Error Estimate in L^2 of a Covolume Method for the Generalized Stokes Problem

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We investigate an L^2-error estimate of a covolume scheme for the Stokes problem recently introduced by Chou (Math Comp 66 (1997), 85–104). We show the error in L^2 norm is of second order provided the exact velocity is in $\underline{H}^3$ and the exact pressure is in H^2. © 2005 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 22: 165–179, 2006

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I. INTRODUCTION

In developing numerical schemes for various PDEs, optimal order of convergence is often a primary concern. For second-order elliptic problems, for example, we expect first-order convergence in H^1 norm and second-order convergence in L^2 norm when piecewise linear element is used. Similar result holds for velocity components for the Stokes problem, when certain basis functions are chosen to satisfy inf-sup condition. These facts are well documented in standard finite element literature [1]. In contrast, the error estimates for finite volume methods or covolume methods are either ad-hoc or lack proper optimal order although they have been used widely among engineers for quite some time because of its local conservation property and simplicity. For a survey of covolume methods and related works, see [2–7] and references therein.

The analysis for the covolume method for various flow problems have been carried out by many authors. For Delaunay-Voronoi type of grid, Cai and McCormick [8], Cai et al. [9] showed certain discrete energy error estimate for elliptic problem and Nicholaides showed first-order convergence in L^2 and H^1 norm of div-curl system [10]. For rectangular grids, Bank and Rose

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[11] analyzed elliptic problem while Porsching [12], Nicolaides [6], and Nicolaides and Wu [13] analyzed the Stokes problem. Recently, Chou [14] introduced a covolume scheme for the Stokes problem using P_1 nonconforming finite element on triangular grid. Also, see Chou and Kwak [15, 16] for rectangular grid using bilinear or rotated bilinear element. For elliptic problem using triangular grids, see Chou and Li [17] where barycenter was used instead of commonly used circumcenter as in Delaunay-Voronoi pair.

The central idea in analyzing various covolume schemes studied by Chou et al. [14–16] is to treat it as a Petrov-Galerkin type by introducing certain transfer operator γ_h from finite element space to the space of piecewise constant functions on covolume, which plays the role of test function space. It turns out that in certain cases [14, 17], the covolume scheme is equivalent to committing a variational crime to finite element formulations and H^1-error estimate follows by analyzing the perturbation terms. However, the optimal order L^2-error estimate for the Stokes problem using the covolume scheme can be found nowhere. The difficulty in showing second-order convergence lies in the framework of the covolume method: The transfer operator γ_h does not have enough approximation property necessary to apply the Aubin-Nitsche duality argument. In this article, we show an L^2-error estimate of the velocity of the covolume scheme introduced in [14] under higher regularity assumption. We overcome the abovementioned difficulty by a careful examination of the difference of two bilinear forms and by using a generalization of the Aubin-Nitsche duality argument.

The rest of the article is organized as follows. In §2, we describe the covolume scheme for the Stokes problem using P_1 nonconforming finite element space. A second-order L^2-error estimate for the velocity is shown in §3. Numerical experiments are shown in §4.

II. NOTATION AND PRELIMINARIES

Consider the generalized Stokes problem in two dimensions for steady flow of a heavily viscous fluid:

$$\alpha_0 \underline{u} - \nu\Delta\underline{u} + \nabla p = \underline{f}, \qquad \text{in } \Omega \subset R^2, \tag{2.1a}$$

$$\operatorname{div} \underline{u} = 0, \qquad \text{in } \Omega, \tag{2.1b}$$

$$\underline{u} = 0, \qquad \text{on } \partial\Omega, \tag{2.1c}$$

where $\alpha_0 \geq 0$, $\nu > 0$. When $\alpha_0 = 0$, we have the Stokes problem, and the case of $\alpha_0 \neq 0$ usually arises as part of the solution process for the Navier-Stokes equation. We shall assume $\nu = 1$ for simplicity. Let $H_0^1(\Omega)$ be the space of weakly differentiable functions with zero trace, $H^i(\Omega)$, $i =$ 1, 2, 3, be the usual Sobolev spaces, and $L_0^2(\Omega)$ be the set of all L^2 functions over Ω with zero integral mean, and underline denote vector-valued functions and spaces.

The approximation of this system using the mixed finite element method is well known ([1, 18]), which we describe briefly.

We need to partition the domain Ω, which for simplicity, will be assumed to be polygonal. Referring to Fig. 1, let $\mathcal{R}_h = \cup K_C$ be a partition of the domain Ω into of triangular elements. The nodes of an element are defined to be the midpoints of its sides. These points are represented in Fig. 2 as P_i, $i = 1, \ldots, 3$. We shall assume throughout the article that the primal partition family $\{\mathcal{R}_h\}$ is regular.

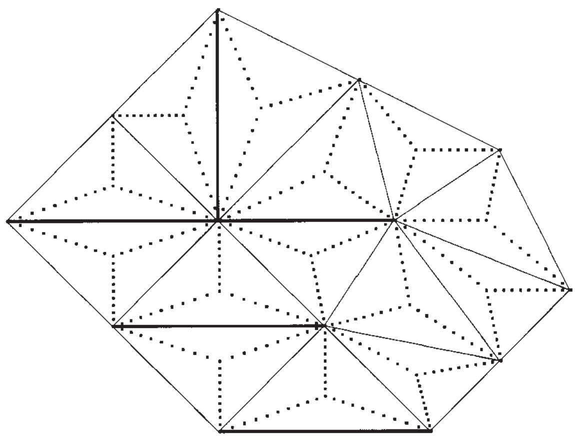


FIG. 1. Primal and dual partition of Ω.

The *trial* and *test* function space $\underline{H}_h$ associated with the approximation of the fluid velocity space $\underline{H}_0^1$ is defined as

$$\underline{H}_h = \{ \underline{v}_h : \underline{v}_h|_K \in P_1^2(K),\ \forall \text{triangle } K \in \mathcal{R}_h;$$

$$\underline{v}_h = \underline{0} \text{ at all boundary nodes and is continuous at all nodes,}\}$$

where P_1 denote the space of linear functions on the triangle K.

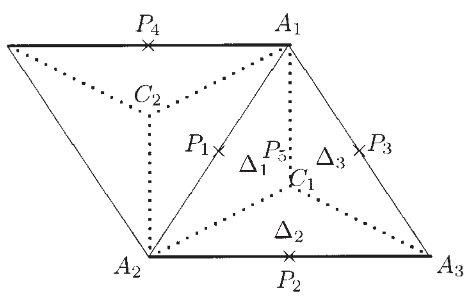
As for the approximation of pressure space $L_h \subset L_0^2(\Omega)$, we define it to be the set of all piecewise constants with respect to the primal partition.

Then the corresponding nonconforming mixed finite element problem is: Given $\underline{f} \in (L^2(\Omega))^2$, find $(\underline{u}_h^*, p_h^*) \in \underline{H}_h \times L_h$ such that

$$a^*(\underline{u}_h^*, \underline{v}_h) + b^*(\underline{v}_h, p_h^*) = (\underline{f}, \underline{v}_h), \qquad \forall \underline{v}_h \in \underline{H}_h, \tag{2.2a}$$

$$b^*(\underline{u}_h^*, q_h) = 0, \qquad \forall q_h \in L_h, \tag{2.2b}$$

where



C_1: Barycenter of $\Delta A_1A_2A_3$.

FIG. 2. Notation of primal element $\Delta A_1A_2A_3$.

$$a^*(\underline{u}_h, \underline{v}_h) = \alpha_0(\underline{u}_h, \underline{v}_h) + \sum_{K\in\mathcal{R}_h} (\nabla\underline{u}_h \cdot \nabla\underline{v}_h), \qquad \forall\underline{u}_h, \underline{v}_h \in \underline{H}_h,$$

$$b^*(\underline{v}_h, q_h) = -\sum_{K\in\mathcal{R}_h} (\text{div } \underline{v}_h, q_h), \qquad \forall\underline{v}_h \in \underline{H}_h, q_h \in L_h.$$

Next, we describe covolume method. The idea is to replace the test function space $\underline{H}_h$ by the space of piecewise constant functions on each covolume that straddles each edge of triangles in primal triangulation.

The trial function space is unchanged, except that we use the mean value $\underline{w}_h^m$ of $\underline{w}_h$ over the edge e_j of K, i.e.,

$$\underline{w}_h^m(P_j) := \frac{1}{|e_j|}\int_{e_j} \underline{w}_h(x)ds$$

as a degree of freedom, where P_j is the midpoint of e_j. We say $\underline{w}_h$ is continuous at node P_j of an edge if

$$\underline{w}_h^m(P_j) := \frac{1}{|e_j|}\int_{e_j} \underline{w}_h|_{\partial K_1}ds = \frac{1}{|e_j|}\int_{e_j} \underline{w}_h|_{\partial K_2}ds.$$

Since $\underline{w}_h^m(P_j) = \underline{w}_h(P_j)$ when $\underline{w}_h$ is a linear function on K, the mean value degree of freedom is equivalent to point value degree of freedom. However, the former is more natural to covolume concepts and its formulations.

To introduce the test function space, we construct the dual partition $\mathcal{R}_h^*$. Divide each triangle of the primal partition into three subtriangles by connecting two vertices and barycenter of a primal element as in Fig. 1. As in Fig. 2, the dual element based at the node P_1 is made up of the two triangles $\Delta A_1C_1A_2$ and $\Delta A_1C_2A_2$. We do the obvious modification at a boundary node. Carrying out the construction for every node in the primal partition, we obtain a dual partition for the domain. We denote the dual element based at node P as K_P^* and the dual partition as $\mathcal{R}_h^* = \cup K_P^*$. Define the associated test function space $\underline{Y}_h$ as the space of piecewise constant vector-valued functions:

$$\underline{Y}_h = \{\underline{z} \in (L^2(\Omega))^2 : \underline{z}|_{K_P^*} \text{ is a constant vector, and } \underline{z}|_{K_P^*} = \underline{0} \text{ on any boundary dual element } K_P^*\}.$$

Denote by χ_j^* the scalar characteristic function associated with the dual element $K_{P_j}^*$, $j = 1, \ldots, N_I$. Here N_I is the number of interior nodes of $\mathcal{R}_h$. We see that, for any $\underline{v}_h \in \underline{Y}_h$,

$$\underline{v}_h(x) = \sum_{j=1}^{N_I} \underline{v}_h(P_j)\chi_j^*(x), \qquad \forall x \in \Omega.$$

Define, for $\underline{u}_h \in \underline{H}_h$ and $\underline{v}_h \in \underline{Y}_h$,

$$a(\underline{u}_h, \underline{v}_h) := \alpha_0(\underline{u}_h, \underline{v}_h) - \sum_{i=1}^{N_I} \int_{\partial K_{P_i}^*} \frac{\partial \underline{u}_h}{\partial \underline{n}} \cdot \underline{v}_h d\sigma \tag{2.3a}$$

$$= \alpha_0(\underline{u}_h, \underline{v}_h) - \sum_{i=1}^{N_I} \underline{v}_h(P_i) \cdot \int_{\partial K_{P_i}^*} \frac{\partial \underline{u}_h}{\partial \underline{n}} d\sigma, \tag{2.3b}$$

where $\partial \underline{u}_h / \partial \underline{n}$ is the vector field containing two normal derivatives of $\underline{u}_h$. Equation (2.3) is motivated by integrating the second term of (2.1a) against a test function and then formally applying Green's formula. Let N_R denote the number of elements in the primal partition and define, for $\underline{u}_h \in \underline{H}_h$, $\underline{v}_h \in \underline{Y}_h$, and $p_h, q_h \in L_h$,

$$b(\underline{v}_h, p_h) := \sum_{i=1}^{N_I} \underline{v}_h(P_i) \cdot \int_{\partial K_{P_i}^*} p_h \underline{n} d\sigma,$$

$$(\underline{f}, \underline{v}_h) = \sum_{i=1}^{N_I} \underline{v}_h(P_i) \cdot \int_{K_{P_i}^*} \underline{f} dx.$$

The covolume scheme of equation (2.1) is: Given $\underline{f} \in (L^2(\Omega))^2$, find $(\underline{u}_h, p_h) \in \underline{H}_h \times L_h$ such that

$$a(\underline{u}_h, \underline{v}_h) + b(\underline{v}_h, p_h) = (\underline{f}, \underline{v}_h), \qquad \forall \underline{v}_h \in \underline{Y}_h, \tag{2.4a}$$

$$b^*(\underline{u}_h, q_h) = 0, \qquad \forall q_h \in L_h. \tag{2.4b}$$

Since $\underline{H}_h$ is nonconforming, the gradient and divergence operator on it must be defined piecewisely:

$$(\nabla_h \underline{w}_h)|_K := \nabla(\underline{w}_h|K),$$

$$(\mathrm{div}_h \underline{w}_h)|_K := \mathrm{div}(\underline{w}_h|K).$$

On the space $\underline{H}_h$, we define

$$|\underline{w}_h|_{1,h}^2 := (\nabla_h \underline{w}_h, \nabla_h \underline{w}_h) = \sum_{K \in \mathcal{R}_h} (\nabla \underline{w}_h, \nabla \underline{w}_h)_K,$$

$$(\nabla \underline{w}_h, \nabla \underline{z}_h)_K := \sum_{i=1}^{2} (D_i \underline{w}_h, D_i \underline{z}_h)_K,$$

$$\|\underline{w}_h\|_{1,h}^2 := |\underline{w}_h|_{1,h}^2 + \|\underline{w}_h\|_0^2,$$

where $(\cdot, \cdot)_K$ is the $L_2(K)^2$ inner product, and D_i denotes the partial derivative. Below, we shall use ∇ for ∇_h and div for div_h when there is no danger of confusion.

III. L^2-ERROR ESTIMATE

In this section, we give an L^2-error estimate for the velocity vector under the assumption $\underline{u} \in \underline{H}^3(\Omega)$ and $p \in H^2(\Omega)$. This higher regularity seems to be inevitable because of the low approximation property of γ_h.

Before proceeding the L^2-error estimate, we need some preliminary results.

First, we introduce an one-to-one transfer operator γ_h from $\underline{H}_h$ onto $\underline{Y}_h$ by

$$\gamma_h \underline{v}_h(x) := \sum_{j=1}^{N_I} \underline{v}_h(P_j)\chi_j^*(x), \qquad \forall x \in \Omega. \tag{3.1}$$

Then, we have that [14]

$$b(\gamma_h \underline{w}_h, q_h) = b^*(\underline{w}_h, q_h), \qquad \forall \underline{w}_h \in \underline{H}_h, q_h \in L_h \tag{3.2}$$

and there exists $C_0 > 0$ independent of h such that

$$\|\gamma_h \underline{w}_h - \underline{w}_h\|_0 \leq C_0 h |\underline{w}_h|_{1,h}, \qquad \forall \underline{w}_h \in \underline{H}_h. \tag{3.3}$$

Due to (3.2), (2.4) can be rewritten as

$$a(\underline{u}_h, \gamma_h \underline{w}_h) + b(\gamma_h \underline{w}_h, p_h) = (\underline{f}, \gamma_h \underline{w}_h), \qquad \forall \underline{w}_h \in \underline{H}_h, \tag{3.4a}$$

$$b(\gamma_h \underline{u}_h, q_h) = 0, \qquad \forall q_h \in L_h. \tag{3.4b}$$

Then the error estimate of this covolume scheme can be carried out by comparing (3.4) with (2.2) to yield the following theorem [14].

Theorem 3.1. *Let the triangular partition family $\mathcal{R}_h$ of the domain Ω be regular, let $(\underline{u}_h, p_h)$ be the solution of the problem* (2.4), *and $(\underline{u}, p)$ be that of the problem* (2.1). *Then there exists a positive constant C independent of h such that*

$$\|\underline{u} - \underline{u}_h\|_{1,h} + \|p - p_h\|_0 \leq Ch(\|\underline{u}\|_2 + \|p\|_1 + 1),$$

provided that $\underline{u} \in \underline{H}_0^1(\Omega) \cap \underline{H}^2(\Omega)$, $p \in H^1(\Omega)$.

The proof of the next Lemma can be found in [18].

Lemma 3.2. *Let m indicate the mean value over an edge e in K. There exists a constant $C >$ 0 independent of K such that*

$$\left| \int_e \phi(v - v^m)\,d\sigma \right| \leq Ch(K)|\phi|_{1,K}|v|_{1,K},$$

for all ϕ, $v \in H^1(K)$.

The following two lemmas and corollary are easy to verify.

Lemma 3.3. *For* $\phi \in H^1(\Omega)$ *and* $\underline{g}_h \in \underline{H}_h$, *we have*

$$\sum_{i=1}^{N_R} \int_{\partial K_i} \phi \underline{g}_h \cdot \underline{n} d\sigma = \sum_{i=1}^{N_I} \int_{e_i} \phi [\underline{g}_h]_{e_i} \cdot \underline{n} d\sigma,$$

where $[\cdot]_{e_i}$ *denotes the jump of a function across* e_i *if* e_i *is an interior edge and denotes the value of a function on* e_i *if* e_i *is a boundary edge and* $\underline{n}$ *is an outward unit normal vector on* ∂K_i *and any unit normal vector on edge* e_i. *Also, we have*

$$\sum_{i=1}^{N_R} \int_{\partial K_i} \frac{\partial \underline{f}}{\partial \underline{n}} \cdot \underline{g}_h d\sigma = \sum_{i=1}^{N_I} \int_{e_i} \frac{\partial \underline{f}}{\partial \underline{n}} \cdot [\underline{g}_h]_{e_i} d\sigma,$$

for $\underline{f} \in \underline{H}^2(\Omega)$ *and* $\underline{g}_h \in \underline{H}_h$.

Corollary. *We have*

$$\left| \sum_{i=1}^{N_R} \int_{\partial K_i} (v - v^m) \underline{z} \cdot \underline{n} d\sigma \right| = \left| \sum_{i=1}^{N_I} \int_{e_i} (v - v^m) [\underline{z}]_{e_i} \cdot \underline{n} d\sigma \right| \leq Ch |v|_1 |\underline{z}|_{1,h},$$

$$\left| \sum_{i=1}^{N_R} \int_{\partial K_i} \left(\frac{\partial \underline{v}}{\partial \underline{n}} - \left(\frac{\partial \underline{v}}{\partial \underline{n}} \right)^m \right) \cdot \underline{z} d\sigma \right| = \left| \sum_{i=1}^{N_I} \int_{e_i} \left(\frac{\partial \underline{v}}{\partial \underline{n}} - \left(\frac{\partial \underline{v}}{\partial \underline{n}} \right)^m \right) \cdot [\underline{z}]_{e_i} d\sigma \right|$$

$$\leq Ch \left| \frac{\partial \underline{v}}{\partial \underline{n}} \right|_1 |\underline{z}|_{1,h} \leq Ch |\underline{v}|_2 \|\underline{z}\|_{1,h},$$

for all $v \in H^1(\Omega)$, $\underline{v} \in \underline{H}^2(\Omega)$, $z \in H_h \oplus H^2(\Omega)$ (*or* $C^0(\Omega)$), *and* $\underline{z} \in \underline{H}_h \oplus \underline{H}^2(\Omega)$ (*or* $(C^0(\Omega))^2$).

Lemma 3.4. *For any* $\underline{w}_h \in \underline{H}_h$ *and any* $\underline{v} \in \underline{H}^2(\Omega) \cap \underline{H}^1_0(\Omega)$, *we have*

$$\int_e [\underline{w}_h]_e d\sigma = 0, \qquad \int_e \underline{f} \cdot [\underline{v}]_e d\sigma = 0,$$

for any edge e *in* Ω *and for any* $\underline{f} \in (L^2(e))^2$.

Lemma 3.5. *For any* $\underline{w}_h \in \underline{H}_h$ *and for all element* K *in* Ω, *we have*

$$\int_K (\underline{w}_h - \gamma_h \underline{w}_h) dx = 0.$$

Proof. Since $\gamma_h \underline{w}_h$ is constant for each subtriangle $\Delta_i = \Delta A_i C_1 A_{i+1}$, $i = 1, 2, 3$, and $\underline{w}_h$ is linear, we have

$$\int_K \gamma_h \underline{w}_h dx = \sum_{i=1}^{3} \int_{\Delta_i} \gamma_h \underline{w}_h dx = \sum_{i=1}^{3} \underline{w}_h(P_i)|\Delta_i| = \frac{|K|}{3} \sum_{i=1}^{3} \underline{w}_h(P_i) = \int_K \underline{w}_h dx. \qquad \blacksquare$$

Lemma 3.6. *For any* $\underline{f}|_{K_i} \in \underline{H}^2(K_i)$ *for all* $K_i \in \mathcal{R}_h$ *and* $\underline{w}_h \in \underline{Y}_h$, *we have*

$$\sum_{i=1}^{N_R} \int_{K_i} \Delta \underline{f} \cdot \underline{w}_h dx = \sum_{i=1}^{N_I} \int_{\partial K_i^*} \frac{\partial \underline{f}}{\partial \underline{n}} \cdot \underline{w}_h d\sigma + \sum_{i=1}^{N_R} \int_{\partial K_i} \frac{\partial \underline{f}}{\partial \underline{n}} \cdot \underline{w}_h d\sigma.$$

Proof. Since $\nabla \underline{w}_h = 0$, using Green's formula over each $K_i \cap K_{P_j}^*$, $j = 1, 2, 3$, we get

$$\begin{aligned}
\sum_{i=1}^{N_R} \int_{K_i} \Delta \underline{f} \cdot, \underline{w}_h dx &= \sum_{i=1}^{N_R} \sum_{j=1}^{3} \int_{K_{P_j}^* \cap K_i} \Delta \underline{f} \cdot \underline{w}_h dx = \sum_{i=1}^{N_R} \sum_{j=1}^{3} \int_{\partial (K_{P_j}^* \cap K_i)} \frac{\partial \underline{f}}{\partial \underline{n}} \cdot \underline{w}_h d\sigma \\
&= \sum_{i=1}^{N_R} \sum_{j=1}^{3} \left(\int_{(\partial K_{P_j}^*) \cap K_i} \frac{\partial \underline{f}}{\partial \underline{n}} \cdot \underline{w}_h d\sigma + \int_{(\partial K_i) \cap K_{P_j}^*} \frac{\partial \underline{f}}{\partial \underline{n}} \cdot \underline{w}_h d\sigma \right) \\
&= \sum_{i=1}^{N_I} \int_{\partial K_{P_j}^*} \frac{\partial \underline{f}}{\partial \underline{n}} \cdot \underline{w}_h d\sigma + \sum_{i=1}^{N_R} \int_{\partial K_i} \frac{\partial \underline{f}}{\partial \underline{n}} \cdot \underline{w}_h d\sigma. \qquad \blacksquare
\end{aligned}$$

Lemma 3.7. *For any* $\underline{z} \in (C^0(\Omega))^2 \cap \underline{H}_0^1(\Omega)$, $\underline{v} \in \underline{H}^3(\Omega)$, *and* $\underline{v}_h, \underline{z}_h \in \underline{H}_h$, *we have*

$$\sum_{i=1}^{N_R} \left(\int_{\partial K_i} \frac{\partial (\underline{v} - \underline{v}_h)}{\partial \underline{n}} \cdot (\underline{z}_h - \gamma_h \underline{z}_h) d\sigma \right) \leq Ch|\underline{v}|_2 |\underline{z}_h - \underline{z}|_{1,h}.$$

Proof. Since $\partial \underline{v}_h / \partial \underline{n}$ is constant along an edge e of the element K_i, and $\partial \underline{v} / \partial \underline{n} \cdot \gamma_h \underline{z}_h$ continuous on any edge e of the element K_i, we have, by Lemma 3.4,

$$\begin{aligned}
\sum_{i=1}^{N_R} \int_{\partial K_i} \frac{\partial (\underline{v} - \underline{v}_h)}{\partial \underline{n}} \cdot (\underline{z}_h - \gamma_h \underline{z}_h) d\sigma &= \sum_{i=1}^{N_R} \int_{\partial K_i} \frac{\partial \underline{v}}{\partial \underline{n}} \cdot (\underline{z}_h - \gamma_h \underline{z}_h) d\sigma = \sum_{i=1}^{N_I} \int_{e_i} \frac{\partial \underline{v}}{\partial \underline{n}} \cdot [\underline{z}_h]_{e_i} d\sigma \\
&= \sum_{i=1}^{N_I} \int_{e_i} \left(\frac{\partial \underline{v}}{\partial \underline{n}} - \left(\frac{\partial \underline{v}}{\partial \underline{n}} \right)^m \right) \cdot [\underline{z}_h]_{e_i} d\sigma = \sum_{i=1}^{N_I} \int_{e_i} \left(\frac{\partial \underline{v}}{\partial \underline{n}} - \left(\frac{\partial \underline{v}}{\partial \underline{n}} \right)^m \right) \cdot [\underline{z}_h - \underline{z}]_{e_i} d\sigma,
\end{aligned}$$

for any $\underline{z} \in (C^0(\Omega))^2 \cap \underline{H}_0^1(\Omega)$. The result follows from Corollary. ■

To prove L^2-error estimate, we introduce a perturbation operator E that is defined by

$$E(\underline{v} - \underline{v}_h, \underline{z}_h) := a^*(\underline{v} - \underline{v}_h, \underline{z}_h) - a(\underline{v} - \underline{v}_h, \gamma_h \underline{z}_h),$$

where $\underline{v} \in \underline{H}^3(\Omega)$, $\underline{v}_h, \underline{z}_h \in \underline{H}_h$.

Lemma 3.8. *For any* $\underline{v} \in \underline{H}^3(\Omega)$, $\underline{v}_h, \underline{z}_h \in \underline{H}_h$, *and* $\underline{z} \in (C^0(\Omega))^2 \cap \underline{H}_0^1$, *we have*

$$|E(\underline{v} - \underline{v}_h, \underline{z}_h)| \leq Ch(\|\underline{v} - \underline{v}_h\|_0 |\underline{z}_h|_{1,h} + \|\underline{v}\|_2 |\underline{z}_h - \underline{z}|_{1,h} + h\|\underline{v}\|_3 |\underline{z}_h|_{1,h}). \tag{3.5}$$

Proof. By using Green's formula over each $K_i \in \mathcal{R}_h$, we get, for $\underline{v} \in \underline{H}^3(\Omega)$ and $\underline{v}_h, \underline{z}_h \in \underline{H}_h$,

$$\begin{aligned} a^*(\underline{v} - \underline{v}_h, \underline{z}_h) &= \alpha_0(\underline{v} - \underline{v}_h, \underline{z}_h) + \sum_{i=1}^{N_R} \int_{K_i} \nabla(\underline{v} - \underline{v}_h) \cdot \nabla \underline{z}_h dx \\ &= \alpha_0(\underline{v} - \underline{v}_h, \underline{z}_h) + \sum_{i=1}^{N_R} \left(\int_{\partial K_i} \frac{\partial(\underline{v} - \underline{v}_h)}{\partial \underline{n}} \cdot \underline{z}_h d\sigma - \int_{K_i} \Delta(\underline{v} - \underline{v}_h) \cdot \underline{z}_h dx \right). \end{aligned} \tag{3.6}$$

Let $\underline{f} = \underline{v} - \underline{v}_h$ and $\underline{w}_h = \gamma_h \underline{z}_h$ in Lemma 3.6, then we see

$$\begin{aligned} a(\underline{v} - \underline{v}_h, \gamma_h \underline{z}_h) &:= \alpha_0(\underline{v} - \underline{v}_h, \gamma_h \underline{z}_h) - \sum_{i=1}^{N_I} \int_{\partial K_{P_i}^*} \frac{\partial(\underline{v} - \underline{v}_h)}{\partial \underline{n}} \cdot \gamma_h \underline{z}_h d\sigma \\ &= \alpha_0(\underline{v} - \underline{v}_h, \gamma_h \underline{z}_h) + \sum_{i=1}^{N_R} \left(\int_{\partial K_i} \frac{\partial(\underline{v} - \underline{v}_h)}{\partial \underline{n}} \cdot \gamma_h \underline{z}_h d\sigma - \int_{K_i} \Delta(\underline{v} - \underline{v}_h) \cdot \gamma_h \underline{z}_h dx \right). \end{aligned} \tag{3.7}$$

Subtracting (3.7) from (3.6), we obtain

$$\begin{aligned} E(\underline{v} - \underline{v}_h, \underline{z}_h) &= \alpha_0(\underline{v} - \underline{v}_h, \underline{z}_h - \gamma_h \underline{z}_h) \\ &+ \sum_{i=1}^{N_R} \left(\int_{\partial K_i} \frac{\partial(\underline{v} - \underline{v}_h)}{\partial \underline{n}} \cdot (\underline{z}_h - \gamma_h \underline{z}_h) d\sigma + \int_{K_i} \Delta(\underline{v} - \underline{v}_h) \cdot (\underline{z}_h - \gamma_h \underline{z}_h) dx \right). \end{aligned} \tag{3.8}$$

We have, from Hölder inequality and (3.3),

$$\alpha_0(\underline{v} - \underline{v}_h, \underline{z}_h - \gamma_h \underline{z}_h) \leq C\|\underline{v} - \underline{v}_h\|_0 \|\underline{z}_h - \gamma_h \underline{z}_h\|_0 \leq Ch\|\underline{v} - \underline{v}_h\|_0 |\underline{z}_h|_{1,h}. \tag{3.9}$$

Next, we estimate the third term on right side (3.8). Since $\Delta \underline{v}_h = 0$, we obtain, by Bramble-Hilbert lemma and Lemma 3.5,

$$\begin{aligned} \sum_{i=1}^{N_R} \int_{K_i} \Delta(\underline{v} - \underline{v}_h) \cdot (\underline{z}_h - \gamma_h \underline{z}_h) dx &= \sum_{i=1}^{N_R} \int_{K_i} (\Delta \underline{v} - (\Delta \underline{v})^m) \cdot (\underline{z}_h - \gamma_h \underline{z}_h) dx \\ &\leq Ch\|\Delta \underline{v}\|_1 \|\underline{z}_h - \gamma_h \underline{z}_h\|_0 \leq Ch^2 \|\underline{v}\|_3 |\underline{z}_h|_{1,h}, \end{aligned} \tag{3.10}$$

where $(\Delta \underline{v})^m$ is a mean value of $\Delta \underline{v}$ over K. From (3.9), Lemma 3.7, and (3.10), we have (3.5). ■

Lemma 3.9. *For any* $p \in H^2(\Omega)$, $p_h \in L_h$, $\underline{\Psi}_h \in \underline{H}_h$ *with* $b^*(\underline{\Psi}_h, q_h) = 0$, $\forall q_h \in L_h$, *and* $\underline{\Psi} \in \underline{H}^2(\Omega)$ *with div* $\underline{\Psi} = 0$, *we have*

$$\left| -\sum_{i=1}^{N_I} \int_{K_{P_i}^*} \nabla p \cdot \gamma_h \underline{\Psi}_h dx \right| \le ch^2 \|p\|_2 |\underline{\Psi}_h|_{1,h} + Ch\|p\|_1 |\underline{\Psi}_h - \underline{\Psi}|_{1,h} + \|\underline{\Psi}_h - \underline{\Psi}\|_{1,h} \|p_h - p\|_0. \quad (3.11)$$

Proof. By adding and subtracting $\underline{\Psi}_h$ and rearranging of domain of integration, we have

$$-\sum_{i=1}^{N_I} \int_{K_{P_i}^*} \nabla p \cdot \gamma_h \underline{\Psi}_h dx = -\sum_{i=1}^{N_I} \int_{K_{P_i}^*} \nabla p \cdot (\gamma_h \underline{\Psi}_h - \underline{\Psi}_h) dx - \sum_{i=1}^{N_I} \int_{K_{P_i}^*} \nabla p \cdot \underline{\Psi}_h dx$$

$$= -\sum_{i=1}^{N_R} \int_{K_i} \nabla p \cdot (\gamma_h \underline{\Psi}_h - \underline{\Psi}_h) dx - \sum_{i=1}^{N_R} \int_{K_i} \nabla p \cdot \underline{\Psi}_h dx.$$

We see, by Lemma 3.5, Bramble-Hilbert lemma, and (3.3), that

$$\left| -\sum_{i=1}^{N_R} \int_{K_i} \nabla p \cdot (\gamma_h \underline{\Psi}_h - \underline{\Psi}_h) dx \right| = \left| -\sum_{i=1}^{N_R} \int_{K_i} (\nabla p - (\nabla p)^m) \cdot (\gamma_h \underline{\Psi}_h - \underline{\Psi}_h) dx \right|$$

$$\le \|\nabla p - (\nabla p)^m\|_0 \|\gamma_h \underline{\Psi}_h - \underline{\Psi}_h\|_0 \le Ch^2 \|p\|_2 |\underline{\Psi}_h|_{1,h}. \quad (3.12)$$

Also, by Green's Theorem, continuity of p and the fact that $b^*(\underline{\Psi}_h, p_h) = 0$, we have

$$-\sum_{i=1}^{N_R} \int_{K_i} \nabla p \cdot \underline{\Psi}_h dx = -\sum_{i=1}^{N_R} \int_{\partial K_i} p \underline{\Psi}_h \cdot \underline{n} d\sigma + \sum_{i=1}^{N_R} \int_{K_i} p \operatorname{div} \underline{\Psi}_h dx$$

$$= -\sum_{i=1}^{N_I} \int_{e_i} p[\underline{\Psi}_h \cdot \underline{n}]_{e_i} d\sigma + \sum_{i=1}^{N_R} \int_{K_i} \operatorname{div} \underline{\Psi}_h (p - p_h) dx.$$

Since div $\underline{\Psi} = 0$, we have, by Corollary,

$$\left| -\sum_{i=1}^{N_R} \int_{K_i} \nabla p \cdot \underline{\Psi}_h dx \right| = \left| -\sum_{i=1}^{N_I} \int_{e_i} (p - (p)^m)[(\underline{\Psi}_h - \underline{\Psi}) \cdot \underline{n}]_{e_i} d\sigma \right|$$

$$+ \left| \sum_{i=1}^{N_R} \int_{K_i} \operatorname{div}(\underline{\Psi}_h - \underline{\Psi})(p - p_h) dx \right| \le Ch\|p\|_1 |\underline{\Psi}_h - \underline{\Psi}|_{1,h} + \|\operatorname{div}(\underline{\Psi}_h - \underline{\Psi})\|_0 \|p_h - p\|_0$$

$$\le Ch\|p\|_1 |\underline{\Psi}_h - \underline{\Psi}|_{1,h} + \|\underline{\Psi}_h - \underline{\Psi}\|_{1,h} \|p_h - p\|_0.$$

From this and (3.12), we obtain (3.11). ∎

Lemma 3.10. *Let* $\underline{u} \in \underline{H}^3(\Omega)$ *and* $p \in H^2(\Omega)$ *be the solution of* (2.1). *Then for any* $\underline{u}_h$, $\underline{\Psi}_h \in \underline{H}_h$ *with* $b^*(\underline{\Psi}_h, q_h) = 0$, *for all* $q_h \in L_h$, *and* $p_h \in L_h$, *we have*

$$\begin{aligned}|a^*(\underline{\Psi}, \underline{u} - \underline{u}_h)| \leq C\|\underline{\Psi} - \underline{\Psi}_h\|_{1,h}(\|\underline{u} - \underline{u}_h\|_{1,h} + \|p_h - p\|_0) \\ + |\underline{\Psi} - \underline{\Psi}_h|_{1,h}(\|\underline{u}\|_2 + \|p\|_1) + |\underline{\Psi}_h|_{1,h}\|\underline{u} - \underline{u}_h\|_{1,h}] + Ch^2|\underline{\Psi}_h|_{1,h}(\|\underline{u}\|_3 + \|p\|_2).\end{aligned}$$

Proof. Integrating the first equation of (2.1) against any $\underline{v}_h \in \underline{Y}_h$ over each covolume K_P^*, we see that

$$a(\underline{u}, \underline{v}_h) + b(\underline{v}_h, p) = (\underline{f}, \underline{v}_h). \tag{3.13}$$

Subtracting (2.4a) from (3.13), we see that

$$a(\underline{u} - \underline{u}_h, \underline{v}_h) = b(\underline{v}_h, p_h) - b(\underline{v}_h, p) = b(\underline{v}_h, p_h) - \sum_{i=1}^{N_I} \int_{K_{P_i}^*} \nabla p \cdot \underline{v}_h dx. \tag{3.14}$$

By the definition of $a^*(\cdot, \cdot)$, $E(\cdot, \cdot)$, and (3.14), we have

$$\begin{aligned}a^*(\underline{\Psi}, \underline{u} - \underline{u}_h) &= a^*(\underline{\Psi} - \underline{\Psi}_h, \underline{u} - \underline{u}_h) + a^*(\underline{u} - \underline{u}_h, \underline{\Psi}_h) \\ &= a^*(\underline{\Psi} - \underline{\Psi}_h, \underline{u} - \underline{u}_h) + E(\underline{u} - \underline{u}_h, \underline{\Psi}_h) + a(\underline{u} - \underline{u}_h, \gamma_h\underline{\Psi}_h) \\ &a^*(\underline{\Psi} - \underline{\Psi}_h, \underline{u} - \underline{u}_h) + E(\underline{u} - \underline{u}_h, \underline{\Psi}_h) + b(\gamma_h\underline{\Psi}_h, p_h) - \sum_{i=1}^{N_I} \int_{K_{P_i}^*} \nabla p \cdot \gamma_h\underline{\Psi}_h dx.\end{aligned}$$

From this, the boundedness of the bilinear form a^*, Lemma 3.8 and 3.9, we obtain the Lemma. ■

Now we are ready to use the duality argument. For that purpose, we first consider the following: Given $\underline{g} \in (L^2(\Omega))^2$, we let $(\underline{\Psi}, \chi)$ be the solution of the generalized Stokes problem

$$\alpha_0\underline{\Psi} - \Delta\underline{\Psi} + \nabla\chi = \underline{g}, \qquad \text{in } \Omega, \tag{3.15a}$$

$$\operatorname{div} \underline{\Psi} = 0, \qquad \text{in } \Omega, \tag{3.15b}$$

$$\underline{\Psi} = \underline{0}, \qquad \text{on } \partial\Omega. \tag{3.15c}$$

Then, by the regularity property of the Stokes problems [18] for the convex polygonal domain Ω, we have $(\underline{\Psi}, \chi) \in \underline{H}_0^2(\Omega) \times H^1(\Omega)$ and

$$\|\underline{\Psi}\|_2 + \|\chi\|_1 \leq C\|\underline{g}\|_0. \tag{3.16}$$

If we let $(\underline{\Psi}_h, \chi_h) \in \underline{H}_h \times L_h$ to be the solution of nonconforming mixed finite element problem

$$a^*(\underline{\Psi}_h, \underline{v}_h) + b^*(\underline{v}_h, \chi_h) = (\underline{g}, \underline{v}_h), \qquad \forall \underline{v}_h \in \underline{H}_h, \qquad b^*(\underline{\Psi}_h, q_h) = 0, \qquad \forall q_h \in L_h,$$

we have the following well-known error estimate about nonconforming mixed finite element for the Stokes problem [18]:

$$|\underline{\Psi} - \underline{\Psi}_h|_{1,h} + \|\chi - \chi_h\|_0 \le Ch(\|\underline{\Psi}\|_2 + \|\chi\|_1), \tag{3.17a}$$

$$\|\underline{\Psi} - \underline{\Psi}_h\|_0 \le Ch^2(\|\underline{\Psi}\|_2 + \|\chi\|_1). \tag{3.17b}$$

Proposition 3.11 (Generalized Aubin-Nitche's Duality Argument). *Let* $\underline{u} \in \underline{H}^3(\Omega)$ *and* $p \in H^2(\Omega)$ *be the solution of* (2.1). *Then for any* $\underline{u}_h \in \underline{H}_h$ *with* $b^*(\underline{u}_h, q_h) = 0$, $\forall q_h \in L_h$, *and* $\underline{g} \in (L^2(\Omega))^2$ *satisfying* (3.15), *we have*

$$|(\underline{u} - \underline{u}_h, \underline{g})| \le [Ch(\|\underline{u} - \underline{u}_h\|_{1,h} + \|p - p_h\|_0) + Ch^2(\|\underline{u}\|_3 + \|p\|_2)]\|\underline{g}\|_0. \tag{3.18}$$

Proof. Multiplying (3.15a) by $\underline{u} - \underline{u}_h$ and using Green's formula over each K_i, we obtain

$$\begin{aligned}(\underline{u} - \underline{u}_h, \underline{g}) &= \int_\Omega (\alpha_0\underline{\Psi} - \Delta\underline{\Psi} + \nabla\chi)\cdot(\underline{u} - \underline{u}_h)dx \\ &= a^*(\underline{\Psi}, \underline{u} - \underline{u}_h) - \sum_{i=1}^{N_R}\int_{\partial K_i}\frac{\partial\underline{\Psi}}{\partial\underline{n}}\cdot(\underline{u} - \underline{u}_h)d\sigma + \sum_{i=1}^{N_R}\left(\int_{\partial K_i}\chi(\underline{u} - \underline{u}_h)\cdot\underline{n}d\sigma\right. \\ &\qquad \left. - \int_{K_i}\chi\,\mathrm{div}(\underline{u} - \underline{u}_h)dx\right).\end{aligned} \tag{3.19}$$

From Lemma 3.10 and (3.17a), we see

$$\begin{aligned}a^*(\underline{\Psi}, \underline{u} - \underline{u}_h) \le{}& Ch(\|\underline{\Psi}\|_2 + \|\chi\|_1)(\|\underline{u} - \underline{u}_h\|_{1,h} + \|p - p_h\|_0) \\ &+ Ch^2(\|\Psi\|_2 + \|\chi\|_1)(\|\underline{u}\|_3 + \|p\|_2),\end{aligned} \tag{3.20}$$

where the inequality $|\underline{\Psi}_h|_{1,h} \le |\underline{\Psi} - \underline{\Psi}_h|_{1,h} + |\underline{\Psi}|_1 \le \|\underline{\Psi}\|_2 + \|\chi\|_1$ was used.

For the second term in (3.19), we see that, from Lemma 3.3, Lemma 3.4, Corollary,

$$\begin{aligned}\sum_{i=1}^{N_R}\int_{\partial K_i}\frac{\partial\underline{\Psi}}{\partial\underline{n}}\cdot(\underline{u} - \underline{u}_h)d\sigma &= \sum_{i=1}^{N_I}\int_{e_i}\frac{\partial\underline{\Psi}}{\partial\underline{n}}[\underline{u} - \underline{u}_h]_{e_i}d\sigma \\ &= \sum_{i=1}^{N_I}\int_{e_i}\left(\frac{\partial\underline{\Psi}}{\partial\underline{n}} - \left(\frac{\partial\underline{\Psi}}{\partial\underline{n}}\right)^m\right)[\underline{u} - \underline{u}_h]_{e_i}d\sigma \le Ch\|\underline{\Psi}\|_2|\underline{u} - \underline{u}_h|_{1,h}.\end{aligned} \tag{3.21}$$

Using the fact that div $\underline{u} = 0$ and $b^*(\underline{u}_h, \chi_h) = 0$ and using Lemma 3.3, Lemma 3.4, Corollary, and (3.17), we have

TABLE I. L^2 error with $\alpha_0 = 0.0$.

h	Covolume $\|\underline{u} - \underline{u}_h\|_{0,h}(\rho_h)$	Mixed FEM $\|\underline{u} - \underline{u}_h\|_{0,h}(\rho_h)$
1/4	0.573651	0.599566
1/8	0.216305(2.6520)	0.221553(2.7062)
1/16	0.064714(3.3425)	0.065847(3.3647)
1/32	0.017265(3.7483)	0.017532(3.7558)
1/64	0.004428(3.8991)	0.004493(3.9021)

$$\sum_{i=1}^{N_R}\left(\int_{\partial K_i}\chi(\underline{u}-\underline{u}_h)\cdot\underline{n}d\sigma-\int_{K_i}\chi\,\mathrm{div}(\underline{u}-\underline{u}_h)dx\right)=\sum_{i=1}^{N_I}\int_{e_i}(\chi-(\chi)^m)[(\underline{u}-\underline{u}_h)\cdot\underline{n}]_{e_i}d\sigma$$

$$-\sum_{i=1}^{N_R}\int_{K_i}(\chi-\chi_h)\mathrm{div}(\underline{u}-\underline{u}_h)dx\le Ch\|\chi\|_1|\underline{u}-\underline{u}_h|_{1,h}+\|\chi-\chi_h\|_0|\underline{u}-\underline{u}_h|_{1,h},$$

and thus, we have

$$\sum_{i=1}^{N_R}\left(\int_{\partial K_i}\chi(\underline{u}-\underline{u}_h)\cdot\underline{n}d\sigma-\int_{K_i}\chi\,\mathrm{div}(\underline{u}-\underline{u}_h)dx\right)\le Ch(\|\underline{\Psi}\|_2+\|\chi\|_1)|\underline{u}-\underline{u}_h|_{1,h}. \qquad (3.22)$$

Collecting (3.20)–(3.22), we obtain the desired result using (3.16). ■

Now we state the main result of this article.

Theorem 3.12. *Let the triangular partition family $\mathcal{R}_h$ of the domain Ω be regular, let $(\underline{u}_h, p_h)$ be the solution of the problem* (2.4), *and $(\underline{u}, p)$ be that of the problem* (2.1). *Then there exists a positive constant C, independent of h, such that*

$$\|\underline{u}-\underline{u}_h\|_0\le Ch^2(\|\underline{u}\|_3+\|p\|_2+1),$$

provided that $\underline{u}\in\underline{H}_0^1(\Omega)\cap\underline{H}^3(\Omega)$, $p\in H^2(\Omega)$.

Proof. From Proposition 3.11 and Theorem 3.1, we have

$$|(\underline{u}-\underline{u}_h,\underline{g})|\le Ch^2\|\underline{g}\|_0(\|\underline{u}\|_3+\|p\|_2+1).$$

TABLE II. L^2 error with $\alpha_0 = 1.0$.

h	Covolume $\|\underline{u} - \underline{u}_h\|_{0,h}(\rho_h)$	Mixed FEM $\|\underline{u} - \underline{u}_h\|_{0,h}(\rho_h)$
1/4	0.569075	0.593256
1/8	0.215386(2.6421)	0.220419(2.6915)
1/16	0.064570(3.3357)	0.065669(3.3565)
1/32	0.017239(3.7456)	0.017499(3.7527)
1/64	0.004422(3.8985)	0.004486(3.9008)

TABLE III. L^2 error with $\alpha_0 = 10.0$.

h	Covolume $\|\underline{u} - \underline{u}_h\|_{0,h}(\rho_h)$	Mixed FEM $\|\underline{u} - \underline{u}_h\|_{0,h}(\rho_h)$
1/4	0.532489	0.543780
1/8	0.207765(2.5629)	0.211146(2.5754)
1/16	0.063378(3.2782)	0.064219(3.2879)
1/32	0.017026(3.7224)	0.017234(3.7263)
1/64	0.004377(3.8899)	0.004429(3.8912)

Taking supremum over $\underline{g}$, we get

$$\|\underline{u} - \underline{u}_h\|_0 \leq Ch^2(\|\underline{u}\|_3 + \|p\|_2 + 1). \qquad \blacksquare$$

IV. NUMERICAL EXPERIMENTS

For the numerical verification of our theory, we have chosen one of the usual artificial test problem on the unit square, $\Omega = (0, 1) \times (0, 1)$, with the exact solution

$$u_1(x_1, x_2) = -256x_1^2(x_1 - 1)^2x_2(x_2 - 1)(2x_2 - 1),$$

$$u_2(x_1, x_2) = -u_1(x_2, x_1),$$

$$p(x_1, x_2) = 150(x_1 - 1/2)(x_2 - 1/2).$$

Tables I–III represent the numerical results according to h and α_0. As a reference, we compare it with standard mixed nonconforming finite element method. In these tables, $\|\cdot\|_{0,h}$ is the discrete L^2-error and $\rho_h = \|\underline{u} - \underline{u}_h\|_{0,2h}/\|\underline{u} - \underline{u}_h\|_{0,h}$. For all the cases we tested, both schemes exhibit similar error behavior.

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