx = 0, \quad \forall \mathbf{g} \in \Psi_h(\mathcal{P}). \tag{2.15}$$

Define bilinear forms (for vector variables)

$$\mathbf{a}_h^P(\mathbf{u}, \mathbf{v}) := (\mathcal{K} \mathbf{\Pi}_k^0 \mathbf{u}, \mathbf{\Pi}_k^0 \mathbf{v})_{\mathcal{P}} + \mathbf{S}^P(\mathbf{u} - \mathbf{\Pi}_k^0 \mathbf{u}, \mathbf{v} - \mathbf{\Pi}_k^0 \mathbf{v}) \tag{2.16}$$

and

$$\mathbf{a}_h(\mathbf{u}, \mathbf{v}) = \sum_{\mathcal{P}} \mathbf{a}_h^P(\mathbf{u}, \mathbf{v}), \tag{2.17}$$

where $\mathbf{S}^P(\mathbf{u}, \mathbf{v})$ is any bilinear form that scales with the inner product $(\mathcal{K} \cdot, \cdot)_{\mathcal{P}}$.

For $k \geq 0$, the MVEM is: Find $(\tilde{\mathbf{u}}_h, \tilde{p}_h) \in \mathbf{V}_h^k \times W_h^k$ such that

$$\mathbf{a}_h(\tilde{\mathbf{u}}_h, \mathbf{v}_h) - (\tilde{p}_h, \text{div } \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h^k, \tag{2.18a}$$

$$(\text{div } \tilde{\mathbf{u}}_h, q_h) = (g, q_h), \quad \forall q_h \in W_h^k. \tag{2.18b}$$

The following error estimates are given in [13].

Theorem 2.1. Under the assumptions above, the problem (2.18a,b) has a unique solution $(\tilde{\mathbf{u}}_h, \tilde{p}_h)$ and the following error estimates hold.

$$\|p - \tilde{p}_h\|_0 \leq Ch^{k+1} (\|\mathbf{u}\|_{k+1} + \|p\|_{k+1}),$$

$$\|\mathbf{u} - \tilde{\mathbf{u}}_h\|_0 \leq Ch^{k+1} \|\mathbf{u}\|_{k+1},$$

$$\|\mathbf{u} - \mathbf{\Pi}_k^F \mathbf{u}\|_0 \leq Ch^{k+1} \|\mathbf{u}\|_{k+1},$$

$$\|\text{div}(\mathbf{u} - \tilde{\mathbf{u}}_h)\|_0 \leq Ch^{k+1} |g|_{k+1}.$$

3. Nonconforming virtual element methods

We briefly describe NCVEM introduced in [33], [25].

We need a broken Sobolev space

$$H^1(\mathcal{T}_h) = \{ q \in L^2(\Omega) : q|_{\mathcal{P}} \in H^1(\mathcal{P}), \quad \forall \mathcal{P} \in \mathcal{T}_h \},$$

with a broken norm

$$\|q\|_{1,h}^2 = \sum_{\mathcal{P} \in \mathcal{T}_h} \|q\|_{1,\mathcal{P}}^2.$$

For each $f \in \mathcal{E}_h$, we associate a unit vector \mathbf{n}_f at f . We define the jump $[q]_f$ for $q \in H^1(\mathcal{T}_h)$ as follows:

$$[q]_f(\mathbf{x}) := \lim_{\delta \rightarrow 0^+} (q(\mathbf{x} - \delta \mathbf{n}_f) - q(\mathbf{x} + \delta \mathbf{n}_f)), \quad \text{if } f \in \mathcal{E}_h^o,$$

$$[q]_f(\mathbf{x}) := \lim_{\delta \rightarrow 0^+} (q(\mathbf{x} - \delta \mathbf{n}_{\partial \Omega})), \quad \text{if } f \in \mathcal{E}_h^{\partial},$$

where $\mathbf{n}_{\partial \Omega}$ is an outward unit vector on $\partial \Omega$. For positive integers $r = k + 1, (k \geq 0)$, we let

$$H^{1,nc}(\mathcal{T}_h; r) = \left\{ q \in H^1(\mathcal{T}_h) : \int_f [q]_f m d\sigma = 0, m \in P_{r-1}(f), \forall f \in \mathcal{E}_h^o \right\}. \tag{3.1}$$

In order to utilize the nonconforming virtual element space in the next section, we need to use an extended version of VEM as in [25]. The reason is to compute L^2 -projection onto the space P_r .

The local space for NCVEM on each $\mathcal{P} \in \mathcal{T}_h$ is defined as

$$N_h^r(\mathcal{P}) = \{ q \in W_h^r(\mathcal{P}) : (q - \mathbf{\Pi}_r^* q, m)_{\mathcal{P}} = 0, \forall m \in P_{r-1}(\mathcal{P}) \cup P_r(\mathcal{P}) \}, \tag{3.2}$$

where $W_h^r(\mathcal{P})$ is an auxiliary space defined by

$$W_h^r(\mathcal{P}) = \left\{ q \in H^1(\mathcal{P}) : \frac{\partial q}{\partial \mathbf{n}} \in P_{r-1}(f), \forall f \subset \partial \mathcal{P}, \Delta q \in P_r(\mathcal{P}) \right\},$$

and $\mathbf{\Pi}_r^*$ is a certain projection onto P_r that can be computed from the degrees of freedom. For example, one can use the elliptic projection $\mathbf{\Pi}_r^\nabla$ [7,25].

The global nonconforming virtual element space N_h^r is defined as

$$N_h^r = \left\{ q \in H^{1,nc}(\mathcal{T}_h; r) : q|_{\mathcal{P}} \in N_h^r(\mathcal{P}), \forall \mathcal{P} \in \mathcal{T}_h, \int_f q m d\sigma = 0, \forall m \in P_{r-1}(f), \forall f \subset \mathcal{E}_h^o \right\}. \tag{3.3}$$

The global d.o.f.s are given by the followings:

$$\begin{aligned} \mu_{f,\alpha}(q) &= \frac{1}{|f|} \int_f q m_\alpha d\sigma, \forall m_\alpha \in M_{r-1}(f), f \in \mathcal{E}_h^o, \\ \mu_{p,\alpha}(q) &= \frac{1}{|p|} \int_p q m_\alpha dx, \forall m_\alpha \in M_{r-2}(p), p \in \mathcal{T}_h. \end{aligned} \tag{3.4}$$

We define the usual elliptic bilinear forms (for scalar variables) $a^P : H^1(\mathcal{P}) \times H^1(\mathcal{P}) \rightarrow \mathbb{R}$ and $a : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$ as:

$$\begin{aligned} a^P(p, q) &= \int_p \mathcal{K} \nabla p \cdot \nabla q dx, \quad \forall p, q \in H^1(\mathcal{P}), \\ a(p, q) &= \sum_{p \in \mathcal{T}_h} a^P(p, q), \quad \forall p, q \in H^1(\Omega). \end{aligned}$$

Now we define a discrete bilinear form $a_h^P(\cdot, \cdot) : N_h^r \times N_h^r \rightarrow \mathbb{R}$:

$$a_h^P(p_h, q_h) = (\mathcal{K} \Pi_{r-1}^0 \nabla p_h, \Pi_{r-1}^0 \nabla q_h)_p + S^P((I - \Pi_r^0)p_h, (I - \Pi_r^0)q_h), \tag{3.5}$$

where S^P is any stabilizing term satisfying

$$C_* a_h^P(q_h, q_h) \leq S^P(q_h, q_h) \leq C^* a_h^P(q_h, q_h), \forall q_h \in \ker(\Pi_r^0).$$

We let

$$a_h(p_h, q_h) = \sum_p a_h^P(p_h, q_h), \forall p_h, q_h \in N_h^r.$$

Now the NCVEM of order $r \geq 1$ is defined as in [25]: Find $p_h \in N_h^r$ such that

$$a_h(p_h, q_h) = (\Pi_{r-1}^0 g, q_h), \tag{3.6}$$

The following optimal error estimate for (3.6) is given in Theorems 6.2 and 6.3 [25].

Theorem 3.1. *Let p and p_h be the solutions of (2.1) and (3.6). Assume $p \in H^{r+1}(\Omega)$, $g \in H^{r-1}(\Omega)$. Then, there exists a constant $C > 0$ independent of h such that*

$$\|p - p_h\|_0 + h \|p - p_h\|_{1,h} \leq Ch^{r+1} \|p\|_{r+1}.$$

Remark 3.1.

1. If we use $\Pi_{\max(r-2,0)}^0 g$ on the right hand side of (3.6), we can still get H^1 error estimate like

$$\|p - p_h\|_{1,h} \leq Ch^r (\|p\|_{r+1} + \|g\|_{r-1}),$$

but we do not get optimal L^2 -error estimate.

2. As is well known in VEM community, there are two choices of bilinear forms. We used the more general form (3.5) which works for variable coefficient. For constant coefficient \mathcal{K} , the form using elliptic projection

$$(\mathcal{K} \nabla \Pi^\nabla p_h, \nabla \Pi^\nabla q_h) + S^P((I - \Pi^\nabla)p_h, (I - \Pi^\nabla)q_h)$$

defined in [7,33] can be used.

4. Mixed virtual volume methods

Let $k \geq 0$. Assume that we have some $H(div)$ virtual element space \mathbf{V}_h^k and NCVEM space N_h^{k+1} (to be associated with \mathbf{V}_h^k).

We denote the number of edges (faces) of each element \mathcal{P} by n_p . We note the following type of Euler's formula:

$$\sum_{p \in \mathcal{T}_h} n_p = \sum_{p \in \mathcal{T}_h} \sum_{f \subset \partial p} 1 = 2 \sum_{f \in \mathcal{E}_h^o} 1 + \sum_{f \in \mathcal{E}_h^d} 1 = 2\#\mathcal{E}_h^o + \#\mathcal{E}_h^d. \tag{4.1}$$

Now we introduce our mixed virtual volume method (MVVM) for all order $k \geq 0$: Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h^k \times N_h^{k+1}$ which satisfies on every element $\mathcal{P} \in \mathcal{T}_h$,

$$\int_p \mathbf{u}_h \cdot \nabla \chi + a_h^P(p_h, \chi) = 0, \forall \chi \in N_h^{k+1}(\mathcal{P}), \tag{4.2a}$$

$$\int_p (\mathbf{u}_h + \mathcal{K} \Pi_k^0 \nabla p_h) \cdot \mathbf{v} dx = 0, \forall \mathbf{v} \in \mathcal{G}_k(\mathcal{P})^\perp, (k \geq 1) \tag{4.2b}$$

$$\int_p \operatorname{div} \mathbf{u}_h \phi dx = \int_p \Pi_k^0 g \phi dx, \forall \phi \in P_k(\mathcal{P}). \tag{4.2c}$$

From (4.2c) we have

$$\operatorname{div} \mathbf{u}_h = \Pi_k^0 g. \tag{4.3}$$

We see from (2.10), (2.11), (2.12) (using (2.7)), that the dimension of $\mathbf{V}_h^k \times N_h^{k+1}$ is

$$\pi_{k,d-1} \#\mathcal{E}_h + (\pi_{k,d} - 1 + d\pi_{k,d} - \pi_{k+1,d} + 1) \#\mathcal{T}_h + \pi_{k,d-1} \#\mathcal{E}_h^o + \pi_{k-1,d} \#\mathcal{T}_h \tag{4.4}$$

while the number of equations in (4.2) is

$$\begin{aligned} & \sum_{p \in \mathcal{T}_h} (\dim N_h(\mathcal{P}) - 1 + \dim \mathcal{G}_k^\perp(\mathcal{P}) + \dim P_k(\mathcal{P})) \\ &= \sum_{p \in \mathcal{T}_h} [\pi_{k,d-1} n_p + \pi_{k-1,d} - 1 + d\pi_{k,d} - \pi_{k+1,d} + 1 + \pi_{k,d}]. \end{aligned} \tag{4.5}$$

Using Euler's formula (4.1), we see

$$\#\mathcal{E}_h + \#\mathcal{E}_h^o = \sum_{p \in \mathcal{T}_h} n_p.$$

Hence we see (4.4) and (4.5) are equal, and hence (4.2) is a square system. Integration by parts gives

$$-\int_p \mathbf{u}_h \cdot \nabla \chi dx = -\int_{\partial p} \mathbf{u}_h \cdot \mathbf{n} \chi ds + \int_p \operatorname{div} \mathbf{u}_h \chi dx. \tag{4.6}$$

Summing over all \mathcal{P} , we have, by (4.2a) and (4.3)

$$\begin{aligned} a_h(p_h, \chi) &= -\sum_p \int_p \mathbf{u}_h \cdot \nabla \chi dx \\ &= -\sum_p \int_{\partial p} \mathbf{u}_h \cdot \mathbf{n} \chi ds + \sum_p \int_p \Pi_k^0 g \chi dx. \end{aligned} \tag{4.7}$$

Now assume $\chi \in N_h^{k+1}$. Since χ has continuous moments up to degree k across internal edges (faces) and has vanishing moments on $\partial\Omega$, we obtain

$$a_h(p_h, \chi) = (\Pi_k^0 g, \chi), \quad \chi \in N_h^{k+1}. \tag{4.8}$$

This is exactly NCVEM of order $k + 1$. Thus we have shown that our mixed virtual volume scheme is equivalent to the NCVEM.

Remark 4.1.

1. For $k \geq 1$, we can replace the test space in (4.2c) by $P_{k-1}(\mathcal{P})$ to obtain a scheme that corresponds to BDM like MVEM [12], for which we lose one order in $H(div)$ -norm.
2. We can allow each polygon to have different number of edges (faces). Similarly, the first term of the right hand side of (4.5) has to be changed exactly the same way. Thus the system is still a square system.

4.1. Recovery of \mathbf{u}_h and L^2 -projection

We see from (4.6), (4.3) and (4.2a) that for any $\chi \in N_h^{k+1}(\mathcal{P})$

$$\int_{\partial\mathcal{P}} \mathbf{u}_h \cdot \mathbf{n} \chi \, ds = (\Pi_k^0 g, \chi)_{\mathcal{P}} - a_h^{\mathcal{P}}(p_h, \chi). \tag{4.9}$$

Hence the moments of $\mathbf{u}_h \cdot \mathbf{n} \in P_k(f)$ can be obtained by choosing the basis functions $\chi \in N_h^{k+1}$ corresponding to the degrees of freedom.

The interior moments can be obtained similarly. Indeed, for $\mathbf{v} \in \mathcal{G}_{k-1}(\mathcal{P}) = \nabla P_k(\mathcal{P})$, we have $\mathbf{v} = \nabla q$, for some $q \in P_k(\mathcal{P})$. Hence from (4.2c), we have

$$\int_{\mathcal{P}} \mathbf{u}_h \cdot \nabla q \, dx = \int_{\partial\mathcal{P}} \mathbf{u}_h \cdot \mathbf{n} q \, d\sigma - \int_{\mathcal{P}} \Pi_k^0 g q \, dx, \tag{4.10}$$

which is computable from the moments of $\mathbf{u}_h \cdot \mathbf{n}$. Meanwhile for $\mathbf{v} \in \mathcal{G}_k(\mathcal{P})^\perp$, we see from (4.2b)

$$\begin{aligned} \int_{\mathcal{P}} \mathbf{u}_h \cdot \mathbf{v} \, dx &= - \int_{\mathcal{P}} \mathcal{K} \Pi_k^0 \nabla p_h \cdot \mathbf{v} \, dx \\ &= - \int_{\mathcal{P}} \nabla p_h \cdot \Pi_k^0(\mathcal{K} \mathbf{v}) \, dx \\ &= - \int_{\partial\mathcal{P}} p_h \Pi_k^0(\mathcal{K} \mathbf{v}) \cdot \mathbf{n} \, d\sigma + \int_{\mathcal{P}} p_h \operatorname{div}(\Pi_k^0(\mathcal{K} \mathbf{v})) \, dx \end{aligned} \tag{4.11}$$

which is computable from the d.o.f.s of p_h . Hence all the degrees of freedom of \mathbf{u}_h can be computed.

Furthermore, we can find the L^2 -projection of \mathbf{u}_h , at the same time. Since

$$(P_k(\mathcal{P}))^d = \mathcal{G}_k(\mathcal{P}) \oplus \mathcal{G}_k(\mathcal{P})^\perp,$$

and $\mathcal{G}_k = \nabla P_{k+1}(\mathcal{P}) \subset \nabla N^{k+1}(\mathcal{P})$, the same integration by parts as (4.10), (4.11) give enough information to compute the projection of \mathbf{u}_h onto $(P_k(\mathcal{P}))^d$. Indeed,

$$\begin{aligned} \int_{\mathcal{P}} \Pi_k^0 \mathbf{u}_h \cdot \mathbf{v} &= \int_{\mathcal{P}} \mathbf{u}_h \cdot \mathbf{v} \\ &= - \int_{\mathcal{P}} \mathcal{K} \Pi_k^0 \nabla p_h \cdot \mathbf{v} \end{aligned} \tag{4.12}$$

holds for all \mathbf{v} in $(P_k(\mathcal{P}))^d$. Hence one can easily find the projection of \mathbf{u}_h onto $(P_k(\mathcal{P}))^d$.

4.2. Construction of Raviart-Thomas type approximation for $k = 0$

It is clear that when $k = 0$, MVVM and MFEM are equivalent on triangular/tetrahedral grids. In the case of MFEM, it is known [37] that, if \mathcal{K} is piecewise constant, then

$$\mathbf{u}_h = -\mathcal{K} \nabla p_h + \frac{\bar{g}}{d}(\mathbf{x} - \mathbf{x}_p), \quad d = 2, 3, \tag{4.13}$$

where \bar{g} is the average of g on each element \mathcal{P} . The same formula holds for MVVM.

On general grids, we have a similar representation (even when \mathcal{K} is nonconstant). We see from (4.2c), that $\operatorname{div}(\mathbf{u}_h - \frac{\bar{g}}{d}(\mathbf{x} - \mathbf{x}_p)) = 0$, $d = 2, 3$. Hence

$$\mathbf{u}_h = \frac{\bar{g}}{d}(\mathbf{x} - \mathbf{x}_p) + \operatorname{curl} \xi \text{ (rot } \xi \text{ if } d = 2), \text{ for some } \xi \in H^1(\mathcal{P}).$$

Substituting this into (4.2a), and letting $\chi = x, y, z$, we have

$$\Pi_0^0(\operatorname{curl} \xi) = -\Pi_0^0(\mathcal{K} \Pi_0^0(\nabla p_h)) = -\bar{\mathcal{K}} \Pi_0^0(\nabla p_h).$$

Hence

$$\mathbf{u}_h = \frac{\bar{g}}{d}(\mathbf{x} - \mathbf{x}_p) - \bar{\mathcal{K}} \Pi_0^0(\nabla p_h) + \operatorname{curl} \xi - \Pi_0^0(\operatorname{curl} \xi). \tag{4.14}$$

The projection $\Pi_0^0(\nabla p_h)$ can be computed by letting $\chi = x$ (or $\chi = y, z$) as follows:

$$\begin{aligned} \int_{\mathcal{P}} \Pi_0^0(\nabla p_h) \cdot \nabla \chi \, dx &= \int_{\mathcal{P}} \nabla p_h \cdot (1, 0, 0)^T \, dx = \int_{\partial\mathcal{P}} p_h n_x \, d\sigma \\ &= \sum_{f \subset \partial\mathcal{P}} n_x^f |f| \frac{1}{|f|} \int_f p_h \, d\sigma = \sum_{f \subset \partial\mathcal{P}} n_x^f |f| \mu_{f,0}(p_h), \end{aligned}$$

where $\mu_{f,0}(p_h)$ is the degree of freedom of p_h on f , $\mathbf{n} = (n_x, n_y, n_z)$ is the unit outer normal to $\partial\mathcal{P}$, and $\mathbf{n}^f = (n_x^f, n_y^f, n_z^f)$ is its restriction to each f . Hence

$$\begin{aligned} \Pi_0^0(\nabla p_h) &= \frac{1}{|\mathcal{P}|} \left(\sum_f n_x^f |f| \mu_{f,0}(p_h), \sum_f n_y^f |f| \mu_{f,0}(p_h), \sum_f n_z^f |f| \mu_{f,0}(p_h) \right)^T. \end{aligned}$$

We can write (4.14) in the form (when \mathbf{u}_h is smooth)

$$\mathbf{u}_h = \bar{\mathbf{u}}_h + \operatorname{curl} \xi - \Pi_0^0(\operatorname{curl} \xi) = \bar{\mathbf{u}}_h + O(h),$$

where

$$\bar{\mathbf{u}}_h = -\bar{\mathcal{K}} \Pi_0^0(\nabla p_h) + \frac{\bar{g}}{d}(\mathbf{x} - \mathbf{x}_p) \in \mathbf{V}_h. \tag{4.15}$$

Thus we have obtained a lowest order Raviart - Thomas approximation to \mathbf{u}_h on polygonal/polyhedral grids. We believe it is a better approximation than the L^2 -projection $\bar{\mathcal{K}} \Pi_0^0(\mathbf{u}_h)$, because it satisfies $\operatorname{div} \bar{\mathbf{u}}_h = \bar{g}$, while $\operatorname{div}(\bar{\mathcal{K}} \Pi_0^0(\mathbf{u}_h)) = 0$. Indeed, the numerical tests support this assertion (see Table 2).

5. Error estimates

Although our scheme is developed for problems on 2D (polygonal domain) and 3D (polyhedral domain), we focus on 2D case for the error estimates to avoid technicalities. From now on, the symbol \mathcal{P} (resp. f) denotes a polygon (resp. edge). We need some lemmas which can be found in the literature.

Lemma 5.1. (Inverse inequality for VEM [18,27]) *There exists a constant $C > 0$ such that*

$$\|\nabla q\|_0 \leq C h^{-1} \|q\|_0, \forall q \in N_h^{k+1}. \tag{5.1}$$

Lemma 5.2. (Norm equivalence for VEM [18,25,27]) *For any $q \in N_h^{k+1}$, there exists a constant $C > 0$ such that*

$$\frac{1}{C} h_{\mathcal{P}} \|\Xi(q)\|_{\ell^2} \leq \|q\|_{0,\mathcal{P}} \leq C h_{\mathcal{P}} \|\Xi(q)\|_{\ell^2}, \tag{5.2}$$

where $\Xi(q)$ is the vector representing the degrees of freedom of q .

We need the following lemma which are standard for FEM [2], but not for VEM since there is no reference element.

Lemma 5.3. *Let $\phi \in L^2(\mathcal{P})$, $\Phi \in (L^2(\mathcal{P}))^2$ and $\mu \in L^2(\partial\mathcal{P})$. Then the function $\chi \in N_h^{k+1}(\mathcal{P})$ determined by*

$$\int_f \chi q \, d\sigma = \int_f \mu q \, d\sigma, \forall q \in M_k(f), \text{ for all edge } f \text{ of } \mathcal{P} \tag{5.3a}$$

$$\int_{\mathcal{P}} \chi^m \, dx = \int_{\mathcal{P}} \phi^m \, dx, \forall m \in M_{k-1}(\mathcal{P}) \tag{5.3b}$$

satisfies

$$\|\chi\|_{0,\mathcal{P}} \leq C(\|\phi\|_{0,\mathcal{P}} + h^{1/2} \|\mu\|_{\partial\mathcal{P}}). \tag{5.4}$$

Similarly, the function $\mathbf{v} \in \mathbf{V}_h^k(\mathcal{P})$ determined by the degrees of freedom

$$\int_f \mathbf{v} \cdot \mathbf{n} \xi \, ds = \int_f \mu \xi \, ds, \forall \xi \in M_k(f), \text{ for all edge } f \text{ of } \mathcal{P}, \tag{5.5a}$$

$$\int_{\mathcal{P}} \mathbf{v} \cdot \mathbf{g} \, dx = \int_{\mathcal{P}} \boldsymbol{\phi} \cdot \mathbf{g} \, dx, \forall \mathbf{g} \in \boldsymbol{\Psi}_h(\mathcal{P}) \tag{5.5b}$$

satisfies

$$\|\mathbf{v}\|_{0,\mathcal{P}} \leq C(\|\boldsymbol{\phi}\|_{0,\mathcal{P}} + h^{1/2}\|\mu\|_{\partial\mathcal{P}}). \tag{5.6}$$

Proof. We only prove (5.6), since the proof of (5.4) is similar. It is well known that the square of L^2 -norm of a function $\mathbf{v} \in \mathbf{V}_h^k(\mathcal{P})$ scales like

$$|\mathcal{P}| \sum_{i=1}^{n_k} |\Xi_i(\mathbf{v})|^2,$$

where $\Xi_i(\mathbf{v})$ is the i -th d.o.f. of \mathbf{v} and n_k is the dim $\mathbf{V}_h^k(\mathcal{P})$. In other words, if $\{\boldsymbol{\phi}_j\}_{j=1}^{n_k}$ is the canonical basis functions such that $\Xi_i(\boldsymbol{\phi}_j) = \delta_{ij}$, then

$$\mathbf{v} = \sum_{i=1}^{n_k} \Xi_i(\mathbf{v}) \boldsymbol{\phi}_i.$$

It can be easily verified that the scaled monomials $\xi \in M_k(f)$ and $\mathbf{g} \in (M_k(\mathcal{P}))^2$ satisfy

$$\|\xi\|_f^2 = O(|f|), \|\mathbf{g}\|_{0,\mathcal{P}}^2 = O(|\mathcal{P}|). \tag{5.7}$$

Hence for any $\mathbf{v} \in \mathbf{V}_h^k$, we have

$$\begin{aligned} \|\mathbf{v}\|_{0,\mathcal{P}}^2 &\leq C|\mathcal{P}| \sum_{i=1}^{n_k} |\Xi_i(\mathbf{v})|^2 \\ &= C|\mathcal{P}| [\sum (\text{edge d.o.f.s})^2 + \sum (\text{interior d.o.f.s})^2] \\ &\leq C|\mathcal{P}| \left[\sum_{f \subset \partial\mathcal{P}} \sum_{\xi} \left(\frac{1}{|f|} \int_f \mathbf{v} \cdot \mathbf{n} \xi \, d\sigma \right)^2 + \sum_{\mathbf{g}} \left(\frac{1}{|\mathcal{P}|} \int_{\mathcal{P}} \mathbf{v} \cdot \mathbf{g} \, dx \right)^2 \right] \\ &\leq C \frac{|\mathcal{P}|}{|f|^2} \sum_{f \subset \partial\mathcal{P}} \sum_{\xi} \|\mu\|_f^2 \|\xi\|_f^2 + C|\mathcal{P}|^{-1} \sum_{\mathbf{g}} \|\boldsymbol{\phi}\|_{0,\mathcal{P}}^2 \|\mathbf{g}\|_{0,\mathcal{P}}^2 \text{ (by (5.5))} \\ &\leq C \frac{|\mathcal{P}|}{|f|} \sum_{f \subset \partial\mathcal{P}} \|\mu\|_f^2 + C \sum_{\mathbf{g}} \|\boldsymbol{\phi}\|_{0,\mathcal{P}}^2 \text{ (by (5.7))} \\ &\leq C(h\|\mu\|_{\partial\mathcal{P}}^2 + \|\boldsymbol{\phi}\|_{0,\mathcal{P}}^2). \quad \square \end{aligned}$$

Theorem 5.4. Let (\mathbf{u}_h, p_h) be the solution of the system (4.2). Then there exists a constant C independent of h such that

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leq Ch^{k+1}(\|\mathbf{u}\|_{k+1} + \|g\|_k), \tag{5.8a}$$

$$\|\operatorname{div}(\mathbf{u} - \mathbf{u}_h)\|_0 \leq Ch^{k+1}|g|_{k+1}, \tag{5.8b}$$

provided that $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$ and $g \in H^{k+1}(\Omega)$.

Proof. We shall prove (5.8b) first. We see from (4.3) that

$$\|\operatorname{div}(\mathbf{u} - \mathbf{u}_h)\|_0 \leq \|g - \Pi_k^0 g\|_0 \leq Ch^{k+1}|g|_{k+1}.$$

Next we prove (5.8a). By the triangle inequality

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leq \|\mathbf{u} - \Pi_k^F \mathbf{u}\|_0 + \|\Pi_k^F \mathbf{u} - \mathbf{u}_h\|_0,$$

and the approximation property of Π_k^F (Theorem 2.1), it suffices to estimate

$$\|\Pi_k^F \mathbf{u} - \mathbf{u}_h\|_0.$$

For the sake of simplicity we assume $\mathcal{K} = 1$. (Similar estimate holds as long as \mathcal{K} is sufficiently smooth.) Let p_π be an arbitrary function in $P_{k+1}(\mathcal{P})$. Then, clearly we have $a_h^P(p_\pi, \chi) = a^P(p_\pi, \chi)$ for all $\chi \in N_h^{k+1}(\mathcal{P})$. From (4.2a), we see

$$\begin{aligned} \int_{\mathcal{P}} (\mathbf{u} - \mathbf{u}_h) \cdot \nabla \chi \, dx &= -a^P(p, \chi) + a_h^P(p_h, \chi) \\ &= -a^P(p - p_\pi, \chi) + a_h^P(p_h - p_\pi, \chi). \end{aligned} \tag{5.9}$$

Let $\chi \in H^1(\mathcal{P})$ be the solution of

$$\Delta \chi = 0 \text{ in } \mathcal{P},$$

$$\chi = (\Pi_k^F \mathbf{u} - \mathbf{u}_h) \cdot \mathbf{n} \text{ on } \partial\mathcal{P}.$$

Since χ is completely determined by the moments $\int_f \chi q_k \, d\sigma = \int_f (\Pi_k^F \mathbf{u} - \mathbf{u}_h) \cdot \mathbf{n} q_k \, d\sigma, \forall q_k \in P_k(f), \forall f \subset \partial\mathcal{P}$, we see $\chi \in N_h^{k+1}(\mathcal{P})$. Hence by (5.4) we have

$$\|\chi\|_{0,\mathcal{P}} \leq Ch^{1/2} \|(\Pi_k^F \mathbf{u} - \mathbf{u}_h) \cdot \mathbf{n}\|_{\partial\mathcal{P}}. \tag{5.10}$$

Now by the definition of Π_k^F , (5.9), and (4.2)

$$\begin{aligned} \int_{\partial\mathcal{P}} \chi^2 \, d\sigma &= \int_{\partial\mathcal{P}} \chi (\Pi_k^F \mathbf{u} - \mathbf{u}_h) \cdot \mathbf{n} \, d\sigma \\ &= \int_{\partial\mathcal{P}} \chi (\mathbf{u} - \mathbf{u}_h) \cdot \mathbf{n} \, d\sigma \\ &= \int_{\mathcal{P}} (\mathbf{u} - \mathbf{u}_h) \cdot \nabla \chi \, dx + \int_{\mathcal{P}} \operatorname{div}(\mathbf{u} - \mathbf{u}_h) \chi \, dx \\ &= -a^P(p - p_\pi, \chi) + a_h^P(p_h - p_\pi, \chi) + ((I - \Pi_k^0)g, \chi)_\mathcal{P}. \end{aligned}$$

Now by the approximation property of p_π and p_h , the inverse inequality, and (5.10),

$$\begin{aligned} \|(\Pi_k^F \mathbf{u} - \mathbf{u}_h) \cdot \mathbf{n}\|_{\partial\mathcal{P}}^2 &\leq C(\|p - p_\pi\|_{1,\mathcal{P}} + \|p_h - p_\pi\|_{1,\mathcal{P}}) \|\chi\|_{1,\mathcal{P}} \\ &\quad + Ch^k \|g\|_{k,\mathcal{P}} \|\chi\|_{0,\mathcal{P}} \\ &\leq C(2\|p - p_\pi\|_{1,\mathcal{P}} + \|p_h - p\|_{1,\mathcal{P}}) \|\chi\|_{1,\mathcal{P}} \\ &\quad + Ch^k \|g\|_{k,\mathcal{P}} \|\chi\|_{0,\mathcal{P}} \\ &\leq Ch^{k+1} \|p\|_{k+2,\mathcal{P}} \|\chi\|_{1,\mathcal{P}} + Ch^k \|g\|_{k,\mathcal{P}} \|\chi\|_{0,\mathcal{P}} \\ &\leq Ch^k (\|p\|_{k+2,\mathcal{P}} + \|g\|_{k,\mathcal{P}}) \|\chi\|_{0,\mathcal{P}}, \\ &\quad \text{(by the inverse inequality)} \\ &\leq Ch^k \|g\|_{k,\mathcal{P}} \|\chi\|_{0,\mathcal{P}}, \text{ (by the regularity assumption)} \\ &\leq Ch^{k+1/2} \|g\|_{k,\mathcal{P}} \|(\Pi_k^F \mathbf{u} - \mathbf{u}_h) \cdot \mathbf{n}\|_{\partial\mathcal{P}}. \end{aligned}$$

Hence

$$\|(\Pi_k^F \mathbf{u} - \mathbf{u}_h) \cdot \mathbf{n}\|_{\partial\mathcal{P}} \leq Ch^{k+1/2} \|g\|_{k,\mathcal{P}}. \tag{5.11}$$

On the other hand, from equations (2.2), (4.2a,b) we see

$$\int_{\mathcal{P}} (\mathbf{u} + \mathcal{K} \nabla p) \cdot \mathbf{v} \, dx = 0, \forall \mathbf{v} \in (L^2(\mathcal{P}))^2,$$

$$\int_{\mathcal{P}} (\mathbf{u}_h + \mathcal{K} \Pi_k^0 \nabla p_h) \cdot \mathbf{v} \, dx = 0, \forall \mathbf{v} \in \boldsymbol{\Psi}_h(\mathcal{P}).$$

Subtracting, we have (since \mathcal{K} is constant)

$$\begin{aligned} \int_{\mathcal{P}} (\mathbf{u} - \mathbf{u}_h) \cdot \mathbf{v} \, dx &= - \int_{\mathcal{P}} (\mathcal{K} \nabla p - \mathcal{K} \Pi_k^0 \nabla p_h) \cdot \mathbf{v} \, dx, \forall \mathbf{v} \in \boldsymbol{\Psi}_h(\mathcal{P}) \\ &= - \int_{\mathcal{P}} (\mathcal{K} \nabla p - \mathcal{K} \nabla p_h) \cdot \mathbf{v} \, dx, \forall \mathbf{v} \in \boldsymbol{\Psi}_h(\mathcal{P}). \end{aligned}$$

Let $\mu = (\Pi_k^F \mathbf{u} - \mathbf{u}_h) \cdot \mathbf{n}$ and $\boldsymbol{\phi} = \mathcal{K} \nabla p - \mathcal{K} \nabla p_h$. Then $\sigma = \Pi_k^F \mathbf{u} - \mathbf{u}_h \in \mathbf{V}_h^k(\mathcal{P})$ is the solution of

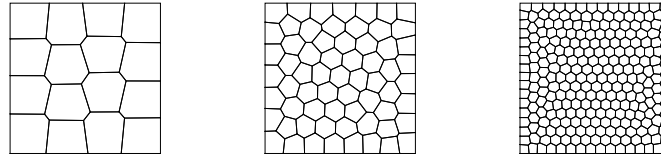


Fig. 1. Polygonal mesh \mathcal{T}_h with 16 (left), 64 (middle), and 256 (right) elements.

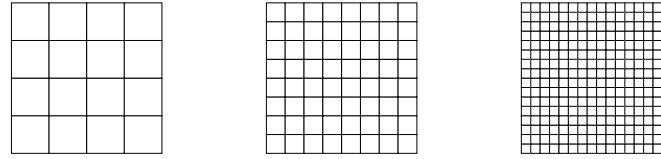


Fig. 2. Uniform rectangle mesh \mathcal{T}_h with 16 (left), 64 (middle), and 256 (right) elements.

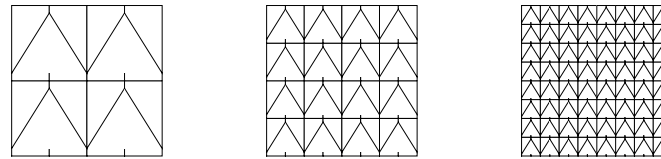


Fig. 3. Trapezoidal meshes \mathcal{T}_h with 16 (left), 64 (middle), and 256 (right) elements.

$$\int_f \boldsymbol{\sigma} \cdot \mathbf{n} q ds = \int_f \mu q ds, \forall q \in M_k(f), \text{ for all edges(faces) of } \mathcal{P},$$

$$\int_{\mathcal{P}} \boldsymbol{\sigma} \cdot \mathbf{g} dx = \int_{\mathcal{P}} \boldsymbol{\phi}(\mathcal{P}) \cdot \mathbf{g} dx, \forall \mathbf{g} \in \boldsymbol{\Psi}_h.$$

Then by (5.6), (5.11), and the approximation property of p_h , we have

$$\begin{aligned} \|\Pi_k^F \mathbf{u} - \mathbf{u}_h\|_{0,\mathcal{P}} &\leq \|\mathcal{K}\|_{\infty} \|\nabla p - \nabla p_h\|_{0,\mathcal{P}} + Ch^{1/2} \|(\Pi_k^F \mathbf{u} - \mathbf{u}_h) \cdot \mathbf{n}\|_{\partial \mathcal{P}} \\ &\leq Ch^{k+1} (\|p\|_{k+2,\mathcal{P}} + \|g\|_{k,\mathcal{P}}). \end{aligned}$$

By the triangle inequality and approximation property of $\Pi_k^F \mathbf{u}$, the proof is complete. \square

6. Numerical experiments

In this section, we present some numerical results in two dimensional case. The exact solutions on $\Omega = [0, 1]^2$ are chosen as

$$p(x, y) = x(1 - x)y(1 - y),$$

$$\mathbf{u}(x, y) = -\mathcal{K}((1 - 2x)y(1 - y), x(1 - x)(1 - 2y)).$$

Example 6.1 (Numerical results of MVVM on polygonal meshes). The scheme is tested on the sequential polygonal meshes \mathcal{T}_h in Fig. 1. On each polygon, the maximum number of vertexes is set to six. We report the error between the exact solution and the L^2 -projection of \mathbf{u}_h with $\mathcal{K} = 1 + 0.5 \sin(x)$, for $k = 0, 1, 2, 3$ in Table 1. Here, the L^2 -projection of \mathbf{u}_h is obtained using equation (4.12). We observe results optimal for all cases.

Example 6.2 (Comparison between L^2 -projection and Raviart-Thomas type reconstruction). In the lowest order case, we compare the errors of L^2 -projection of \mathbf{u}_h and Raviart-Thomas type reconstruction (4.15). Here, we set $\mathcal{K} = 1$. The result on polygonal mesh is reported in Table 2. We observe that the Raviart-Thomas type reconstruction is more accurate. We perform similar comparison on the uniform rectangle mesh (Fig. 2) and trapezoidal mesh (Fig. 3). Table 3 shows the errors on the uniform rectangle mesh and Table 4 shows those on the trapezoidal mesh. We see that the convergence rates are optimal for all cases. However, the Raviart-Thomas type reconstruction is marginally better.

Table 1

L^2 -errors between exact solution and the L^2 -projection of \mathbf{u}_h for orders $k = 0, 1, 2, 3$. Table at left top, right top, left bottom and right bottom correspond to the case $k = 0, k = 1, k = 2$ and $k = 3$ respectively.

N of Elt.	$\ u - \Pi_0^0 u_h\ _0$	order	N of Elt.	$\ u - \Pi_1^0 u_h\ _0$	order
$2^2 \times 2^2$	6.494E-02		$2^2 \times 2^2$	2.964E-02	
$2^3 \times 2^3$	3.250E-02	0.998	$2^3 \times 2^3$	6.261E-03	1.981
$2^4 \times 2^4$	1.578E-02	1.042	$2^4 \times 2^4$	1.162E-03	2.041
$2^5 \times 2^5$	7.853E-03	1.007	$2^5 \times 2^5$	2.304E-04	2.001
$2^6 \times 2^6$	3.895E-03	1.012	$2^6 \times 2^6$	5.040E-05	2.014

N of Elt.	$\ u - \Pi_2^0 u_h\ _0$	order	N of Elt.	$\ u - \Pi_3^0 u_h\ _0$	order
$2^2 \times 2^2$	4.158E-03		$2^2 \times 2^2$	7.678E-05	
$2^3 \times 2^3$	6.260E-04	2.732	$2^3 \times 2^3$	8.684E-06	3.144
$2^4 \times 2^4$	6.369E-05	3.297	$2^4 \times 2^4$	5.552E-07	3.967
$2^5 \times 2^5$	5.719E-06	3.477	$2^5 \times 2^5$	3.368E-08	4.043
$2^6 \times 2^6$	6.326E-07	3.176	$2^6 \times 2^6$	2.109E-09	3.997

Table 2

L^2 -errors of $\Pi_0^0 \mathbf{u}_h$ (left) and Raviart-Thomas type reconstruction $\tilde{\mathbf{u}}_h$ (right) on polygonal meshes.

N of Elt.	$\ u - \Pi_0^0 u_h\ _{L^2(\Omega)}$	order	$\ u - \tilde{u}_h\ _{L^2(\Omega)}$	order
$2^2 \times 2^2$	4.303E-02		3.171E-02	
$2^3 \times 2^3$	2.241E-02	0.941	1.628E-02	0.962
$2^4 \times 2^4$	1.111E-02	1.012	7.930E-03	1.037
$2^5 \times 2^5$	5.575E-03	0.995	4.011E-03	0.983
$2^6 \times 2^6$	2.784E-03	1.002	1.988E-03	1.013

Table 3

L^2 -errors of $\Pi_0^0 \mathbf{u}_h$ (left) and Raviart-Thomas type reconstruction $\tilde{\mathbf{u}}_h$ (right) on uniform rectangular meshes.

N of Elt.	$\ u - \Pi_0^0 u_h\ _{L^2(\Omega)}$	order	$\ u - \tilde{u}_h\ _{L^2(\Omega)}$	order
$2^2 \times 2^2$	4.129E-02		2.880E-02	
$2^3 \times 2^3$	2.077E-02	0.991	1.476E-02	0.965
$2^4 \times 2^4$	1.033E-02	1.008	7.320E-03	1.011
$2^5 \times 2^5$	5.154E-03	0.003	3.647E-03	1.005
$2^6 \times 2^6$	2.576E-03	1.001	1.822E-03	1.002

Table 4
 L^2 -errors of $\Pi_0^0 \mathbf{u}_h$ (left) and Raviart-Thomas type reconstruction $\tilde{\mathbf{u}}_h$ (right) on trapezoidal meshes.

N of Elt.	$\ u - \Pi_0^0 \mathbf{u}_h\ _{L^2(\Omega)}$	order	$\ u - \tilde{\mathbf{u}}_h\ _{L^2(\Omega)}$	order
$2^2 \times 2^2$	4.910E-02		3.999E-02	
$2^3 \times 2^3$	2.547E-02	0.947	2.366E-02	0.757
$2^4 \times 2^4$	1.314E-02	0.955	1.269E-02	0.898
$2^5 \times 2^5$	6.683E-03	0.975	6.541E-03	0.957
$2^6 \times 2^6$	3.369E-03	0.988	3.314E-03	0.981

7. Conclusion

In this work, we develop mixed virtual volume methods (MVVM) of all orders on polygonal/polyhedral meshes. For the primary variable we use the nonconforming virtual element space, and for the velocity variable we use the $H(\text{div})$ conforming virtual element space. The proposed method is the first success to compute $H(\text{div})$ -conforming velocity variables through the NCVEM. We show that the MVVM is equivalent to the NCVEM for all orders. Once the primary variable is obtained from solving the (SPD) system arising from NCVEM, the velocity variable can be computed locally. Thus, the whole procedure can be implemented efficiently, avoiding a saddle point problem. The optimal error estimates in 2D are proved and some numerical results supporting our analysis are presented. The detailed analysis and numerical tests for 3D case will be given in a future article. The extension to the curvilinear case seems possible using the results on curved domains in [15,29]. However, some technical details have to be verified together with numerical works. It is left for a future investigation.

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