International Journal of Pure and Applied Mathematics
Volume 76 No. 4 2012, 571-592
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

A NEW FAMILY OF MIXED FINITE ELEMENT SPACES ON A CLASS OF DISTORTED HEXAHEDRA

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Abstract: In this paper, we introduce a new family of mixed finite element spaces for elliptic problem on a domain partitioned by a class of distorted hexahedra. We show an optimal order of approximation for the velocity variable for all $k \geq 0$. For the pressure variable, we suggest a local post-processing technique to obtain an optimal order.

AMS Subject Classification: 65N30, 65N12
Key Words: mixed finite element method, optimal order, hexahedra, local post-processing technique

1. Introduction

In this paper, we consider mixed finite element approximations of Darcy equations. Mixed finite element methods have been widely used to obtain an accurate approximation of the velocity of a fluid in porous media (see [1], [2], [3], [4], [5], [6], [7], [8], [9]). It is known that classical mixed finite element spaces such as the Raviart-Thomas [1], Brezzi-Douglas-Marini [2] and Brezzi-

Received: January 20, 2012

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Douglas-Fortin-Marini [3] spaces perform well for meshes consisting of triangles or rectangles in two dimensions. However, for general quadrilateral meshes, not all the classical mixed finite element spaces are optimal. Arnold et al. [10] have recently shown that only Raviart-Thomas spaces are optimal in L^2-norm. In three dimensions, most spaces [7], [8], [9] are designed for cubes or parallelepiped. Study for distorted hexahedra are rare. Naff et al. [11], [12], [13] have shown that even the Raviart-Thomas-Nedelec mixed finite element spaces [7], [8] do not show optimal orders of approximation.

The purpose of this paper is to introduce a new family of mixed finite element spaces when the domain consists of distorted hexahedra having only one pair of quadrilateral faces that are parallel but not necessarily congruent. As usual, shape functions are defined through mappings from a reference cube: Scalar functions are defined by a composition with a trilinear map; The vectors are mapped by the Piola transformation. It can be shown that, when the vectors in the classical Raviart-Thomas-Nedelec ($RTN_{[k]}$) spaces are mapped through the Piola transformation, the mapped spaces do not contain full set of polynomials of degree k. This is the reason why one cannot obtain optimal order of approximation. Our idea is to construct a new set of polynomials $\mathbf{S}_k^*$ on the reference element which would retain the set of all polynomials of degree k in the mapped space. Then we define proper degrees of freedom which determine the element uniquely. We take $\mathbf{S}_k^*$ as the reference vector space $\hat{\mathbf{V}}$. Next, we define the corresponding pressure space $\hat{W}$ by taking the divergence of $\hat{\mathbf{V}}$. Thus we obtain a stable pair $(\hat{\mathbf{V}}, \hat{W})$ for which we show an optimal order of approximation for the velocity variable for all $k \geq 0$. For $k = 0$, our new space is also optimal for the pressure variable. For $k \geq 1$, we use a local post-processing technique to obtain the optimal order. Thus we obtained a new family of mixed finite element spaces which are optimal for both velocity and pressure for all $k \geq 0$. Near the completion of our paper, we have found out that Falk et al. [14] have constructed a similar space for the lowest order case.

The organization of this paper is as follows: In the next section, we describe notations and basic material for mixed methods, focused on hexahedral meshes. In section 3, we define our new spaces and prove the optimal order of approximation for the velocity. Finally, a local post-processing technique for the pressure is presented.

2. The Mixed Finite Element Methods

We introduce notations for various function spaces. Let K be any distorted hexahedron having at least one pair of quadrilateral faces that are parallel but not necessarily congruent (see Fig. 1). We let

$$\begin{aligned}
&P_k(K) = \{\text{polynomials of total degree up to } k\},\\
&\mathbf{P}_k(K) = (P_k(K))^3,\\
&Q_{\ell,m,n}(K) = \{\text{polynomials of degree up to } \ell, m \text{ and } n \\
&\qquad\qquad\qquad\qquad \text{in } x,\ y \text{ and } z, \text{ respectively}\},\\
&Q'_{\ell,m,n}(K) = \text{Span}\{x^i y^j z^k \mid 0 \le i \le \ell,\ 0 \le j \le m,\ 0 \le k \le n,\\
&\qquad\qquad\qquad\qquad (i,j,k) \ne (\ell,m,n)\},\\
&Q_{\ell,m',n'}(K) = \text{Span}\{x^i y^j z^k \mid 0 \le i \le \ell,\ 0 \le j \le m,\\
&\qquad\qquad\qquad\qquad 0 \le k \le n,\ (j,k) \ne (m,n)\}.
\end{aligned}$$

The spaces $Q_{\ell',m',n}(K)$ and $Q_{\ell',m,n'}(K)$ are similarly defined. Define

$$\begin{aligned}
Q''_{\ell,m,n}(K) &\equiv Q_{\ell,m',n'}(K) \cap Q_{\ell',m',n}(K) \cap Q_{\ell',m,n'}(K)\\
&= \text{Span}\{x^i y^j z^k \mid 0 \le i \le \ell,\ 0 \le j \le m,\ 0 \le k \le n,\\
&\qquad (i,j) \ne (\ell,m),\ (j,k) \ne (m,n) \text{ and} (k,i) \ne (n,\ell)\}.
\end{aligned}$$

Finally, we define

$$y^m z^n P_\ell(x) = \{y^m z^n p(x) \mid p \in P_\ell(x)\}.$$

The spaces $x^\ell z^n P_m(y)$ and $x^\ell y^m P_n(z)$ are similarly defined.

Let Ω be a bounded polyhedral domain in $\mathbb{R}^3$ with the boundary $\partial\Omega$. We consider the following second order elliptic boundary value problem:

$$\begin{cases} -\text{div}(\kappa\nabla p) &= f, \quad \text{in } \Omega,\\ \qquad\quad p &= 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $f \in L^2(\Omega)$ and $\kappa = \kappa(\mathbf{x})$ is a symmetric and uniformly positive definite matrix, i.e., there exists two positive constants c_1 and c_2 such that

$$c_1 \xi^T \xi \le \xi^T \kappa(\mathbf{x}) \xi \le c_2 \xi^T \xi, \quad \forall \xi \in \mathbb{R}^3,\ \mathbf{x} \in \overline{\Omega}.$$

Let us introduce a vector variable $\mathbf{u} = -\kappa\nabla p$. Then the problem (1) can be factored to give the following first order system:

$$\begin{cases} \mathbf{u} + \kappa\nabla p &= 0, \quad \text{in } \Omega,\\ \quad \text{div}\,\mathbf{u} &= f, \quad \text{in } \Omega,\\ \qquad\quad p &= 0, \quad \text{on } \partial\Omega. \end{cases} \tag{2}$$

In the mathematical modeling of fluid flow in porous media, $\mathbf{u}$ and p represent the velocity and pressure, respectively. The first equation of (2), which relates $\mathbf{u}$ and p, is called Darcy's law, and the second equation represents the conservation of mass. In the full system of equations for certain porous media problems such as oil reservoir, these equations are coupled with a concentration equation. Since the coupling is only through the velocity variable $\mathbf{u}$, it is important to get a very accurate approximation for the velocity $\mathbf{u}$.

Now we introduce the function spaces

$$\mathbf{V} = H(\mathrm{div}\,,\Omega) = \{\mathbf{v} \in (L^2(\Omega))^3 : \mathrm{div}\,\mathbf{v} \in L^2(\Omega)\},$$
$$W = L^2(\Omega).$$

The weak form of the problem (2) appropriate for the mixed method is to find $(\mathbf{u}, p) \in \mathbf{V} \times W$ such that

$$\begin{cases} (\kappa^{-1}\mathbf{u}, \mathbf{v}) - (p, \mathrm{div}\,\mathbf{v}) &= 0, \qquad \forall \mathbf{v} \in \mathbf{V}, \\ (\mathrm{div}\,\mathbf{u}, q) &= (f, q), \quad \forall q \in W, \end{cases} \tag{3}$$

where $(\cdot,\cdot)$ indicates the inner product in $L^2(\Omega)$ or $(L^2(\Omega))^3$. Then it is well known [15] that this problem has a unique solution pair $(\mathbf{u}, p) \in \mathbf{V} \times W$. Now we consider finite element methods. Assume that we have some finite element subspaces $\mathbf{V}_h \subset \mathbf{V}$ and $W_h \subset W$. Then the mixed finite element approximation $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ is defined as the solution of the following equations:

$$\begin{cases} (\kappa^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \mathrm{div}\,\mathbf{v}_h) &= 0, \qquad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ (\mathrm{div}\,\mathbf{u}_h, q_h) &= (f, q_h), \quad \forall q_h \in W_h. \end{cases} \tag{4}$$

First, we consider the case when Ω is a domain partitioned into a family of tetrahedrons, cubes or parallelepiped. Assume that $\mathscr{T}_h = \{K\}$ is regular and non-degenerate, i.e. $\mathscr{T}_h$ satisfy the following two conditions [16]:

(i) if h_K is the diameter of K then the quantity $h = \max_{K\in\mathscr{T}_h} h_K$ approaches zero.

(ii) if r_K is the radius of the ball inscribed in K, then there exists a constant σ such that $h_K/r_K \le \sigma$.

If the spaces $\mathbf{V}_h$ and W_h are chosen to satisfy the inf-sup condition and the ellipticity on the kernel of the divergence operator, then we have the following error estimates:

$$\| \mathbf{u} - \mathbf{u}_h \|_{H(\mathrm{div}\,)} + \| p - p_h \|_0 \le C(\inf_{\mathbf{v}\in\mathbf{V}_h} \| \mathbf{u} - \mathbf{v} \|_{H(\mathrm{div}\,)} + \inf_{q\in W_h} \| p - q \|_0).$$

Next, we consider general hexahedral grids. Here we describe how the space $\mathbf{V}_h$ are constructed via Piola transformation from a space of shape functions on a reference element. Let $\hat{K}$ be a fixed reference element, typically unit cube with vertices $\hat{\mathbf{x}}_i = (\hat{x}_i, \hat{y}_i, \hat{z}_i)$ and let K be a hexahedron with vertices $\mathbf{x}_i = (x_i, y_i, z_i)$ for $i = 1, \ldots 8$. Here we assume that there exists a one to one map F_K from $\hat{K}$ onto K satisfying

$$F_K(\hat{\mathbf{x}}_i) = \mathbf{x}_i, \quad i = 1, \ldots 8.$$

Let $\hat{\mathbf{V}}(\hat{K}) \subset H(\mathrm{div}\,, \hat{K})$ be a given finite dimensional space of vector fields on $\hat{K}$. We describe how the space $\mathbf{V}_h$ is constructed from $\hat{\mathbf{V}}(\hat{K})$. Define $\mathbf{u} = P_F \hat{\mathbf{u}}$ by

$$\mathbf{u}(\mathbf{x}) = P_F \hat{\mathbf{u}}(\hat{\mathbf{x}}) = |J_K(\hat{\mathbf{x}})|^{-1} DF_K(\hat{\mathbf{x}}) \hat{\mathbf{u}}(\hat{\mathbf{x}}), \quad \hat{\mathbf{u}} \in \hat{\mathbf{V}}(\hat{K}),$$

where $\mathbf{x} = F_K(\hat{\mathbf{x}})$, DF_K is the Jacobian matrix of the mapping F_K and J_K is its determinant. The transformation P_F is called a Piola transformation. Then the Piola transformation has the following well-known properties [1], [17]:

$$\mathrm{div}\, \mathbf{v} = \frac{1}{|J_K|} \hat{\mathrm{div}}\, \hat{\mathbf{v}}, \tag{5}$$

$$\int_K \mathrm{div}\, \mathbf{v} q \; d\mathbf{x} = \int_{\hat{K}} \hat{\mathrm{div}}\, \hat{\mathbf{v}} \hat{q} \; d\hat{\mathbf{x}}, \tag{6}$$

$$\int_{\partial K} \mathbf{v} \cdot \mathbf{n} \; q \; ds = \int_{\partial \hat{K}} \hat{\mathbf{v}} \cdot \hat{\mathbf{n}} \; \hat{q} \; d\hat{s}, \tag{7}$$

where $\mathbf{n}$ and $\hat{\mathbf{n}}$ denote the unit outward normal vectors on ∂K and $\partial \hat{K}$, respectively.

Assume that $\mathscr{T}_h = \{K\}$ is regular and non-degenerate, i.e., each K satisfies the following two conditions [18]:

(i) There exists a constant σ such that $h_K / r_K \leq \sigma$, where h_K is the diameter of K and r_K is the radius of the largest ball inscribed in K.

(ii) There exists a constant $\gamma > 0$ such that $J_K \geq \gamma h_K^3$.

Using the Piola transformation, we define the following space of shape functions on K

$$\mathbf{V}_h(K) = \{\mathbf{v} = P_F \hat{\mathbf{v}} : \hat{\mathbf{v}} \in \hat{\mathbf{V}}(\hat{K})\}. \tag{8}$$

Then the global finite element space is defined naturally as

$$\mathbf{V}_h = \{\mathbf{v} \in \mathbf{V} : \mathbf{v}\,|_K \in \mathbf{V}_h(K), \; \forall K \in \mathscr{T}_h\}. \tag{9}$$

Next, we define the scalar function space W_h. For any scalar function $\hat{q}$ on $\hat{K}$, we let $q = \hat{q} \circ F_K^{-1}$. Then we define

$$W_h(K) = \{q = \hat{q} \circ F_K^{-1} : \hat{q} \in \hat{W}(\hat{K})\}, \tag{10}$$

and define

$$W_h = \{q \in L^2(\Omega) : q \mid_K \in W_h(K),\ \forall K \in \mathscr{T}_h\}. \tag{11}$$

Here we recall a well-known example when $\hat{K}$ is the unit cube. The Raviart-Thomas-Nedelec spaces of index $k \geq 0$ is given by

$$\hat{\mathbf{V}}(\hat{K}) = RTN_{[k]} = Q_{k+1,k,k}(\hat{K}) \times Q_{k,k+1,k}(\hat{K}) \times Q_{k,k,k+1}(\hat{K}),$$
$$\hat{W}(\hat{K}) = Q_{k,k,k}(\hat{K}).$$

The dimensions of $\hat{\mathbf{V}}(\hat{K})$ and $\hat{W}(\hat{K})$ are $3(k+1)^2(k+2)$ and $(k+1)^3$, respectively.

It is well known that, when the elements are partitioned into general hexahedra, we do not obtain an optimal order error estimates even for the lowest element [11], [12]. The reason is that the velocity vector fields mapped by Piola transformation do not contain all polynomials of degree k. Fortunately, for a special class of distorted hexahedra, we can find a set of polynomials which have an optimal order of approximation. Utilizing this set, we can design a new family of mixed finite.

3. Construction of New Mixed Finite Elements

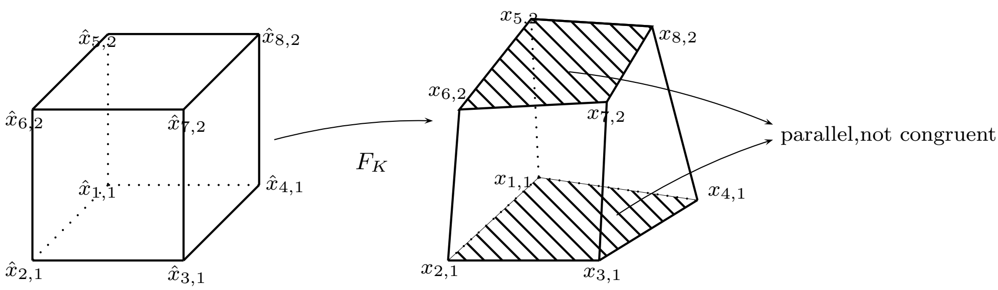

Figure 1: A Transformation onto a distorted hexahedron having two parallel sides

For convenience' sake, we assume that K is a distorted hexahedron where two faces are parallel to the xy-plane. Let $\mathbf{x}_{i,1} = (x_i, y_i, z_1),\ i = 1,2,3,4$, and $\mathbf{x}_{i,2} = (x_i, y_i, z_2),\ i = 5,6,7,8$ denote its vertices (Fig. 1). We use the unit cube $\hat{K} = [0,1] \times [0,1] \times [0,1]$ as the reference element in the $\hat{x}\hat{y}\hat{z}$-space with the vertices

$$\hat{\mathbf{x}}_{1,1} = (0,0,0), \quad \hat{\mathbf{x}}_{2,1} = (1,0,0), \quad \hat{\mathbf{x}}_{3,1} = (1,1,0), \quad \hat{\mathbf{x}}_{4,1} = (0,1,0),$$

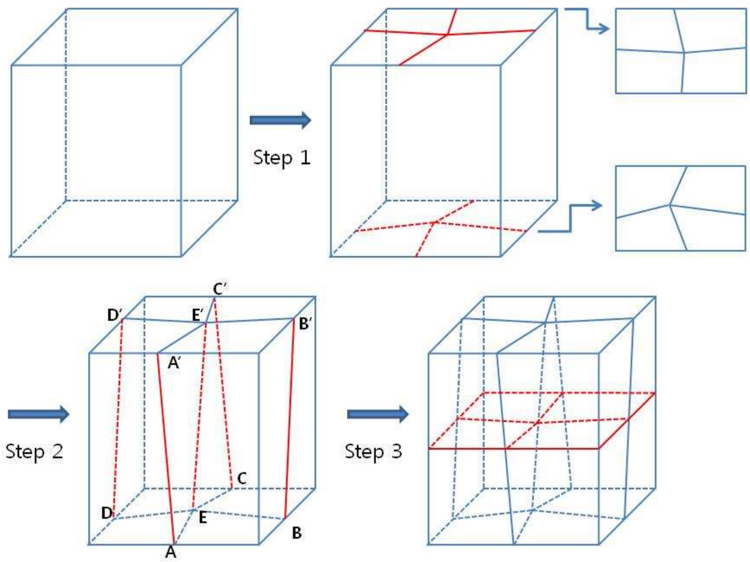

Figure 2: The mesh generation on unit cube

$$\hat{\mathbf{x}}_{5,2} = (0,0,1), \quad \hat{\mathbf{x}}_{6,2} = (1,0,1), \quad \hat{\mathbf{x}}_{7,2} = (1,1,1), \quad \hat{\mathbf{x}}_{8,2} = (0,1,1).$$

First, we consider the mesh generation. We give an example how to generate hexahedral grids satisfying our conditions. We start from a unit cube $[0,1]^3$. First, consider the face $z = 0$. We refine it by four shape regular quadrilaterals. We do the similar refinement for the face $z = 1$. (See the step 1 of Fig. 2). Next, we connect the each points A...E to A′...E′, respectively(See the step 2 of Fig. 2). Then, we slice the box by the plane $z = \frac{1}{2}$ to obtain eight hexahedra(See the step 3 of Fig. 2). Note that the inner faces of these sub hexahedra are not necessarily planar. Repeat the same process to each of the sub hexahedra. Then the hexahedra constructed above look like those in the left side of Figure 3. In this case, we can do the similar method on these hexahedra(See Fig. 3).

We assume there exists a unique invertible transformation F_K which maps $\hat{K}$ onto K:

$$F_K(\hat{\mathbf{x}}) = (F_1(\hat{x},\hat{y},\hat{z}), F_2(\hat{x},\hat{y},\hat{z}), \alpha + \beta\hat{z}), \tag{12}$$

where $F_i(\hat{x},\hat{y},\hat{z}) \in Q_{1,1,1}(\hat{x},\hat{y},\hat{z}),\ i = 1,2$. Also, we assume that the map in (12) is in its most general form, i.e., the coefficients of $F_K(\hat{\mathbf{x}})$ are all nonzero. It is easy to check that the image of $\hat{K}$ under F_K has two faces which are parallel to the xy-plane. Let $\hat{\mathbf{V}}(\hat{K})$ be a finite dimensional space of vector fields on $\hat{K}$. Since $\mathbf{u}(\mathbf{x}) = P_F\hat{\mathbf{u}}(\hat{\mathbf{x}}) = J_K(\hat{\mathbf{x}})^{-1}DF_K(\hat{\mathbf{x}})\hat{\mathbf{u}}(\hat{\mathbf{x}})$ for $\hat{\mathbf{u}} \in \hat{\mathbf{V}}(\hat{K})$, we

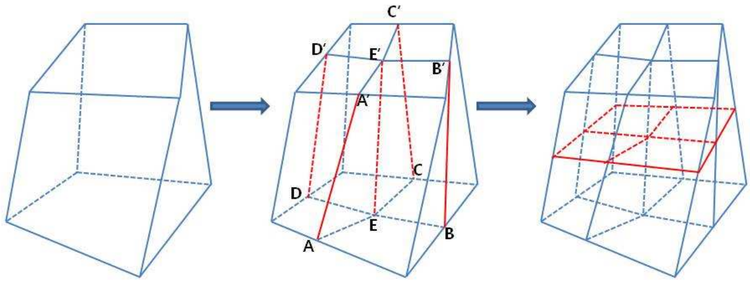

Figure 3: The mesh generation for the distorted hexahedron

can construct the finite element space $\mathbf{V}_h(K)$ from the definition (8). In order to obtain $O(h^{k+1})$ approximation order, it is sufficient that $\mathbf{V}_h(K)$ contains $\mathbf{P}_k(K)$. From this fact, we determine $\hat{\mathbf{V}}(\hat{K})$ by applying the inverse Piola transformation to $\mathbf{P}_k(K)$. We first consider the following vectors for $0 \le i \le k$:

$$\begin{aligned}
\mathbf{s}_{11}^i &= (\hat{x}^{k+2}\hat{y}^{k+1}\hat{z}^i,\ 0,\ 0), & \mathbf{s}_{12}^i &= (0,\ \hat{x}^{k+1}\hat{y}^{k+2}\hat{z}^i,\ 0),\\
\mathbf{s}_{21}^i &= (\hat{x}^{k+2}\hat{y}^i\hat{z}^{k+1},\ 0,\ 0), & \mathbf{s}_{22}^i &= (0,\ 0,\ \hat{x}^{k+1}\hat{y}^i\hat{z}^{k+2}),\\
\mathbf{s}_{31}^i &= (0,\ \hat{x}^i\hat{y}^{k+2}\hat{z}^{k+1},\ 0), & \mathbf{s}_{32}^i &= (0,\ 0,\ \hat{x}^i\hat{y}^{k+1}\hat{z}^{k+2}),
\end{aligned}$$

$$\begin{aligned}
\mathbf{s}_1^i &= (\hat{x}^{k+2}\hat{y}^{k+1}\hat{z}^i,\ -\hat{x}^{k+1}\hat{y}^{k+2}\hat{z}^i,\ 0),\\
\mathbf{s}_2^i &= (\hat{x}^{k+2}\hat{y}^i\hat{z}^{k+1},\ 0,\ -\hat{x}^{k+1}\hat{y}^i\hat{z}^{k+2}),\\
\mathbf{s}_3^i &= (0,\ \hat{x}^i\hat{y}^{k+2}\hat{z}^{k+1},\ -\hat{x}^i\hat{y}^{k+1}\hat{z}^{k+2}).
\end{aligned}$$

Definition 1. We let

$$\mathbf{S}_k' = Q_{k+2,(k+1)',(k+1)'}(\hat{K}) \times Q_{(k+1)',k+2,(k+1)'}(\hat{K}) \times Q_{(k+1)',(k+1)',k+2}(\hat{K}),$$

and let $\mathbf{S}_k^*$ be the subspace of $\mathbf{S}_k'$ where for $i = 1, \ldots, k,\ j = 1, 2, 3$, the elements $\mathbf{s}_{j\ell}^i(\ell = 1, 2)$ are replaced by the elements $\mathbf{s}_j^i$.

Lemma 2. *The image of $\mathbf{S}_k^*$ under the Piola transformation of any F_K contains $\mathbf{P}_k(K)$.*

Proof. It suffices to show that the inverse image of any polynomial $p(\mathbf{x}) \in \mathbf{P}_k(K)$ lies in $\mathbf{S}_k^*$. First, note that

$$F_K(\hat{\mathbf{x}}) = \begin{pmatrix} a_0 + a_1\hat{x} + a_2\hat{y} + a_3\hat{z} + a_4\hat{x}\hat{y} + a_5\hat{y}\hat{z} + a_6\hat{z}\hat{x} + a_7\hat{x}\hat{y}\hat{z} \\ b_0 + b_1\hat{x} + b_2\hat{y} + b_3\hat{z} + b_4\hat{x}\hat{y} + b_5\hat{y}\hat{z} + b_6\hat{z}\hat{x} + b_7\hat{x}\hat{y}\hat{z} \\ \alpha + \beta\hat{z} \end{pmatrix}.$$

Then it can be easily shown that

$$DF_K(\hat{\mathbf{x}}) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & \beta \end{pmatrix},$$

and

$$J_K(\hat{\mathbf{x}})DF_K^{-1}(\hat{\mathbf{x}}) = \begin{pmatrix} \beta A_{22} & -\beta A_{12} & A'_{13} \\ -\beta A_{21} & \beta A_{11} & A'_{23} \\ 0 & 0 & \frac{1}{\beta}J_K(\hat{\mathbf{x}}) \end{pmatrix}.$$

Here $a_i,\ b_i(i = 1, \dots 7),\ c_j,\ d_j,\ e_j(j = 1, 2, 3),\ \alpha,\ \beta,\ \gamma$ are constants and

$$\begin{aligned}
A_{11} &= a_1 + a_4\hat{y} + a_6\hat{z} + a_7\hat{y}\hat{z}, \\
A_{12} &= a_2 + a_4\hat{x} + a_5\hat{z} + a_7\hat{x}\hat{z}, \\
A_{13} &= a_3 + a_5\hat{y} + a_6\hat{x} + a_7\hat{x}\hat{y}, \\
A_{21} &= b_1 + b_4\hat{y} + b_6\hat{z} + b_7\hat{y}\hat{z}, \\
A_{22} &= b_2 + b_4\hat{x} + b_5\hat{z} + b_7\hat{x}\hat{z}, \\
A_{23} &= b_3 + b_5\hat{y} + b_6\hat{x} + b_7\hat{x}\hat{y}, \\
A'_{13} &= A_{12}A_{23} - A_{13}A_{22} = p_1 + c_1\hat{x}\hat{y} + c_2\hat{x}\hat{z} + c_3\hat{x}^2 + \alpha\hat{x}^2\hat{y} - \gamma\hat{x}^2\hat{z}, \quad (13) \\
A'_{23} &= A_{13}A_{21} - A_{11}A_{23} = p_2 + d_1\hat{x}\hat{y} + d_2\hat{y}\hat{z} + d_3\hat{y}^2 - \alpha\hat{x}\hat{y}^2 + \beta\hat{y}^2\hat{z}, \quad (14) \\
J_K &= \beta(A_{11}A_{22} - A_{12}A_{21}) = p_3 + e_1\hat{y}\hat{z} + e_2\hat{x}\hat{z} + e_3\hat{z}^2 + \gamma\hat{x}\hat{z}^2 - \beta\hat{y}\hat{z}^2, \quad (15)
\end{aligned}$$

where $p_j \in P_1(\hat{x}, \hat{y}, \hat{z})$ for $j = 1, 2, 3$. Let $\mathbf{u}(\mathbf{x}) \in \mathbf{P}_k(x, y, z)$. Then $\hat{\mathbf{u}}(\hat{\mathbf{x}}) = P_F^{-1}\mathbf{u}(\mathbf{x}) = J_K(\hat{\mathbf{x}})DF_K^{-1}(\hat{\mathbf{x}})\mathbf{u}(\mathbf{x})$ contains all the elements belonging to the following set:

$$\left\{ \begin{pmatrix} \beta A_{22} & -\beta A_{12} & A'_{13} \\ -\beta A_{21} & \beta A_{11} & A'_{23} \\ 0 & 0 & \frac{1}{\beta}J_K(\hat{\mathbf{x}}) \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \forall\ r_1, r_2, r_3 \in P_k(x, y, z) \right\}. \quad (16)$$

For each polynomial r_j in $P_k(x, y, z),\ j = 1, 2, 3$, we have

$$\begin{aligned} r_j(x, y, z) &= r_j(F_1(\hat{x}, \hat{y}, \hat{z}), F_2(\hat{x}, \hat{y}, \hat{z}), \alpha + \beta\hat{z}) \quad (17) \\ &= q_j(\hat{x}, \hat{y}, \hat{z}), \end{aligned}$$

for some $q_j(\hat{x}, \hat{y}, \hat{z}) \in Q_{k,k,k}(\hat{x}, \hat{y}, \hat{z})$. Noting that

$$\begin{aligned} A_{22},\ A_{12},\ A'_{13} &\in Q_{2,1',1'}(\hat{K}), \\ A_{11},\ A_{21},\ A'_{23} &\in Q_{1',2,1'}(\hat{K}), \end{aligned}$$

$$J_K(\hat{\mathbf{x}}) \in Q_{1',1',2}(\hat{K}),$$

we see that the vectors in (16) belong to $\mathbf{S}'_k$. We now show that any element in the set (16) is contained in $\mathbf{S}^*_k$. To do so, we only need to consider the highest degree terms of vectors in (16). Since those are of the form $(\hat{x}^{k+2}\hat{y}^{k+1}\hat{z}^i,\ -\hat{x}^{k+1}\hat{y}^{k+2}\hat{z}^i,\ 0)$, $(\hat{x}^{k+2}\hat{y}^i\hat{z}^{k+1},\ 0,\ -\hat{x}^{k+1}\hat{y}^i\hat{z}^{k+2})$ and $(0,\ \hat{x}^i\hat{y}^{k+2}\hat{z}^{k+1},\ -\hat{x}^i\hat{y}^{k+1}\hat{z}^{k}$ we see that they belong to $\mathbf{S}^*_k$. □

For the error analysis, we need to define a new space.

Definition 3. We let

$$R_{-1} = 1 \ \text{ and } \ R_k = Q_{(k+1)',(k+1)',k+2}(\hat{K}),$$

for $k \geq 0$.

Lemma 4. *Given any element K, the space R_k for $k \geq 0$ contains all functions of the form*

$$J_K(\hat{\mathbf{x}})p(F_K(\hat{\mathbf{x}})),$$

for all $p \in P_k(K)$.

Proof. Let $p \in P_k(K)$ be arbitrary and consider $J_K(\hat{\mathbf{x}})(p \circ F_K)(\hat{\mathbf{x}})$. Since $J_K(\hat{\mathbf{x}}) \in Q_{1',1',2}(\hat{K})$ and $p(x,y,z) = q(\hat{x},\hat{y},\hat{z})$ for some $q(\hat{x},\hat{y},\hat{z}) \in Q_{k,k,k}(\hat{x},\hat{y},\hat{z})$ by (18), we obtain the desired result. □

By the construction above, we see that $\mathbf{V}_h(K)$ contains $P_k(K)$. Similarly, $\operatorname{div}\mathbf{V}_h(K)$ contains $P_k(K)$. Hence the space $\mathbf{V}_h(K)$ contains enough polynomials to achieve an optimal order of approximation in the $H(\operatorname{div})$-norm. Hence we have the following result.

Theorem 5. *There exists a constant C such that*

$$\inf_{\mathbf{v}\in\mathbf{V}_h} (\| \mathbf{u} - \mathbf{v} \|_0 + \| \operatorname{div}\mathbf{u} - \operatorname{div}\mathbf{v} \|_0) \leq Ch^{k+1}(| \mathbf{u} |_{k+1} + | \operatorname{div}\mathbf{u} |_{k+1}), \quad (18)$$

for all $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$ with $\operatorname{div}\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$.

Now, we introduce new mixed finite element spaces that are based on $\mathbf{S}^*_k$.

Definition 6. We let

$$\hat{\mathbf{V}}(\hat{K}) = \mathbf{S}_k^*,$$
$$\hat{W}(\hat{K}) = Q''_{k+1,k+1,k+1}(\hat{K}),$$

where $Q''_{k+1,k+1,k+1}(\hat{K})$ is the space defined at the beginning of section 2.

Then we see that the dimensions of $\hat{\mathbf{V}}(\hat{K})$ and $\hat{W}(\hat{K})$ are $3(k+1)(k+2)(k+4)$ and $(k+2)^3 - 3(k+1) - 1$, respectively.

Remark 7. This type of element can be used if any one pair of opposite faces are flat quadrilaterals paralleled to yz or zx-plane.

Remark 8. From the definition of above, we know that $\hat{W}(\hat{K})$ contains R_{k-1} but not R_k. For example, we see that $\hat{W}(\hat{K})$ for $k = 0$ is a space containing polynomials of the following form:

$$p = a_0 + a_1\hat{x} + a_2\hat{y} + a_3\hat{z}.$$

Since $R_0 = b_0 + b_1\hat{x} + b_2\hat{y} + b_3\hat{z} + b_4\hat{y}\hat{z} + b_5\hat{x}\hat{z} + b_6\hat{z}^2 + b_7\hat{x}\hat{z}^2 + b_8\hat{y}\hat{z}^2$ for some constants $b_i (i = 0, \ldots, 8)$, $\hat{W}(\hat{K})$ contains R_{-1} only.

Lemma 9. *We have the following property:*

$$\hat{\text{div}}\, \hat{\mathbf{V}}(\hat{K}) = \hat{W}(\hat{K}).$$

Proof. First, we shall investigate the divergence of an element in $\mathbf{S}'_k$. Then as a special case, we show that the the divergence of elements in $\mathbf{S}_k^*$ must belong to $\hat{W}(\hat{K})$. Let $\hat{\mathbf{v}}' = (\hat{v}'_1, \hat{v}'_2, \hat{v}'_3)$ be any element in $\mathbf{S}'_k$. Since $\hat{v}'_1 \in Q_{k+2,(k+1)',(k+1)'}(\hat{K})$,

$$\frac{\partial \hat{v}'_1}{\partial \hat{x}} = q'_1, \quad \text{for some} \quad q'_1 \in Q_{k+1,(k+1)',(k+1)'}.$$

In the same way, we have

$$\frac{\partial \hat{v}'_2}{\partial \hat{y}} = q'_2, \quad \text{for some} \quad q'_2 \in Q_{(k+1)',k+1,(k+1)'}$$

and

$$\frac{\partial \hat{v}'_3}{\partial \hat{z}} = q'_3, \quad \text{for some} \quad q'_3 \in Q_{(k+1)',(k+1)',k+1}.$$

However, from the construction of $\mathbf{S}_k^*$, we see that the following vectors are replaced by divergence free vectors $\mathbf{s}_j^i$ for $i = 1, \ldots, k,\ j = 1, 2, 3$:

$$(\hat{x}^{k+2}\hat{y}^{k+1}\hat{z}^i,\ 0,\ 0),\ (\hat{x}^{k+2}\hat{y}^i\hat{z}^{k+1},\ 0,\ 0),\ (0,\ \hat{x}^i\hat{y}^{k+2}\hat{z}^{k+1},\ 0),$$
$$(0,\ \hat{x}^{k+1}\hat{y}^{k+2}\hat{z}^i,\ 0),\ (0,\ 0,\ \hat{x}^{k+1}\hat{y}^i\hat{z}^{k+2}),\ (0,\ 0,\ \hat{x}^i\hat{y}^{k+1}\hat{z}^{k+2}).$$

Hence for $\hat{\mathbf{v}} = (\hat{v}_1, \hat{v}_2, \hat{v}_3)$ in $\mathbf{S}^*_k$, we see that

$$\frac{\partial \hat{v}_1}{\partial \hat{x}} = q_1,\ \frac{\partial \hat{v}_2}{\partial \hat{y}} = q_2,\ \frac{\partial \hat{v}_3}{\partial \hat{z}} = q_3 \ \text{ for some } \ q_1,\ q_2,\ q_3 \in Q''_{k+1,k+1,k+1}.$$

Therefore we see that $\hat{\text{div}}\,\hat{\mathbf{V}}(\hat{K}) \subseteq \hat{W}(\hat{K})$. The reverse inclusion $\hat{\text{div}}\,\hat{\mathbf{V}}(\hat{K}) \supseteq \hat{W}(\hat{K})$ is clear, thus the proof is complete. □

Now we need to define the degrees of freedom and show the unisolvence. For this purpose, we first consider the following vectors for $0 \le i \le k$:

$$\begin{aligned}
&\boldsymbol{\phi}^i_{11} = (\hat{x}^k\hat{y}^{k+1}\hat{z}^i,\ 0,\ 0), \quad \boldsymbol{\phi}^i_{12} = (0,\ \hat{x}^{k+1}\hat{y}^k\hat{z}^i,\ 0),\\
&\boldsymbol{\phi}^i_{21} = (\hat{x}^k\hat{y}^i\hat{z}^{k+1},\ 0,\ 0), \quad \boldsymbol{\phi}^i_{22} = (0,\ 0,\ \hat{x}^{k+1}\hat{y}^i\hat{z}^k),\\
&\boldsymbol{\phi}^i_{31} = (0,\ \hat{x}^i\hat{y}^k\hat{z}^{k+1},\ 0), \quad \boldsymbol{\phi}^i_{32} = (0,\ 0,\ \hat{x}^i\hat{y}^{k+1}\hat{z}^k),\\
&\boldsymbol{\phi}^i_{1} = (\hat{x}^k\hat{y}^{k+1}\hat{z}^i,\ -\hat{x}^{k+1}\hat{y}^k\hat{z}^i,\ 0),\\
&\boldsymbol{\phi}^i_{2} = (\hat{x}^k\hat{y}^i\hat{z}^{k+1},\ 0,\ -\hat{x}^{k+1}\hat{y}^i\hat{z}^k),\\
&\boldsymbol{\phi}^i_{3} = (0,\ \hat{x}^i\hat{y}^k\hat{z}^{k+1},\ -\hat{x}^i\hat{y}^{k+1}\hat{z}^k).
\end{aligned}$$

Definition 10. We let

$$\boldsymbol{\Psi'}_k = Q_{k,(k+1)',(k+1)'}(\hat{K}) \times Q_{(k+1)',k,(k+1)'}(\hat{K}) \times Q_{(k+1)',(k+1)',k}(\hat{K})$$

and let $\boldsymbol{\Psi}^*_k$ be the subspace of $\boldsymbol{\Psi'}_k$ where for $i = 1, \dots, k,\ j = 1, 2, 3$ the elements $\boldsymbol{\phi}^i_{j\ell}(\ell = 1, 2)$ are replaced by the elements $\boldsymbol{\phi}^i_j$.

Note that the definition of $\boldsymbol{\Psi'}_k$ and $\boldsymbol{\Psi}^*_k$ are similar to those of $\mathbf{S}'_k$ and $\mathbf{S}^*_k$ except that the highest exponent $k + 2$ is replaced by k.

For any $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3) \in \hat{\mathbf{V}}(\hat{K})$, we consider the following degrees of freedom:

$$\int_{\hat{f}} \hat{\mathbf{u}} \cdot \hat{\mathbf{n}}\, \hat{q}\, d\hat{A}, \qquad \hat{q} \in Q'_{k+1,k+1}(\hat{f}), \quad \text{for each face } \hat{f}, \tag{19}$$

$$\int_{\hat{K}} \hat{\mathbf{u}} \cdot \hat{\boldsymbol{\psi}}\; d\hat{\mathbf{x}}, \qquad \hat{\boldsymbol{\psi}} \in \boldsymbol{\Psi}^*_k. \tag{20}$$

The number of conditions is $6\{(k+2)^2 - 1\} + 3\{(k+1)(k+2)^2 - (k+1)\} - 6(k+1) + 3(k+1)$ which is also the dimension of $\hat{\mathbf{V}}(\hat{K})$. See Fig. 4 for an illustration of degrees of freedom when $k = 0$.

Now we show that these choice of $\hat{\mathbf{V}}(\hat{K})$ and degrees of freedom determine a finite element subspace of $H(\text{div}\,)$.

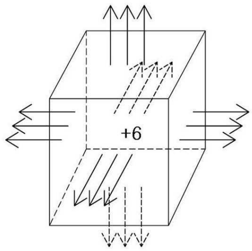

Figure 4: The degrees of freedom for $k = 0$. Three normal components per face and six interior degrees of freedom.

Theorem 11. *A vector function* $\hat{\mathbf{u}} \in \hat{\mathbf{V}}(\hat{K})$ *in the Definition* 6 *is uniquely determined by the degrees of freedom* (19) *and* (20). *Moreover, the space* V_h *of finite elements defined by mapping the element in* $\hat{\mathbf{V}}(\hat{K})$ *using* (8) *is divergence conforming.*

Proof. We verify the unisolvence by showing that if all the quantities (19) and (20) vanish, then $\hat{\mathbf{u}} = 0$. Since $\hat{\mathbf{u}} \cdot \hat{\mathbf{n}} \in Q'_{k+1,k+1}(\hat{f})$, it is clear that $\hat{\mathbf{u}} \cdot \hat{\mathbf{n}} = 0$ on each face and this proves the conformity. Then, $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$ satisfies

$$\hat{u}_1 = \hat{x}(1-\hat{x})\hat{v}_1,\ \hat{u}_2 = \hat{y}(1-\hat{y})\hat{v}_2 \text{ and } \hat{u}_3 = \hat{z}(1-\hat{z})\hat{v}_3,$$

where $\hat{\mathbf{v}} = (\hat{v}_1, \hat{v}_2, \hat{v}_3) \in \mathbf{\Psi}^*_k$. Thus we can take $\hat{\boldsymbol{\psi}} = \hat{\mathbf{v}}$ in condition (20) and this implies $\hat{\mathbf{u}} = 0$. □

For the error estimates, we define a projection operator $\hat{\mathbf{\Pi}} : \mathbf{H}^{k+1}(\hat{K}) \to \hat{\mathbf{V}}(\hat{K})$ satisfying

$$\int_{\hat{f}} (\hat{\mathbf{u}} - \hat{\mathbf{\Pi}}\hat{\mathbf{u}}) \cdot \hat{\mathbf{n}}\, \hat{q}\, d\hat{A} = 0, \qquad \hat{q} \in Q'_{k+1,k+1}(\hat{f}), \quad \text{for each face } \hat{f},$$

$$\int_{\hat{K}} (\hat{\mathbf{u}} - \hat{\mathbf{\Pi}}\hat{\mathbf{u}}) \cdot \hat{\boldsymbol{\psi}}\; d\hat{\mathbf{x}} = 0, \qquad \hat{\boldsymbol{\psi}} \in \mathbf{\Psi}^*_k.$$

Then this operator has the following property:

Lemma 12. *We have*

$$(\hat{\mathrm{div}}\,(\hat{\mathbf{u}} - \hat{\mathbf{\Pi}}\hat{\mathbf{u}}), \hat{q})_K = 0, \quad \forall\, \hat{\mathbf{u}} \in \mathbf{H}^{k+1}(\hat{K}), \ \ \forall\, \hat{q} \in \hat{W}(\hat{K}).$$

Proof. First, we note that $\hat{q}\mid_{\hat{f}} \in Q'_{k+1,k+1}(\hat{f})$ for all $\hat{q} \in \hat{W}(\hat{K})$. Since $\nabla\hat{q} \in \mathbf{\Psi}_k^*$, we have by the definition of $\hat{\mathbf{\Pi}}$,

$$\begin{aligned}
(\hat{\operatorname{div}}\,\hat{\mathbf{\Pi}}\hat{\mathbf{u}}, \hat{q})_K &= \int_{\partial\hat{K}} \hat{\mathbf{\Pi}}\hat{\mathbf{u}} \cdot \hat{\mathbf{n}}\hat{q}\; d\hat{A} - \int_{\hat{K}} \hat{\mathbf{\Pi}}\hat{\mathbf{u}} \cdot \hat{\nabla}\hat{q}\; d\hat{\mathbf{x}} \\
&= \int_{\partial\hat{K}} \hat{\mathbf{u}} \cdot \hat{\mathbf{n}}\hat{q}\; d\hat{A} - \int_{\hat{K}} \hat{\mathbf{u}} \cdot \hat{\nabla}\hat{q}\; d\hat{\mathbf{x}} \\
&= (\hat{\operatorname{div}}\,\hat{\mathbf{u}}, \hat{q})_K.
\end{aligned}$$

□

For an arbitrary element $K = F(\hat{K})$, we define the following projection $\mathbf{\Pi}_K : \mathbf{H}^{k+1}(K) \to P_F\hat{\mathbf{V}}(\hat{K})$ by $\mathbf{\Pi}_K = P_F \circ \hat{\mathbf{\Pi}} \circ P_F^{-1}$. Then we have the following commutative diagram:

$$\begin{array}{ccc}
\mathbf{H}^{k+1}(\hat{K}) & \xrightarrow{\hat{\mathbf{\Pi}}} & \hat{\mathbf{V}}(\hat{K}) \\
{\scriptstyle P_F}\downarrow & & \downarrow{\scriptstyle P_F} \\
\mathbf{H}^{k+1}(K) & \xrightarrow[\mathbf{\Pi}_K]{} & P_F\hat{\mathbf{V}}(\hat{K})
\end{array}$$

Lemma 13. *We have*

$$(\operatorname{div}(\mathbf{u} - \mathbf{\Pi}_K\mathbf{u}), q)_K = 0, \quad \forall \mathbf{u} \in \mathbf{H}^{k+1}(\hat{K}), \quad \forall q \in W_h(K).$$

Proof. From (5) and Lemma 12, we know that

$$\begin{aligned}
(\operatorname{div}\mathbf{\Pi}_K\mathbf{u},\ q)_K &= \int_K \operatorname{div}\mathbf{\Pi}_K\mathbf{u}\; q\; d\mathbf{x} \;=\; \int_K J_K\, \operatorname{div}\mathbf{\Pi}_K\mathbf{u}\; q\; \frac{1}{J_K}\; d\mathbf{x} \\
&= \int_{\hat{K}} \hat{\operatorname{div}}\,\hat{\mathbf{\Pi}}\hat{\mathbf{u}}\; \hat{q}\; d\hat{\mathbf{x}} \;=\; \int_{\hat{K}} \hat{\operatorname{div}}\,\hat{\mathbf{u}}\; \hat{q}\; d\hat{\mathbf{x}} \\
&= \int_{\hat{K}} \frac{1}{J_K}\, \hat{\operatorname{div}}\,\hat{\mathbf{u}}\; \hat{q}\; J_K\; d\hat{\mathbf{x}} \;=\; \int_K \operatorname{div}\mathbf{u}\; q\; d\mathbf{x} \\
&= (\operatorname{div}\mathbf{u},\ q)_K.
\end{aligned}$$

□

Now the global projection operator $\mathbf{\Pi}_h : \mathbf{H}^{k+1}(\Omega) \to \mathbf{V}_h$ is defined by $(\mathbf{\Pi}_h\mathbf{u})\mid_K = \mathbf{\Pi}_K(\mathbf{u}\mid_K)$. Since $\mathbf{V}_h(K)$ contains $\mathbf{P}_k(K)$ by construction, we have the following approximation property of $\mathbf{\Pi}_K$.

Lemma 14. *There is a constant C independent of h such that*

$$\| \mathbf{u} - \mathbf{\Pi}_K \mathbf{u} \|_0 \leq Ch^{k+1} \| \mathbf{u} \|_{k+1},$$

for all $\mathbf{u} \in \mathbf{H}^{k+1}(K)$.

For the divergence, we have the following approximation property:

Lemma 15. *There is a constant C independent of h such that*

$$\| \operatorname{div} (\mathbf{u} - \mathbf{\Pi}_K \mathbf{u}) \|_0 \leq Ch^k \| \operatorname{div} \mathbf{u} \|_k,$$

for all $\mathbf{u} \in \mathbf{H}^k(K)$ *with* $\operatorname{div} \mathbf{u} \in \mathbf{H}^k(K)$.

Proof. Since $\hat{\operatorname{div}}\, \hat{\mathbf{V}}(\hat{K}) \supseteq R_{k-1}$, we have $\operatorname{div} \mathbf{V}(K) \supseteq \mathbf{P}_{k-1}(K)$ by construction of R_{k-1} and (5). Hence we have the result. □

To prove the error estimates, we also need to define an operator Φ_h : $L^2(\Omega) \to W_h$. First, let $\hat{\Phi}$ be the local L^2-projection onto $\hat{W}(\hat{K})$. Then define $\Phi_K : L^2(K) \to W_h(K)$ by $\Phi_K p = (\hat{\Phi}\hat{p}) \circ F_K^{-1}$ with $\hat{p} = p \circ F_K$. Finally, we let $(\Phi_h p) \mid_K = \Phi_K(p \mid_K)$.

Lemma 16. *We have*

$$(\Phi_K p - p,\ \operatorname{div} \mathbf{v})_K = 0, \quad \forall \mathbf{v} \in \mathbf{V}_h(K), \quad \forall p \in L^2(K).$$

Proof. By the definition of Φ_K and (5), we obtain

$$\begin{aligned}
(\Phi_K p,\ \operatorname{div} \mathbf{v})_K &= \int_K (\hat{\Phi}\hat{p}) \circ F_K^{-1} \operatorname{div} \mathbf{v}\ d\mathbf{x} \ = \ \int_{\hat{K}} (\hat{\Phi}\hat{p})\ \frac{1}{J_K} \hat{\operatorname{div}}\, \hat{\mathbf{v}}\ J_K\ d\hat{\mathbf{x}} \\
&= \int_{\hat{K}} \hat{p}\ \hat{\operatorname{div}}\, \hat{\mathbf{v}}\ d\hat{\mathbf{x}} \ = \ \int_K p\ J_K \operatorname{div} \mathbf{v}\ \frac{1}{J_K} d\mathbf{x} \\
&= \int_K p \operatorname{div} \mathbf{v}\ d\mathbf{x} \ = \ (p,\ \operatorname{div} \mathbf{v})_K.
\end{aligned}$$

□

From Lemma 13 and Lemma 16, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathbf{H}^{k+1}(K) & \xrightarrow{\operatorname{div}} & L^2(K) \\
\mathbf{\Pi}_K \big\downarrow & & \big\downarrow \Phi_K \\
\mathbf{V}_h(K) & \xrightarrow[\operatorname{div}]{} & W_h(K)
\end{array}$$

From this diagram, one can easily derive the inf-sup condition[2], [19], [20].

Lemma 17. *We have*

$$\| \operatorname{div}(\mathbf{u}-\mathbf{u}_h) \|_0 \le C \| \operatorname{div}(\mathbf{u}-\Pi_h\mathbf{u}) \|_0, \quad \forall \mathbf{u} \in H(\operatorname{div},\Omega).$$

Proof. Subtracting (4) from (3), we have

$$(\kappa^{-1}(\mathbf{u}-\mathbf{u}_h),\mathbf{v}_h) - (p-p_h, \operatorname{div}\mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{21}$$

$$(\operatorname{div}(\mathbf{u}-\mathbf{u}_h), q_h) = 0, \quad \forall q_h \in W_h. \tag{22}$$

We define

$$s(\mathbf{x}) = \begin{cases} \mid J_K \mid \operatorname{div}\mathbf{v}(\mathbf{x}), & \mathbf{x} \in K,\ \mathbf{v} \in \mathbf{V}_h, \\ 0, & \mathbf{x} \in \Omega \setminus K, \end{cases}$$

then $s \in W_h$. From (22) we have

$$(\operatorname{div}(\mathbf{u}-\mathbf{u}_h), \mid J_K \mid \operatorname{div}\mathbf{v}) = 0.$$

Choosing $\mathbf{v} = \mathbf{u}_h$, we have

$$(\operatorname{div}\mathbf{u}_h, \mid J_K \mid \operatorname{div}\mathbf{u}_h) = (\operatorname{div}\mathbf{u}, \mid J_K \mid \operatorname{div}\mathbf{u}_h).$$

Hence

$$\| \mid J_K \mid^{\frac{1}{2}} \operatorname{div}\mathbf{u}_h \|_0^2 \le \| \mid J_K \mid^{\frac{1}{2}} \operatorname{div}\mathbf{u} \|_0 \, \| \mid J_K \mid^{\frac{1}{2}} \operatorname{div}\mathbf{u}_h \|_0,$$

and so

$$\| \mid J_K \mid^{\frac{1}{2}} \operatorname{div}\mathbf{u}_h \|_0 \le \| \mid J_K \mid^{\frac{1}{2}} \operatorname{div}\mathbf{u} \|_0 .$$

Therefore

$$\| \operatorname{div}\mathbf{u}_h \|_0 \le C \| \operatorname{div}\mathbf{u} \|_0 .$$

Choosing $\mathbf{v} = \mathbf{\Pi}_h\mathbf{u} - \mathbf{u}_h$, we also have

$$(\operatorname{div}\mathbf{u}_h, \mid J_K \mid \operatorname{div}(\mathbf{\Pi}_h\mathbf{u}-\mathbf{u}_h)) = (\operatorname{div}\mathbf{u}, \mid J_K \mid \operatorname{div}(\mathbf{\Pi}_h\mathbf{u}-\mathbf{u}_h)).$$

Subtracting $(\operatorname{div}\mathbf{\Pi}_h\mathbf{u}, \mid J_K \mid \operatorname{div}(\mathbf{\Pi}_h\mathbf{u}-\mathbf{u}_h))$ from both sides, we have

$$(\operatorname{div}(\mathbf{u}_h-\mathbf{\Pi}_h\mathbf{u}), \mid J_K \mid \operatorname{div}(\mathbf{\Pi}_h\mathbf{u}-\mathbf{u}_h)) = (\operatorname{div}(\mathbf{u}-\mathbf{\Pi}_h\mathbf{u}), \mid J_K \mid \operatorname{div}(\mathbf{\Pi}_h\mathbf{u}-\mathbf{u}_h)),$$

and so

$$\| \mid J_K \mid^{\frac{1}{2}} \operatorname{div}(\mathbf{\Pi}_h\mathbf{u}-\mathbf{u}_h) \|_0^2 \le \| \mid J_K \mid^{\frac{1}{2}} \operatorname{div}(\mathbf{u}-\mathbf{\Pi}_h\mathbf{u}) \|_0 \, \| \mid J_K \mid^{\frac{1}{2}} \operatorname{div}(\mathbf{\Pi}_h\mathbf{u}-\mathbf{u}_h) \|_0 .$$

Hence we obtain

$$\| \operatorname{div}(\mathbf{\Pi}_h\mathbf{u}-\mathbf{u}_h) \|_0 \le C \| \operatorname{div}(\mathbf{u}-\mathbf{\Pi}_h\mathbf{u}) \|_0 .$$

By the triangle inequality, we immediately obtain the result. □

We have the following error estimates:

Theorem 18. *Let* $(\mathbf{u}_h, p_h)$ *be the solution of* (4). *Then*

$$\| \mathbf{u} - \mathbf{u}_h \|_0 \leq Ch^{k+1} \| \mathbf{u} \|_{k+1}, \tag{23}$$

$$\| \operatorname{div}(\mathbf{u} - \mathbf{u}_h) \|_0 \leq Ch^{k} \| \operatorname{div} \mathbf{u} \|_{k}, \tag{24}$$

$$\| p - p_h \|_0 \leq Ch^{k+\delta_{0,k}} \| p \|_{k+1+\delta_{0,k}}, \tag{25}$$

where $\delta_{0,k}$ *is the Kronecker delta.*

Proof. From (22) and Lemma 16 we see that

$$\begin{aligned} c \| \mathbf{u} - \mathbf{u}_h \|_0^2 &\leq (\kappa^{-1}(\mathbf{u} - \mathbf{u}_h), \mathbf{u} - \mathbf{u}_h) \\ &= (\kappa^{-1}(\mathbf{u} - \mathbf{u}_h), \mathbf{\Pi}_h \mathbf{u} - \mathbf{u}_h) + (\kappa^{-1}(\mathbf{u} - \mathbf{u}_h), \mathbf{u} - \mathbf{\Pi}_h \mathbf{u}) \\ &= (p - p_h, \operatorname{div}(\mathbf{\Pi}_h \mathbf{u} - \mathbf{u}_h)) + (\kappa^{-1}(\mathbf{u} - \mathbf{u}_h), \mathbf{u} - \mathbf{\Pi}_h \mathbf{u}) \\ &= (\Phi_h p - p_h, \operatorname{div}(\mathbf{\Pi}_h \mathbf{u} - \mathbf{u}_h)) + (\kappa^{-1}(\mathbf{u} - \mathbf{u}_h), \mathbf{u} - \mathbf{\Pi}_h \mathbf{u}) \\ &= (\kappa^{-1}(\mathbf{u} - \mathbf{u}_h), \mathbf{u} - \mathbf{\Pi}_h \mathbf{u}) \\ &= C \| \mathbf{u} - \mathbf{u}_h \|_0 \| \mathbf{u} - \mathbf{\Pi}_h \mathbf{u} \|_0, \end{aligned}$$

where c and C are independent of h and $\mathbf{u}$. Therefore, by Lemma 14, we obtain the first estimate of the theorem:

$$\| \mathbf{u} - \mathbf{u}_h \|_0 \leq c \| \mathbf{u} - \mathbf{\Pi}_h \mathbf{u} \|_0 \leq Ch^{k+1} \| \mathbf{u} \|_{k+1} .$$

To estimate $\| p - p_h \|_0$, we use duality argument. Let ψ be the solution of Dirichlet problem

$$\begin{cases} -\operatorname{div}(\kappa \nabla \psi) = \Phi_h p - p_h, & \text{in } \Omega, \\ \psi = 0, & \text{on } \partial\Omega. \end{cases}$$

Then, by (21) and (22),

$$\begin{aligned} (\Phi_h p - p_h,\ \Phi_h p - p_h) &= -(\Phi_h p - p_h, \operatorname{div}(\kappa \nabla \psi)) \\ &= -(\Phi_h p - p_h, \operatorname{div} \mathbf{\Pi}_h(\kappa \nabla \psi)) \\ &= -(\kappa^{-1}(\mathbf{u} - \mathbf{u}_h), \mathbf{\Pi}_h(\kappa \nabla \psi)) \\ &= (\kappa^{-1}(\mathbf{u} - \mathbf{u}_h), \kappa \nabla \psi - \mathbf{\Pi}_h(\kappa \nabla \psi)) + (\kappa^{-1}(\mathbf{u} - \mathbf{u}_h), -\kappa \nabla \psi) \\ &= (\kappa^{-1}(\mathbf{u} - \mathbf{u}_h), \kappa \nabla \psi - \mathbf{\Pi}_h(\kappa \nabla \psi)) + (\mathbf{u} - \mathbf{u}_h, -\nabla \psi) \\ &= (\kappa^{-1}(\mathbf{u} - \mathbf{u}_h), \kappa \nabla \psi - \mathbf{\Pi}_h(\kappa \nabla \psi)) + (\operatorname{div}(\mathbf{u} - \mathbf{u}_h), \psi) \\ &= (\kappa^{-1}(\mathbf{u} - \mathbf{u}_h), \kappa \nabla \psi - \mathbf{\Pi}_h(\kappa \nabla \psi)) + (\operatorname{div}(\mathbf{u} - \mathbf{u}_h), \psi - \Phi_h \psi). \end{aligned}$$

Using elliptic regularity and approximation property of $\mathbf{\Pi}_h$, we obtain

$$\| \Phi_h p - p_h \|_0 \leq C(h \| \mathbf{u} - \mathbf{u}_h \|_0 + h \| \operatorname{div}(\mathbf{u} - \mathbf{u}_h) \|_0).$$

Since $\hat{\operatorname{div}}\, \hat{\mathbf{V}}(\hat{K}) \supseteq R_{k-1}$, it follows from Lemma 15 that

$$\| \operatorname{div}(\mathbf{u} - \mathbf{\Pi}_h \mathbf{u}) \|_0 \leq Ch^k \| \operatorname{div} \mathbf{u} \|_k .$$

By Lemma 17, we have the second estimate of the theorem

$$\| \operatorname{div}(\mathbf{u} - \mathbf{u}_h) \|_0 \leq C' \| \operatorname{div}(\mathbf{u} - \mathbf{\Pi}_h \mathbf{u}) \|_0 \leq Ch^k \| \operatorname{div} \mathbf{u} \|_k .$$

Hence, for $k = 0$ we obtain

$$\| \Phi_h p - p_h \|_0 \leq Ch \| \mathbf{u} \|_1 .$$

For $k \geq 1$ and $0 \leq t \leq k$, we obtain

$$\| \Phi_h p - p_h \|_0 \leq C' h^{t+1} (\| \mathbf{u} \|_{t+1} + \| \operatorname{div} \mathbf{u} \|_t) \leq Ch^{t+1} \| \mathbf{u} \|_{t+1} . \qquad (26)$$

Choosing $t = k - 1$, we obtain

$$\| \Phi_h p - p_h \|_0 \leq Ch^k \| \mathbf{u} \|_k .$$

Since $W_h(K)$ space does not fully contain the space $P_k(x, y, z)$ except when $k = 0$, we only have $\| p - \Phi_h p \|_0 \leq Ch^{k+\delta_{0,k}} \| p \|_{k+1}$. Then the final estimate of the theorem follows directly from the triangle inequality. □

Remark 19. We note that the pressure estimate of Theorem 18 is not optimal for $k \geq 1$. The loss of order results from the fact that the pressure space $W_h(K)$ does not contain enough polynomials. However, we introduce a local post-processing technique in the next section to obtain an optimal order in the pressure.

4. Post-Processing for the Pressure

In this section, we present a local post-processing technique which provides an optimal approximation order for the pressure. Let

$$W_h^\sharp = \{ q \in L^2(\Omega) \mid q|_K \in Q_{k,k,k}(K), \ \forall K \in \mathscr{T}_h \}.$$

We define the approximation $p_h^\sharp \in W_h^\sharp(K)$ locally on each element $K \in \mathscr{T}_h$ as the solution to the following equation:

$$\int_K \kappa \nabla p_h^\sharp \cdot \nabla q \, d\mathbf{x} = -\int_K \mathbf{u}_h \cdot \nabla q \; d\mathbf{x}, \quad \forall q \in W_h^\sharp(K), \tag{27}$$

$$\int_K p_h^\sharp \; d\mathbf{x} = \int_K p_h \; d\mathbf{x}. \tag{28}$$

This technique has been studied on an affine element[21], [22], but we need to modify it. Let $\hat{\Phi}^\sharp : L^2(\hat{K}) \to \hat{W}^\sharp(\hat{K})$ be the L^2-projection, we let $\Phi_K^\sharp : L^2(K) \to W_h^\sharp(K)$ be the local projection operator defined by $\Phi_K^\sharp p = (\hat{\Phi}^\sharp \hat{p}) \circ F_K^{-1}$. If we define the weighted norm $\| \cdot \|_{0,\kappa} := (\kappa \cdot, \cdot)^{1/2}$ and the weighted semi-norm $| \cdot |_{1,\kappa} := (\kappa \nabla \cdot, \nabla \cdot)^{1/2}$, then we have the following error estimate.

Theorem 20. *Suppose that* $(\mathbf{u}, p)$ *is the solution of* (3) *and* $p_h^\sharp$ *is defined by* (27) *and* (28)*. Then we have*

$$\| \, p - p_h^\sharp \, \|_0 \le C h^{k+1} \, \| \, \mathbf{u} \, \|_{k+1} \, .$$

Proof. Let $q = \Phi_K^\sharp p - p_h^\sharp \in W_h^\sharp$. Then using (27), we have

$$\begin{aligned} | \, q \, |_{1,\kappa,K}^2 &= (\kappa \nabla (\Phi_K^* p - p_h^\sharp), \nabla q)_K \\ &= (\kappa \nabla (\Phi_K^\sharp p - p), \nabla q)_K + (\kappa \nabla (p - p_h^\sharp), \nabla q)_K \\ &= (\kappa \nabla (\Phi_K^\sharp p - p), \nabla q)_K + (-\mathbf{u} + \mathbf{u}_h, \nabla q)_K. \end{aligned}$$

By Cauchy-Schwartz inequality and norm equivalence, we have

$$| \, q \, |_{1,K} \le C(\, | \, \Phi_K^\sharp p - p \, |_{1,K} + \| \, \mathbf{u} - \mathbf{u}_h \, \|_{0,K} \,).$$

Let $q' = q - \bar{q}$ with $\bar{q} = \frac{1}{Vol(K)} \int_K q \; d\mathbf{x}$. Then by Poincare inequality, we have

$$\| \, q' \, \|_{0,K} \le Ch \, | \, q \, |_{1,K} \, .$$

From (28) and $\int_K \Phi_K^\sharp p \; d\mathbf{x} = \int_K \Phi_K p \; d\mathbf{x}$, we obtain

$$\begin{aligned} \| \, \bar{q} \, \|_{0,K} &\le Ch \, \| \, \bar{q} \, \|_\infty \\ &= Ch \, | \, \frac{1}{Vol(K)} \int_K (\Phi_K^\sharp p - p_h^\sharp) \; d\mathbf{x} \, | \\ &\le C \, \| \, \Phi_K p - p_h \, \|_{0,K} \, . \end{aligned}$$

Then we have by the triangle inequality,

$$
\begin{aligned}
\| p - p_h^\sharp \|_{0,K} &= \| p - \Phi_K^\sharp p + \Phi_K^\sharp p - p_h^\sharp \|_{0,K} \\
&= \| p - \Phi_K^\sharp p + q' + \bar{q} \|_{0,K} \\
&\le \| p - \Phi_K^\sharp p \|_{0,K} + \| q' \|_{0,K} + \| \bar{q} \|_{0,K} \\
&\le \| p - \Phi_K^\sharp p \|_{0,K} + Ch(\, | \Phi_K^\sharp p - p |_{1,K} + \| \mathbf{u} - \mathbf{u}_h \|_{0,K} \,) + C \| \Phi_K p - p_h \|_0
\end{aligned}
$$

The desired estimate follows from the approximation property of $\Phi_K^\sharp$, (23) and (26). □

Acknowledgments

This paper has been supported by 2012 Hannam University Research Fund.

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