Analysis and Convergence of a MAC-like Scheme for the Generalized Stokes Problem

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We introduce a MAC-like scheme (a covolume method on rectangular grids) for approximating the generalized Stokes problem on an axiparallel domain. Two staggered grids are used in the derivation of the discretization. The velocity is approximated by conforming bilinears over rectangular elements, and the pressure by piecewise constants over macro-rectangular elements. The error in the velocity in the H^1 norm and the pressure in the L^2 norm are shown to be of first order, provided that the exact velocity is in H^2 and the exact pressure in H^1, and that the partition family of the domain is regular. © 1997 John Wiley & Sons, Inc.

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I. INTRODUCTION

Finite element, finite difference, and finite volume methods are the three main methods employed to numerically solve fundamental field equations of fluid mechanics. Among these three classes of methods, the finite volume method seems to be most intuitive since it is based most frequently on local conservation of mass, momentum, or energy over control volumes. The programming effort in implementing the finite volume method also seems to be simpler than the finite element method. In particular, the MAC (marker and cell) method of Harlow and Welch [1] on rectangular grids and its variants on unstructured grids have demonstrated their reliability and robustness in dealing with heat transfer problems. However, unlike in the finite element method, the theoretical analysis of a MAC-like method is usually *ad hoc*. One reason for this might be that the velocity approximants sought in a MAC-like method are often only the normal components of velocities at the inter-elements or cell interfaces of the partition of the flow domain. Another reason might be that the starting discretization procedure is usually done on the governing PDEs instead of a weak formulation in terms of inner products.

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A consistent relation between the problem domain partition(s) and the discretization is often not so obvious as in the finite element methodology. Attempts to improve this situation were made by Porsching [2] and Chou [3, 4] in which the dual network model approach was adopted to solve two-phase fluid problems. The emphasis of these articles was on a conservation of mass or energy through the design of primal and dual partitions. However, no convergence analysis was done for the *full* discretized systems. Another approach was taken by Nicolaides [5, 6] where rigorous analysis was given to the so-called covolume methods. The partitions used were the Delaunay–Voronoi mesh systems, which differ from those used in the above articles. Nicolaides' approach represents a major advance, because the usual vector operators (div, curl, laplacian, etc.) were generalized to irregular networks. (See also Choudhury and Nicolaides [7]). As for the implementation issues resulting from his methodology, Hall et al. [8, 9] have demonstrated that covolume methods can be effectively implemented by their dual variable method (DVM) [10]. See the review article by Nicolaides, Porsching, and Hall [11] for the status of the covolume methods up to 1995, and Chou and Li [12] Chou and Kwak [13, 14] for some recent results.

The purpose of this article is to introduce and analyze a covolume method on rectangular grids, along the line mentioned in the first approach above. The analysis of the covolume method using this approach on unstructured triangular grids can be found in Chou [15].

Consider the generalized Stokes problem in two dimensions for steady flow of a heavily viscous fluid:

$$\alpha_0 \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } \Omega \subset R^2, \tag{1.1}$$

$$\text{div } \mathbf{u} = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\alpha_0 \geq 0, \nu > 0$. When $\alpha_0 = 0$, we have the Stokes problem, and the case of $\alpha_0 \neq 0$ usually arises as part of the solution process for the Navier–Stokes equation. We shall assume $\nu = 1$ in this article, since $\nu \mathbf{u}$ can be used as a transformed variable. Let $H_0^1(\Omega)$ be the space of weakly differentiable functions with zero trace, $H^i(\Omega), i = 1, 2$ be the usual Sobolev spaces, and $L_0^2(\Omega)$ be the set of all L^2 functions over Ω with zero integral mean. $|\cdot|_1$ and $\|\cdot\|_0$ denote the usual $(H^1(\Omega))^2$ seminorm and the L^2 norm, respectively. Define the bilinear forms:

$$\tilde{a}(\mathbf{u}, \mathbf{v}) := \sum_{i,j} \left(\frac{\partial u_i}{\partial x_j}, \frac{\partial v_i}{\partial x_j} \right) + \alpha_0(\mathbf{u}, \mathbf{v}) \quad \mathbf{u}, \mathbf{v} \in \mathbf{H}_0^1 := H_0^1(\Omega)^2 \tag{1.4}$$

$$\tilde{b}(\mathbf{v}, q) := -(q, \text{div } \mathbf{v}), \quad \mathbf{v} \in \mathbf{H}_0^1, q \in L_0^2, \tag{1.5}$$

where $(\cdot, \cdot)$ is the L^2 inner product. The weak formulation associated with (1.1)–(1.3) is: Find $(\mathbf{u}, p) \in \mathbf{H}_0^1 \times L_0^2$ such that

$$\tilde{a}(\mathbf{u}, \mathbf{v}) + \tilde{b}(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0^1, \tag{1.6}$$

$$\tilde{b}(\mathbf{u}, q) = 0 \quad \forall q \in L_0^2. \tag{1.7}$$

The approximation of this system using the mixed finite element method is well documented in Brezzi et al. [16]. We now describe a MAC method. The method is motivated by the MAC technique for incompressible flow problems and will be viewed as a Petrov–Galerkin method as far as error analysis is concerned.

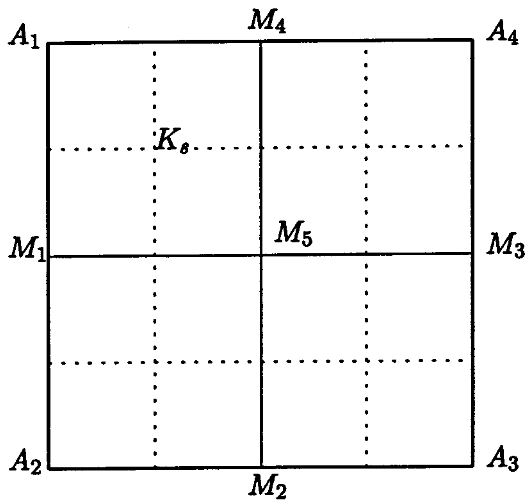

FIG. 1. Macro-element K, subrectangle K_s and its dual partition.

First we need to partition the problem domain, which is assumed to be a polygon whose sides are parallel to the coordinate axes. Let $\mathcal{R}_h = \cup K_B$ be a partition of the domain Ω into a union of macro-rectangular elements, where K_B stands for the macro-rectangle with center B and is made up of four subrectangles, i.e., $K_B = \cup_{s=1}^4 K_s$. For instance in Fig. 1, with $B = M_5$, the associated macro-element is made up of four subrectangles $\square A_1M_1M_5M_4, \square M_4M_5M_3A_4$, etc. The nodes of a macro-element are defined to be its vertices, the midpoints of its sides, and its center. These points are represented in Fig. 1 as $A_i, i = 1, \ldots, 4; M_i, i = 1, \ldots, 4$, and M_5. We define $h := \max h_K$, where h_K is the diameter of the macro-element K. We shall assume throughout the article that the primal partition family $\{\mathcal{R}_h\}$ is regular: there exists a positive constant C independent of h such that

$$Ch^2 \leq |K| \leq h^2, \quad \forall K \in \mathcal{R}_h,$$

where $|K|$ is the area of K. The *trial* function space $\mathbf{H}_h$ associated with the approximation to the fluid velocity space $\mathbf{H}_0^1$ is defined as

$$\mathbf{H}_h = \left\{ \begin{array}{ll} \mathbf{v}_h \in \mathbf{H}_0^1\colon \mathbf{v}_h|_{K_s} \in Q_1^2(K_s), & \forall K_s \text{ subrectangles of } K \in \mathcal{R}_h; \\ \mathbf{v}_h = \mathbf{0} \text{ at all boundary nodes}, & \end{array} \right\} \tag{1.8}$$

where $Q_1(K_s)$ denote the space of bilinears on K_s. The choice of macro-elements is motivated by the fact that the usual rectangular elements do not admit a nonzero divergence-free velocity; the so-called locking phenomenon.

Next we construct the dual partition $\mathcal{R}_h^*$ and the test function space associated with it. With the primal partition given, we can further subdivide the domain Ω by adding horizontal and vertical grid lines through the midpoints of the subrectangles of macro-elements. In Fig. 1 these lines are dashed. The dual grid is defined as a union of rectangles based at the nodes of the macro-elements. For example, in Fig. 1, the dual element based at the interior node M_5 is made up of the dashed rectangle whose vertices are the centers of the four subrectangles K_s from the macro-element K_{M_5}. We do the obvious modification at a boundary node. Carrying out the construction for every node in the primal partition generates a dual partition for the domain. We denote the dual element based at P as K_P^* and the dual partition as $\mathcal{R}_h^* = \cup K_P^*$. Define the associate *test function* *space* $\mathbf{Y}_h$ as the space of certain piecewise constant vector functions:

$$\mathbf{Y}_h = \{q \in (L^2(\Omega))^2\colon q|_{K_P^*} \text{ is a constant vector,} \quad \text{and } q|_{K_P^*} = \mathbf{0} \text{ on any boundary dual element } K_P^*\}.$$

Denote by χ_j^* the scalar characteristic function associated with the dual element $K_{P_j}^*, j = 1, \ldots, N_I$. Here N_I is the number of interior nodes of $\mathcal{R}_h$. We see that, for any $\mathbf{v}_h \in \mathbf{Y}_h$,

$$\mathbf{v}_h(x) = \sum_{j=1}^{N_I} \mathbf{v}_h(P_j)\chi_j^*(x) \quad \forall x \in \Omega. \tag{1.9}$$

As for the approximate pressure space $L_h \subset L_0^2(\Omega)$, we define it to be the set of all piecewise constants with respect to the primal partition, because in the MAC scheme the pressure is assigned at the centers of rectangular elements (the macro-elements in our case). Finally, our test and trial function spaces should reflect the fact that in a MAC method the momentum Eq. (1.1) is integrated over the dual element and the continuity Eq. (1.2) over the primal element. This can be achieved by defining the following bilinear forms. Define $a^S\colon \mathbf{H}_h \times \mathbf{Y}_h \to R$,

$$a^S(\mathbf{u}_h, \mathbf{v}_h) := -\sum_{i=1}^{N_I} \int_{\partial K_{P_i}^*} \frac{\partial \mathbf{u}_h}{\partial \mathbf{n}} \cdot \mathbf{v}_h \, d\sigma \tag{1.10}$$

$$= -\sum_{i=1}^{N_I} \mathbf{v}_h(P_i) \cdot \int_{\partial K_{P_i}^*} \frac{\partial \mathbf{u}_h}{\partial \mathbf{n}} \, d\sigma. \tag{1.11}$$

Equation (1.10) is motivated by integrating the second term of (1.1) against a test function and then formally applying the second Green's identity. A more intuitive way of interpreting this definition is to use (1.9) and bilinearity:

$$a^S(\mathbf{u}_h, \mathbf{v}_h) := \sum_{j=1}^{N_i} a^S(\mathbf{u}_h, \mathbf{v}_h(P_j)\chi_j^*), \tag{1.12}$$

where

$$a^S(\mathbf{u}_h, \mathbf{v}_h(P_j)\chi_j^*) := -\mathbf{v}_h(P_j) \cdot \int_{\partial K_{P_j}^*} \frac{\partial \mathbf{u}_h}{\partial \mathbf{n}} d\sigma, \tag{1.13}$$

and the last equation is obtained by integrating the second term of (1.1) over a typical dual element (control volume) and applying the Divergence theorem. Let N_R denote the number of rectangles in the primal partition. For $\mathbf{u}_h \in \mathbf{H}_h, \mathbf{v}_h \in \mathbf{Y}_h, p_h, q_h \in L_h$, and $\mathbf{f} \in (L^2(\Omega))^2$, define the following bilinear forms:

$$a^N(\mathbf{u}_h, \mathbf{v}_h) := \alpha_0(\mathbf{u}_h, \mathbf{v}_h), \tag{1.14}$$

$$a(\mathbf{u}_h, \mathbf{v}_h) := a^S(\mathbf{u}_h, \mathbf{v}_h) + a^N(\mathbf{u}_h, \mathbf{v}_h), \tag{1.15}$$

$$b(\mathbf{v}_h, p_h) := \sum_{i=1}^{N_I} \mathbf{v}_h(P_i) \cdot \int_{\partial K_{P_i}^*} p_h \mathbf{n} \, d\sigma, \tag{1.16}$$

$$c(\mathbf{u}_h, q_h) := -\sum_{k=1}^{N_R} q_h(B_k) \int_{K_{B_k}} \text{div } \mathbf{u}_h \, dx, \tag{1.17}$$

$$(\mathbf{f}, \mathbf{v}_h) := \sum_{i=1}^{N_I} \mathbf{v}_h(P_i) \cdot \int_{K^*_{P_i}} \mathbf{f} \, dx. \tag{1.18}$$

The weak formulation of the approximate problem to Eqs. (1.6)–(1.7) is: Find $(\mathbf{u}_h, p_h) \in \mathbf{H}_h \times L_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{Y}_h, \tag{1.19}$$

$$c(\mathbf{u}_h, q_h) = 0 \quad \forall q_h \in L_h. \tag{1.20}$$

Note that there are as many unknowns as equations.

It turns out that we can reformulate this system as a saddle-point problem in the manner of Eqs. (1.6)–(1.7). Convergence analysis can, thus, be done in the framework of the conforming mixed finite element method. We outline how the convergence analysis is done. Introduce the one-to-one *transfer* operator γ_h from $\mathbf{H}_h$ *onto* $\mathbf{Y}_h$ by

$$\gamma_h \mathbf{u}_h(x) := \sum_{j=1}^{N_I} \mathbf{u}_h(P_j) \chi^*_j(x) \quad \forall x \in \Omega. \tag{1.21}$$

Define the following bilinear forms:

$$A^S(\mathbf{z}_h, \mathbf{w}_h) := a^S(\mathbf{z}_h, \gamma_h \mathbf{w}_h) \quad \forall \mathbf{z}_h, \mathbf{w}_h \in \mathbf{H}_h, \tag{1.22}$$

$$A^N(\mathbf{z}_h, \mathbf{w}_h) := a^N(\mathbf{z}_h, \gamma_h \mathbf{w}_h) \quad \forall \mathbf{z}_h, \mathbf{w}_h \in \mathbf{H}_h, \tag{1.23}$$

$$A(\mathbf{z}_h, \mathbf{w}_h) := A^S(\mathbf{z}_h, \mathbf{w}_h) + A^N(\mathbf{z}_h, \mathbf{w}_h), \tag{1.24}$$

$$B(\mathbf{w}_h, q_h) := b(\gamma_h \mathbf{w}_h, q_h) \quad \forall \mathbf{w}_h \in \mathbf{H}_h; \quad \forall q_h \in L_h. \tag{1.25}$$

It is shown in Section II that the two bilinear forms B and c are identical.

Thus, the approximation problem (1.19)–(1.20) becomes: Find $(\mathbf{u}_h, p_h) \in \mathbf{H}_h \times L_h$ such that

$$A(\mathbf{u}_h, \mathbf{w}_h) + B(\mathbf{w}_h, p_h) = (\mathbf{f}, \gamma_h \mathbf{w}_h) \quad \forall \mathbf{w}_h \in \mathbf{H}_h, \tag{1.26}$$

$$B(\mathbf{u}_h, q_h) = 0 \quad \forall q_h \in L_h. \tag{1.27}$$

We derive the main error estimate result in Theorem 3.1, which states that there exists a constant $C > 0$ independent of h such that

$$|\mathbf{u}_h - \mathbf{u}|_1 + \|p - p_h\|_0 \leq Ch(\|\mathbf{u}\|_2 + \|p\|_1 + 1),$$

provided that $\mathbf{u} = \mathbf{H}^2(\Omega)$ and $p \in H^1(\Omega)$, and that the primal partition family is regular. We note that the estimate is optimal both in order and in regularity; the regularity assumption on the exact pressure is minimal, because most covolume methods assume $p \in H^2(\Omega)$ [6]. The basic ideas of the proof of the main theorem are as follows. In Lemma 2.2, we show that $A^S(\mathbf{z}_h, \mathbf{w}_h)$ is symmetric, and it differs from $(\nabla \mathbf{z}_h, \nabla \mathbf{w}_h)$ only by a symmetric quadrature error term. Due to this fact, the bilinear form A can be analyzed effectively. We introduce a family of nearby

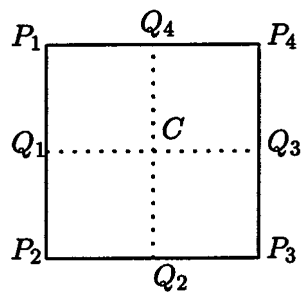

FIG. 2. A subrectangle K_s.

symmetric approximation problems (3.2)–(3.3) to (1.26)–(1.27) in the sense that the convergence of its solutions to the exact solution of the Stokes problem can be obtained by invoking the standard theory of the mixed finite element method. We then compare the nearby symmetric system with (1.26)–(1.27). Since these systems are both finite dimensional, proper inverse estimates can be used.

II. SADDLE-POINT FORM AND INF-SUP CONDITION

In this section we prove several important properties of the following bilinear forms:

$$a^S(\mathbf{z}_h, \mathbf{v}_h) = -\sum_{i=1}^{N_I} \mathbf{v}_h(P_i) \cdot \int_{\partial K_{P_i}^*} \frac{\partial \mathbf{z}_h}{\partial \mathbf{n}} d\sigma, \tag{2.1}$$

$$b(\mathbf{v}_h, p_h) = \sum_{i=1}^{N_I} \mathbf{v}_h(P_i) \cdot \int_{\partial K_{P_i}^*} p_h \mathbf{n} \, d\sigma, \tag{2.2}$$

$$c(\mathbf{z}_h, q_h) = -\sum_{k=1}^{N_R} q_h(B_k) \int_{K_{B_k}} \operatorname{div} \mathbf{z}_h \, dx, \tag{2.3}$$

$$(\mathbf{f}, \mathbf{v}_h) = \sum_{i=1}^{N_I} \mathbf{v}_h(P_i) \cdot \int_{K_{P_i}^*} \mathbf{f} \, dx. \tag{2.4}$$

The following simple lemma of line integral conversion from dual to primal elements will be used often throughout the article.

Lemma 2.1. *For each subelement K_s of a macro-element in the primal partition, let us further divide K_s into four smaller subrectangles as shown in Fig. 2. Let g be a continuous function in the interior of each of these subrectangles. Then*

$$\sum_{i=1}^{N} \int_{\partial K_{P_i}^*} g(x) \, d\sigma = \sum_{K \in \mathcal{R}_h} I_K = \sum_{K \in \mathcal{R}_h} \sum_{s=1}^{4} I_{K_s}, \tag{2.5}$$

where N is the number of nodes in $\mathcal{R}_h$ and

$$\begin{aligned} I_{K_s} &= \int_{Q_1CQ_4} g(x)\,d\sigma + \int_{Q_2CQ_1} g(x)\,d\sigma + \int_{Q_3CQ_2} g(x)\,d\sigma + \int_{Q_4CQ_3} g(x)\,d\sigma \\ &= \sum_{j=1}^{4} \int_{Q_jC+CQ_{j-1}} g(x)\,d\sigma. \end{aligned}$$

Here and below we adopt the convention $Q_{j+4} = Q_j$, $j = 1, 2, 3, 4$ when a subindex is out of bound.

Proof. The proof is straightforward. ∎

We next show that $A^S(\mathbf{z}_h, \mathbf{w}_h)$ is symmetric and it differs from $(\nabla \mathbf{z}_h, \nabla \mathbf{w}_h)$ only by a quadrature term.

Lemma 2.2. *With the above notation, we have*

$$A^S(\mathbf{z}_h, \mathbf{w}_h) = A^S(\mathbf{w}_h, \mathbf{z}_h) = \sum_{K \in \mathcal{R}_h} I_K = \sum_{K \in \mathcal{R}_h} \sum_{s=1}^{4} I_{K_s},$$

where

$$I_{K_s} = \frac{1}{4} h_x^s h_y^s \sum_{i=2,4} (\mathbf{z}_{x_1}^i \cdot \mathbf{w}_{x_1}^i + \mathbf{z}_{x_1}^C \cdot \mathbf{w}_{x_1}^C) + \frac{1}{4} h_y^s h_x^s \sum_{i=1,3} (\mathbf{z}_{x_2}^i \cdot \mathbf{w}_{x_2}^i + \mathbf{z}_{x_2}^C \cdot \mathbf{w}_{x_2}^C). \quad (2.6)$$

Here $\mathbf{z}_x$ stands for the partial derivative with respect to x; $\mathbf{z}_x^i := \mathbf{z}_x(Q_i)$; h_x^s is the width of the K_s, and h_y^s the height of K_s. In fact,

$$A^S(\mathbf{z}_h, \mathbf{w}_h) = (\nabla \mathbf{z}_h, \nabla \mathbf{w}_h) + Q(\mathbf{z}_h, \mathbf{w}_h), \quad (2.7)$$

where

$$Q(\mathbf{z}_h, \mathbf{w}_h) := \frac{1}{24} \sum_{K_s} [h_x^s (h_y^s)^3 + (h_x^s)^3 h_y^s](\mathbf{z}_{x_1x_2} \cdot \mathbf{w}_{x_1x_2}), \quad \forall \mathbf{z}_h, \mathbf{w}_h \in \mathbf{H}_h. \quad (2.8)$$

Proof. By Lemma 2.1 with $g = -\mathbf{w}_h(P_j) \cdot \frac{\partial \mathbf{z}_h}{\partial \mathbf{n}}$ we have

$$\begin{aligned} -I_{K_s} &= \sum_{j=1}^{4} \mathbf{w}_h(P_j) \cdot \int_{Q_jC+CQ_{j-1}} \frac{\partial \mathbf{z}_h}{\partial \mathbf{n}} d\sigma \\ &= \sum_{j=1}^{4} \mathbf{w}_h(P_j) \cdot \int_{Q_jC} \frac{\partial \mathbf{z}_h}{\partial \mathbf{n}} d\sigma - \mathbf{w}_h(P_{j+1}) \cdot \int_{Q_jC} \frac{\partial \mathbf{z}_h}{\partial \mathbf{n}} d\sigma \\ &= \sum_{j=1}^{4} \int_{Q_jC} \frac{\partial \mathbf{z}_h}{\partial x_k} \cdot (\mathbf{w}_h(P_j) - \mathbf{w}_h(P_{j+1}))\, dx_{\hat{k}}, \end{aligned}$$

where $\hat{k} = j(\text{mod } 2)$, $k = (j+1)(\text{mod } 2)$. We have

$$\int_{Q_jC} \frac{\partial \mathbf{z}_h}{\partial x_k} \cdot (\mathbf{w}_h(P_j) - \mathbf{w}_h(P_{j+1}))\, dx_{\hat{k}} = \int_{Q_jC} \frac{\partial \mathbf{z}_h}{\partial x_k} \cdot \frac{\partial \mathbf{w}_h}{\partial x_k}(Q_j) h_k \, dx_{\hat{k}},$$

where $h_1 = h_x^s, h_2 = h_y^s$. Since $\mathbf{z}_h$ is linear along Q_2Q_4, the terms corresponding to $j = 2, 4$ become

$$\frac{h_x^s h_y^s}{4}\left[\left(\frac{\partial \mathbf{z}_h}{\partial x_1}(Q_2) + \frac{\partial \mathbf{z}_h}{\partial x_1}(C)\right)\cdot\frac{\partial \mathbf{w}_h}{\partial x_1}(Q_2) + \left(\frac{\partial \mathbf{z}_h}{\partial x_1}(C) + \frac{\partial \mathbf{z}_h}{\partial x_1}(Q_4)\right)\cdot\frac{\partial \mathbf{w}_h}{\partial x_1}(Q_4)\right],$$

which is the first term of (2.6) by the linearity of $\frac{\partial \mathbf{w}_h}{\partial x_1}$. Similarly, putting $j = 1, 3$ we obtain the second term of (2.6).

Noting that $\mathbf{z}_{x_i}\cdot\mathbf{w}_{x_i}$ is a function of a single variable and applying to the integral of $\mathbf{z}_{x_i}\cdot\mathbf{w}_{x_i}$ over K_s, the error formula for the composite trapezoidal rule with two subintervals

$$\int_a^b f(x)\,dx = \frac{1}{4}(f(a) + 2f(m) + f(b)) * (b-a) - \frac{(b-a)^3}{48}f'',$$

we can easily derive (2.7). ∎

Next we show that the bilinear form A is bounded.

Lemma 2.3. *There exists a positive constant C independent of h such that*

$$|A(\mathbf{z},\mathbf{w})| \le C|\mathbf{z}|_1|\mathbf{w}|_1, \quad \forall \mathbf{z},\mathbf{w} \in \mathbf{H}_h. \tag{2.9}$$

Proof. Since

$$A^S(\mathbf{z},\mathbf{w}) \le A^S(\mathbf{z},\mathbf{z})^{1/2}A^S(\mathbf{w},\mathbf{w})^{1/2},$$

to show the boundedness of A it suffices to prove

$$A^S(\mathbf{z},\mathbf{z}) \le C|\mathbf{z}|_1^2.$$

In view of (2.7) we shall show that

$$|Q(\mathbf{z},\mathbf{z})| \le C|\mathbf{z}|_1^2. \tag{2.10}$$

Define

$$\begin{aligned} |\mathbf{z}|_{2,K_s}^2 &:= \int_{K_s} \mathbf{z}_{x_1x_2}^2 + \mathbf{z}_{x_1x_1}^2 + \mathbf{z}_{x_2x_2}^2\,dx \\ &= \int_{K_s} \mathbf{z}_{x_1x_2}^2\,dx \quad \forall \mathbf{z} \in \mathbf{H}_h, \\ |\mathbf{z}|_{1,K_s}^2 &:= \int_{K_s} \mathbf{z}_{x_1}^2 + \mathbf{z}_{x_2}^2\,dx. \end{aligned}$$

Thus, there exists a constant independent of K_s such that

$$|\mathbf{z}|_{2,K_s} \le Ch^{-1}|\mathbf{z}|_{1,K_s}. \tag{2.11}$$

This inverse estimate can be proved by the common scaling argument ([17], p. 25). Noting that $\mathbf{z}_{x_1x_2}$ is constant over K_s we have, with $h_x := h_x^s, h_y := h_y^s$,

$$(h_x h_y^3 + h_x^3 h_y)\mathbf{z}_{x_1x_2}^2 = (h_y^2 + h_x^2)|\mathbf{z}|_{2,K_s}^2 \le (h_y^2 + h_x^2)Ch^{-2}|\mathbf{z}|_{1,K_s}^2 \le C|\mathbf{z}|_{1,K_s}^2.$$

Summing over K_s derives (2.10) and, hence,

$$A^S(\mathbf{z},\mathbf{z}) \le C|\mathbf{z}|_1^2.$$

∎

We shall also need a particular bound for the quadrature error term Q. In reference to Fig. 1, let us further divide each subrectangle (K_s) into two triangles (T_s^+, T_s^-) by connecting the diagonal (M_1M_4) with positive slope. Call the resulting triangulation $\mathcal{T}_h$. For $T \in \mathcal{T}_h$ let $P_1(T)$ denote the space of all linear polynomials defined on T, and let

$$\mathbf{S}_h := \{\mathbf{v}_h \in \mathbf{H}_0^1(\Omega)\colon \mathbf{v}_h|_T \in P_1^2(T) \quad \forall T \in \mathcal{T}_h\}. \tag{2.12}$$

Recall, with $h_x := h_x^s$ and $h_y := h_y^s$,

$$\begin{aligned} Q(\mathbf{z}_h,\mathbf{w}_h) &:= \frac{1}{24}\sum_{K_s}(h_xh_y^3 + h_x^3h_y)(\mathbf{z}_{x_1x_2}\cdot\mathbf{w}_{x_1x_2}), \quad \forall \mathbf{z}_h,\mathbf{w}_h \in \mathbf{H}_h \\ &= \frac{1}{48}\sum_{T_s\in\mathcal{T}_h}(h_xh_y^3 + h_x^3h_y)(\mathbf{z}_{x_1x_2}\cdot\mathbf{w}_{x_1x_2}), \quad \forall \mathbf{z}_h,\mathbf{w}_h \in \mathbf{H}_h. \end{aligned}$$

Hence, we can extend the definition of Q to $\mathbf{H}_h + \mathbf{S}_h$.

Lemma 2.4. *There exists a constant C independent of h such that*

$$|Q(\mathbf{z}_h - \mathbf{v}_h, \mathbf{w}_h)| \le C|\mathbf{z}_h - \mathbf{v}_h|_1|\mathbf{w}_h|_1, \forall \mathbf{z}_h, \mathbf{w}_h \in \mathbf{H}_h, \mathbf{v}_h \in \mathbf{S}_h. \tag{2.13}$$

Proof. Observe that

$$|Q(\mathbf{z}_h - \mathbf{v}_h, \mathbf{w}_h)| \le |Q(\mathbf{z}_h - \mathbf{v}_h, \mathbf{z}_h - \mathbf{v}_h)|^{1/2}|Q(\mathbf{w}_h,\mathbf{w}_h)|^{1/2}.$$

Now argue, as in the proof of the last lemma, to bound the right side. Bound the second factor with the inverse estimate

$$|\mathbf{w}|_{2,K_s} \le Ch^{-1}|\mathbf{w}|_{1,K_s}, \tag{2.14}$$

and the first factor with

$$|\mathbf{t}|_{2,T_s} \le Ch^{-1}|\mathbf{t}|_{1,T_s}, \tag{2.15}$$

where $\mathbf{t}$ is bilinear on the triangle T_s. Substituting $\mathbf{z}_h - \mathbf{v}_h$ for $\mathbf{t}$ and summing give the result. ∎

Lemma 2.5. *The bilinear form A is coercive on $\mathbf{H}_h$: there exists a positive constant C independent of h such that*

$$A(\mathbf{w}_h,\mathbf{w}_h) \ge C|\mathbf{w}_h|_1^2.$$

Also, there exists a constant $C_1 > 0$ independent of h such that

$$C_1\|\mathbf{w}_h\|_0^2 \le (\mathbf{w}_h, \gamma_h\mathbf{w}_h) \quad \forall \mathbf{w}_h \in \mathbf{H}_h. \tag{2.16}$$

Proof. By Lemma 2.2, it suffices to show

$$A^N(\mathbf{w}_h,\mathbf{w}_h) = \alpha_0(\mathbf{w}_h, \gamma_h\mathbf{w}_h) \ge 0. \tag{2.17}$$

Referring to Fig. 2,

$$(\mathbf{w}_h, \gamma_h\mathbf{w}_h) = \sum_K \int_K \mathbf{w}_h \cdot \gamma_h\mathbf{w}_h\,dx = \sum_K\sum_{s=1}^4 \tilde{I}_{K_s}.$$

Using four-point quadrature for bilinears on each of the dotted rectangles whose side has length $h_x^s/2$ or $h_y^s/2$,

$$\begin{aligned}
\tilde{I}_{K_s} &= \sum_{j=1}^{4} \mathbf{w}_h(P_j) \int_{\Box_j} \mathbf{w}_h(x)\, dx \\
&= \frac{h_x^s h_y^s}{16} \sum_{j=1}^{4} \mathbf{w}_h(P_j)[\mathbf{w}_h(P_j) + \mathbf{w}_h(Q_j) + \mathbf{w}_h(C) + \mathbf{w}_h(Q_{j+3})] \\
&= \frac{h_x^s h_y^s}{64} \sum_{j=1}^{4} \mathbf{w}_h(P_j)[9\mathbf{w}_h(P_j) + 3\mathbf{w}_h(P_{j+1}) + \mathbf{w}_h(P_{j+2}) + 3\mathbf{w}_h(P_{j+3})] \\
&\geq \frac{h_x^s h_y^s}{64} \left(\sum_{j=1}^{4} 9\mathbf{w}_h(P_j)^2 - \sum_{j=1}^{4} \left(\frac{3}{2} + \frac{1}{2} + \frac{3}{2} \right) (\mathbf{w}_h(P_j)^2 + \mathbf{w}_h(P_{j+1})^2) \right) \\
&= \frac{h_x^s h_y^s}{32} \sum_{j=1}^{4} \mathbf{w}_h(P_j)^2 \\
&\geq \frac{h_x^s h_y^s}{32} \max_{1 \leq j \leq 4} \mathbf{w}_h(P_j)^2 \\
&= \frac{h_x^s h_y^s}{32} \max_{x \in K_s} \mathbf{w}_h(x)^2,
\end{aligned}$$

where, in the first inequality, we have used $|ab| \leq \frac{1}{2}(a^2 + b^2)$.

Now summing over $s = 1, \ldots, 4$ and then over K, we have

$$(\mathbf{w}_h, \gamma_h \mathbf{w}_h) = \sum_K \sum_{s=1}^{4} \tilde{I}_{K_s} \geq \frac{1}{32}(\mathbf{w}_h, \mathbf{w}_h).$$

■

Lemma 2.6.

$$B(\mathbf{w}_h, q_h) = b(\gamma_h \mathbf{w}_h, q_h) = c(\mathbf{w}_h, q_h) \quad \forall \mathbf{w}_h \in \mathbf{H}_h, q_h \in L_h.$$

Proof. Let K_s be the subrectangle with vertices $P_1, \ldots, P_4$. Then with $g = q_h \mathbf{w}_h(P_j) \cdot \mathbf{n}$ in Lemma 2.1, we have

$$I_{K_s} = \mathbf{w}_h(P_1) \cdot \int_{Q_1C+CQ_4} q_h \mathbf{n}\, d\sigma + \mathbf{w}_h(P_2) \cdot \int_{Q_2C+CQ_1} q_h \mathbf{n}\, d\sigma + \mathbf{w}_h(P_3) \cdot \int_{Q_3C+CQ_2} q_h \mathbf{n}\, d\sigma + \mathbf{w}_h(P_4) \cdot \int_{Q_4C+CQ_3} q_h \mathbf{n}\, d\sigma.$$

Applying the Divergence theorem to the zero integral of div$(q_h \mathbf{w}_h(P_j))$ on each dotted subrectangle of K_s, we have

$$I_{K_s} = 0 - \sum_{j=1}^{4} \int_{Q_j P_{j+1} + P_{j+1} Q_{j+1}} q_h \mathbf{w}_h(P_{j+1}) \cdot \mathbf{n}\, d\sigma.$$

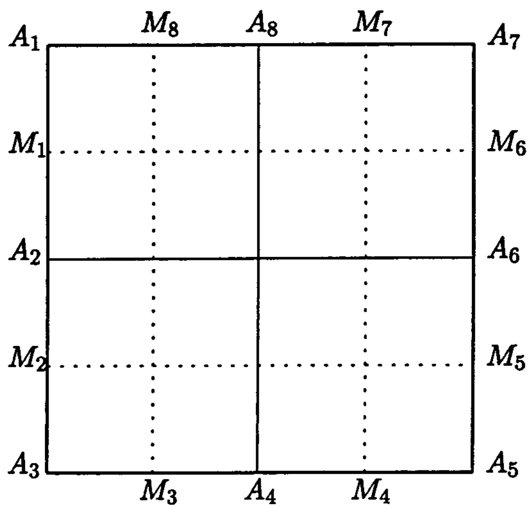

FIG. 3. Renumbering the macro-element K.

Renumbering the nodes of a macro-element as in Fig. 3, we see

$$\begin{aligned}\sum_{s=1}^{4} I_{K_s} &= -\sum_{j=1}^{8} q_h \int_{M_jA_{j+1}+A_{j+1}M_{j+1}} \mathbf{w}_h(A_{j+1})\cdot\mathbf{n}\,d\sigma \\ &= -\sum_{j=1}^{8} q_h \left(\int_{A_jM_j} \mathbf{w}_h(A_j)\cdot\mathbf{n}\,d\sigma + \int_{M_jA_{j+1}} \mathbf{w}_h(A_{j+1})\cdot\mathbf{n}\,d\sigma\right) \\ &= -\sum_{j=1}^{8} q_h \int_{A_jA_{j+1}} \mathbf{w}_h\cdot\mathbf{n}\,d\sigma \\ &= -\int_{\partial K} q_h\mathbf{w}_h\cdot\mathbf{n}\,d\sigma \\ &= -\int_K q_h \operatorname{div}\mathbf{w}_h\,dx = c(\mathbf{w}_h,q_h)|_K,\end{aligned}$$

where the third equality follows from

$$\begin{aligned}\int_{A_jA_{j+1}} \mathbf{w}_h\cdot\mathbf{n}\,d\sigma &= \frac{\mathbf{w}_h(A_j)+\mathbf{w}_h(A_{j+1})}{2}\cdot\mathbf{n}_{M_j}h_y^s \\ &= \int_{A_jM_j} \mathbf{w}_h(A_j)\cdot\mathbf{n}\,d\sigma + \int_{M_jA_{j+1}} \mathbf{w}_h(A_{j+1})\cdot\mathbf{n}\,d\sigma.\end{aligned}$$

■

Lemma 2.7. *There exists a positive constant β independent of h such that*

$$\sup_{\mathbf{w}_h\neq\mathbf{0}} \frac{B(\mathbf{w}_h,q_h)}{|\mathbf{w}_h|_1} \geq \beta\|q_h\|_0. \tag{2.18}$$

Proof. Let P_h be the Ritz–Galerkin projection from $\mathbf{H}_0^1$ to $\mathbf{H}_h$ defined by $(\nabla(P_h\mathbf{z} - \mathbf{z}), \nabla\mathbf{z}_h) = 0$, $\forall\mathbf{z}_h \in \mathbf{H}_h$. Define an ''interpolation'' operator π_h: $\mathbf{H}_0^1 \to \mathbf{H}_h$ as follows. For $K \in \mathcal{R}_h$ (cf. Fig. 3), the values of $\pi_h\mathbf{z}$ at the corners and the center of K are given by

$$\pi_h\mathbf{z}(A_j) = P_h\mathbf{z}(A_j), j = 1, 3, 5, 7, \quad \text{and } \pi_h\mathbf{z}(c) = P_h\mathbf{z}(c), \tag{2.19}$$

where $c = B_K$, the center of K. The remaining values of $\pi_h\mathbf{z}$ are determined by the conditions

$$\int_{A_{j-1}A_{j+1}} \pi_h\mathbf{z}\, d\sigma = \int_{A_{j-1}A_{j+1}} \mathbf{z}\, d\sigma, j = 2, 4, 6, 8. \tag{2.20}$$

By a well known theorem ([18], p. 220), we know that, given $q_h \in L_0^2(\Omega)$, there exists a $\mathbf{w} \in \mathbf{H}_0^1$ such that

$$-\text{div }\mathbf{w} = q_h,$$

and

$$\|\mathbf{w}\|_1 \le C\|q_h\|_0.$$

Noting that B can be defined on $\mathbf{H}_0^1 + \mathbf{H}_h$, we see that, with $A_9 := A_1$,

$$\begin{aligned}
B(\mathbf{w}, q_h) &= -\sum q_h(B_k) \int_{K_{B_k}} \text{div }\mathbf{w}\, dx \\
&= -\sum q_h(B_k) \sum_{j=1}^{4} \int_{A_{2j-1}A_{2j+1}} \mathbf{w}\cdot\mathbf{n}_{2j}\, d\sigma \\
&= -\sum q_h(B_k) \sum_{j=1}^{4} \int_{A_{2j-1}A_{2j+1}} \pi_h\mathbf{w}\cdot\mathbf{n}_{2j}\, d\sigma \\
&= -\sum q_h(B_k) \int_{K_{B_k}} \text{div}(\pi_h\mathbf{w})\, dx \\
&= B(\pi_h\mathbf{w}, q_h).
\end{aligned}$$

Assuming that π_h is bounded for the moment, we have

$$\begin{aligned}
\frac{B(\pi_h\mathbf{w}, q_h)}{|\pi_h\mathbf{w}|_1} &= \frac{B(\mathbf{w}, q_h)}{|\pi_h\mathbf{w}|_1} \\
&\ge \frac{B(\mathbf{w}, q_h)}{C|\mathbf{w}|_1} \\
&= \frac{\|q_h\|_0^2}{C|\mathbf{w}|_1} \ge \frac{1}{C}\|q_h\|_0.
\end{aligned}$$

It remains to show that π_h is bounded. Let $\mathbf{p}_j, \tilde{\mathbf{p}}_j \in \mathbf{H}_h$ be the Lagrange nodal basis functions based at A_j for $j = 1, \ldots, 8$ so that, when restricted to a macro-element K,

$$\mathbf{p}_j(A_i) = (\delta_{ij}, 0)^t; \quad \tilde{\mathbf{p}}_j(A_i) = (0, \delta_{ij})^t, \quad i = 1, \ldots, 8.$$

Then on K we have

$$\pi_h\mathbf{z} = P_h\mathbf{z} + \sum_{i=1}^{4}[\alpha_i\mathbf{p}_{2i} + \tilde{\alpha}_i\tilde{\mathbf{p}}_{2i}],$$

where

$$\begin{aligned}\alpha_i &:= \frac{\int_{A_{i-1}A_{i+1}}(\mathbf{z}-P_h\mathbf{z})\,d\sigma\cdot\mathbf{e}_1}{\int_{A_{i-1}A_{i+1}}\mathbf{p}_{2i}\,d\sigma\cdot\mathbf{e}_1}\\ &= \frac{1}{|A_{i-1}A_{i+1}|}\int_{A_{i-1}A_{i+1}}(\mathbf{z}-P_h\mathbf{z})\,d\sigma\cdot\mathbf{e}_1,\\ \tilde{\alpha}_i &:= \frac{\int_{A_{i-1}A_{i+1}}(\mathbf{z}-P_h\mathbf{z})\,d\sigma\cdot\mathbf{e}_2}{\int_{A_{i-1}A_{i+1}}\mathbf{p}_{2i}\,d\sigma\cdot\mathbf{e}_2}\\ &= \frac{1}{|A_{i-1}A_{i+1}|}\int_{A_{i-1}A_{i+1}}(\mathbf{z}-P_h\mathbf{z})\,d\sigma\cdot\mathbf{e}_2,\end{aligned}$$

with $\mathbf{e}_i$ denoting the two unit natural coordinate vectors. We can, thus, bound the coefficients α_i and $\tilde{\alpha}_i$ using the trace theorem and the boundedness of P_h. Let $\|\cdot\|_{m,K}$ and $|\cdot|_{m,K}$ denote the $\mathbf{H}^m(K)$ norm and seminorm, respectively. Let $\hat{K}$ be the standard reference square $[0,1]\times[0,1]$, and let F: $\hat{K}\to K$ be the unique affine transformation $\mathbf{x}=B_K\hat{\mathbf{x}}+\mathbf{b}$ such that the four corners of the unit square are sent to the four corners of K in a counterclockwise way. Here B_K is the two-by-two diagonal matrix that represents the compressions along the coordinate directions, and $\mathbf{b}$ is the lower-left corner of K. Denote by $\hat{e}$ any edge of $\hat{K}$ and define $\hat{\mathbf{z}}:=\mathbf{z}\circ F\in\mathbf{H}^1(\hat{K})$. Then

$$\begin{aligned}|\alpha_i| &\le \frac{1}{|A_{i-1}A_{i+1}|}\int_{A_{i-1}A_{i+1}}|\mathbf{z}-P_h\mathbf{z}|\,d\sigma\\ &\le \int_{\hat{e}}|\hat{\mathbf{z}}-\widehat{P_h\mathbf{z}}|\,d\hat{\sigma}\\ &\le \|\hat{\mathbf{z}}-\widehat{P_h\mathbf{z}}\|_{1,\hat{K}}\\ &\le C_1h^{-1}\{\|\mathbf{z}-P_h\mathbf{z}\|_{0,K}^2+h^2|\mathbf{z}-P_h\mathbf{z}|_{1,K}^2\}^{1/2}.\end{aligned}$$

It is also routine to bound $\mathbf{p}_{2i},\tilde{\mathbf{p}}_{2i}$ in the $\mathbf{H}^1$ norm via the standard reference element. For instance, we may derive

$$|\mathbf{p}_{2i}|_{1,K}\le C_{2i}.$$

Thus,

$$\left|\sum_i\alpha_i\mathbf{p}_i\right|_{1,K}\le\sum_i|\alpha_i||\mathbf{p}_{2i}|_{1,K}\le\sum_i C_ih^{-1}\{\|\mathbf{z}-P_h\mathbf{z}\|_{0,K}^2+h^2|\mathbf{z}-P_h\mathbf{z}|_{1,K}^2\}^{1/2}.$$

Similar bounds hold for $\tilde{\alpha}_i,\tilde{\mathbf{p}}_i$. Hence,

$$|\pi_h\mathbf{z}-P_h\mathbf{z}|_{1,K}^2\le C_5h^{-2}\{\|\mathbf{z}-P_h\mathbf{z}\|_{0,K}^2+h^2|\mathbf{z}-P_h\mathbf{z}|_{1,K}^2\}.$$

Summing over K and then using the approximation property and the boundedness of P_h, we derive

$$|\pi_h\mathbf{z}-P_h\mathbf{z}|_1\le C|\mathbf{z}|_1,$$

for a constant $C>0$ independent of h. An application of the triangle inequality now completes the proof. ∎

The following lemma can be proven easily.

Lemma 2.8. *There exists a positive constant C_0 independent of h such that*

$$\|\gamma_h \mathbf{w}_h - \mathbf{w}_h\|_0 \le C_0 h |\mathbf{w}_h|_1 \quad \forall \mathbf{w}_h \in \mathbf{H}_h. \tag{2.21}$$

III. ERROR ESTIMATES

We now prove the main theorem of this article.

Theorem 3.1. *Let the primal partition family of the domain Ω be regular, let $\{\mathbf{u}_h, p_h\}$ be the solution of the problem* (1.26)–(1.27), *and $\{\mathbf{u}, p\}$ solve the problem* (1.6)–(1.7). *Then there exists a positive constant C independent of h such that*

$$|\mathbf{u} - \mathbf{u}_h|_1 + \|p - p_h\|_0 \le Ch(\|\mathbf{u}\|_2 + \|p\|_1 + 1), \tag{3.1}$$

provided that $\mathbf{u} \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega), p \in H^1(\Omega)$.

Proof. By a standard existence and uniqueness theorem ([18], p. 251) on saddle point problems, Lemmas 2.5 and 2.7 guarantee the existence and uniqueness of the solution $\{\mathbf{u}_h, p_h\}$. We first introduce an auxiliary symmetric Stokes approximation problem to (1.6)–(1.7): Find $(\tilde{\mathbf{u}}_h, \tilde{p}) \in \mathbf{H}_h \times L_h$ such that

$$(\nabla \tilde{\mathbf{u}}_h, \nabla \mathbf{w}_h) + \alpha_0(\tilde{\mathbf{u}}_h, \mathbf{w}_h) + B(\mathbf{w}_h, \tilde{p}_h) = (\mathbf{f}, \mathbf{w}_h) \quad \forall \mathbf{w}_h \in \mathbf{H}_h, \tag{3.2}$$

$$B(\tilde{\mathbf{u}}_h, q_h) = 0 \quad \forall q_h \in L_h. \tag{3.3}$$

This scheme fits into the standard mixed method [16] and we have the following convergence result:

$$|\mathbf{u} - \tilde{\mathbf{u}}_h|_1 + \|p - \tilde{p}_h\|_0 \le Ch(\|\mathbf{u}\|_2 + \|p\|_1), \tag{3.4}$$

provided that $\mathbf{u} \in \mathbf{H}^2(\Omega), p \in H^1(\Omega)$. On the other hand, the system (1.26), (1.27) yields

$$A^S(\mathbf{u}_h, \mathbf{w}_h) + \alpha_0(\mathbf{u}_h, \gamma_h \mathbf{w}_h) + B(\mathbf{w}_h, p_h) = (\mathbf{f}, \gamma_h \mathbf{w}_h) \quad \forall \mathbf{w}_h \in \mathbf{H}_h, \tag{3.5}$$

$$B(\mathbf{u}_h, q_h) = 0 \quad \forall q_h \in L_h. \tag{3.6}$$

Subtracting (3.3) from (3.6) gives

$$B(\mathbf{u}_h - \tilde{\mathbf{u}}_h, q_h) = 0 \quad \forall q_h \in L_h. \tag{3.7}$$

Subtracting (3.2) from (3.5) gives

$$\begin{aligned}(\nabla(\mathbf{u}_h - \tilde{\mathbf{u}}_h), \nabla \mathbf{w}_h) &+ \alpha_0(\mathbf{u}_h, \gamma_h \mathbf{w}_h) - \alpha_0(\tilde{\mathbf{u}}_h, \mathbf{w}_h) + B(\mathbf{w}_h, p_h - \tilde{p}_h) \\ &= (\mathbf{f}, \gamma_h \mathbf{w}_h) - (\mathbf{f}, \mathbf{w}_h) - Q(\mathbf{u}_h, \mathbf{w}_h) \quad \forall \mathbf{w}_h \in \mathbf{H}_h,\end{aligned} \tag{3.8}$$

where Q is defined in (2.8). Define

$$\tilde{\mathbf{e}}_h := \mathbf{u}_h - \tilde{\mathbf{u}}_h.$$

Replace the $\mathbf{w}_h$ in (3.8) with $\tilde{\mathbf{e}}_h$ and use (3.7), Lemma 2.2 to obtain

$$|\tilde{\mathbf{e}}_h|_1^2 + \alpha_0(\tilde{\mathbf{e}}_h, \gamma_h \tilde{\mathbf{e}}_h) = (\mathbf{f}, \gamma_h \tilde{\mathbf{e}}_h - \tilde{\mathbf{e}}_h) + \alpha_0(\tilde{\mathbf{u}}_h, \tilde{\mathbf{e}}_h - \gamma_h \tilde{\mathbf{e}}_h) - Q(\mathbf{u}_h, \tilde{\mathbf{e}}_h). \tag{3.9}$$

Observe that

$$-Q(\mathbf{u}_h, \tilde{\mathbf{e}}_h) = -Q(\tilde{\mathbf{e}}_h, \tilde{\mathbf{e}}_h) - Q(\tilde{\mathbf{u}}_h, \tilde{\mathbf{e}}_h) \leq |Q(\tilde{\mathbf{u}}_h, \tilde{\mathbf{e}}_h)|.$$

Using this inequality along with (2.16) on the second term of the left-hand side of (3.9), Lemma 2.8, and $\|\tilde{\mathbf{u}}_h\|_0 \leq M$, we obtain

$$|\tilde{\mathbf{e}}_h|_1^2 \leq \|\mathbf{f}\|_0 C_0 h |\tilde{\mathbf{e}}_h|_1 + C_0 \alpha_0 M h |\tilde{\mathbf{e}}_h|_1 + |Q(\tilde{\mathbf{u}}_h, \tilde{\mathbf{e}}_h)|. \tag{3.10}$$

It remains to bound $Q(\tilde{\mathbf{u}}_h, \tilde{\mathbf{e}}_h)$. Let $\mathbf{z}_h$ be a piecewise linear function such that $Q(\mathbf{z}_h, \mathbf{w}_h) = 0 \ \forall \mathbf{w}_h \in \mathbf{H}_h$ and $|\mathbf{u} - \mathbf{z}_h|_1 \leq C_5 h \|\mathbf{u}\|_2$. Such a function can be found by choosing the continuous interpolant of $\mathbf{u}$ that is in $\mathbf{S}_h$ [cf. Eq. (2.12)]. Using this and Lemma 2.4,

$$\begin{aligned}
|Q(\tilde{\mathbf{u}}_h, \tilde{\mathbf{e}}_h)| &= |Q(\tilde{\mathbf{u}}_h - \mathbf{z}_h, \tilde{\mathbf{e}}_h)| \\
&\leq C |\tilde{\mathbf{u}}_h - \mathbf{z}_h|_1 |\tilde{\mathbf{e}}_h|_1 \\
&\leq C\{|\tilde{\mathbf{u}}_h - \mathbf{u}|_1 + |\mathbf{u} - \mathbf{z}_h|_1\} |\tilde{\mathbf{e}}_h|_1 \\
&\leq C_2 h |\tilde{\mathbf{e}}_h|_1.
\end{aligned}$$

Hence,

$$|\tilde{\mathbf{e}}_h|_1 \leq Ch, \tag{3.11}$$

where C depends on $\mathbf{f}, \mathbf{u}$, but not on h. Combining this with (3.4) and using the triangle inequality gives

$$|\mathbf{u} - \mathbf{u}_h|_1 \leq Ch(\|\mathbf{u}\|_2 + \|p\|_1 + 1). \tag{3.12}$$

We can use the inf-sup condition on Eq. (3.8) and the same techniques in deriving (3.10) to obtain

$$\beta \|p_h - \tilde{p}_h\|_0 \leq C_1 h + \sup_{\mathbf{w}_h \neq \mathbf{0}} \frac{|Q(\mathbf{u}_h, \mathbf{w}_h)|}{|\mathbf{w}_h|_1}.$$

The second term on the right can be estimated by the type of arguments in deriving (3.11) and by (3.12). Thus, we obtain

$$\|p_h - \tilde{p}_h\|_0 \leq C_2 h(\|\mathbf{u}\|_2 + \|p\|_1 + 1).$$

An application of the triangle inequality then proves (3.1). ∎

Remark 3.1. Note that we can symmetrize the problem (1.26)–(1.27) by replacing $(\gamma_h \mathbf{v}_h, \mathbf{w}_h)$ by $\frac{1}{2}[(\gamma_h \mathbf{v}_h, \mathbf{w}_h) + (\mathbf{v}_h, \gamma_h \mathbf{w}_h)]$ and still obtain the same optimal error estimate in the above theorem.

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