# Comparison of multigrid performance for higher order scheme with 5-point scheme

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**Abstract**

We consider a multigrid algorithm for higher order finite difference scheme for the Poisson problem on rectangular domain. Several smoothers including Jacobi, Red-black Gauss-Seidel are tested and compared. Since higher order scheme gives much more accurate result then 5-point scheme, one may use small number of levels with higher order scheme and thus the overall cost is reduced quite a lot. The numerical experiment compares the two cases.
**Key words.** multigrid methods, higher-order finite difference method.
**AMS(MOS) subject classifications.** primary 65N30, secondary 65F10.

## 1 Introduction

The purpose of this paper is to consider the convergence of a multigrid algorithm for a higher order finite difference scheme and compare overall computational cost with five point scheme.

Multigrid methods have been proven very fast and robust for most of elliptic problems with many types of discretizations. See [1],[2],[3], for example. For a various convergence proofs, we refer to [5], [6],[8],[9],[4].

Usually, multigrid algorithm consists of a sequence $M_1 \subset, \cdots, \subset M_J$ of nested finite dimensional spaces, smoothing process for each level and some transfer operator between the spaces. First, a smoothing(presmoothing)

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is performed on the finest level, then the residual is passed onto the next coarser level, where the slowly varying component is relaxed, the process repeated until down to the coarsest grid, where the problem is solved exactly. The resulting quantity then is passed back to the finer levels, sometimes with additional smoothing(postsmoothing). Such an algorithm is typical and called a V-cycle, while other variants, such as using smoothing as we go down only(backslash-cycle) or many smoothings for each level are possible. Another variation is to use two correction step at each level. This is called W-cycle. Usually W-cycle converges faster(with more work) and easier to prove its convergence. However, V-cycle is simpler and advocated by practitinoers and the convergence theory is mathematically more challenging.

Commonly used discretization method for simple elliptic problems is 5-point formula, which has second order accuracy. However to achieve very accuracy, one needs to refine the meshes, which sometimes require large memory and time. In most laboratory job, one takes maximum 8 levels, which mounts to $256 \times 256$ grids. This is quite a memory and time consuming for a model problem. For realistic problems, one may need more levels and the problems are coupled with other systems, which requires still faster algorithms for elliptic part. Instead, higher order scheme requires lesser number of levels to achieve the same accuracy. To resolve this problem, one adapts higher order accuracy[11] which has $h^4$ order accuracy. However, when the right hand side is smooth, one can use newly developed higher order scheme in[16] to have $h^6$ order accuracy. This is an enormous gain, since one can use only one third number of levels than five point formula to get the same accuracy.

For finite difference case, there is a common belief that the sum of the order of prolongation and restriction should be greater than 2 [10] so that one uses for example bilinear prolongation with trivial restriction [15] or variants of such algorithms. However, we believe that the restriction operator as the adjoint of prolongation is most natural. It makes the whole algorithm symmetric and also fits the theory developed by Bramble et al.[9],[8]. The performance of such nonsymmetric operator pairs will be shown in later section.

## 2 Multigrid Method for 9 point compact scheme

In this section, we briefly describe 9 point compact scheme and introduce multigrid algorithm for this scheme. We first consider the following Poisson

Equation :

$$\begin{cases} -\Delta u(x,y) & = f(x,y) \quad \text{in} \quad \Omega \\ \quad u(x,y) & = g(x,y) \quad \text{on} \quad \partial\Omega. \end{cases} \tag{2.1}$$

Here, $\Omega$ can be any region in $\mathbb{R}^2$ covered by squares. For simplicity, we assume $\Omega$ is the unit square. For $k = 1, 2, \ldots, J$, let $h_k = 2^{-k}$ be a mesh size of level $k$. Define $\Omega_k$ be a space of points $(x_i, y_j) = (ih_k, jh_k)$ for $i, j = 0, 1, \ldots, 2^k$ and $V_k$ be a vector space of function evaluated at $\Omega_k$. A fourth-order compact scheme is written as follows:

$$\begin{aligned} &\frac{1}{6h^2}\left[\,20u_{i,j} - 4(u_{i-1,j} + u_{i,j-1} + u_{i+1,j} + u_{i,j+1})\right. \\ &\quad \left.-(u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1})\,\right] \\ &\quad = \frac{1}{12}\left[f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} + 8f_{i,j}\right]. \end{aligned} \tag{2.2}$$

When the solution of the equation (2.1) is sufficiently smooth(having continuous sixth-order partial derivatives) one may use the newly developed scheme, whose convergence rate is of $O(h^6)$. The stencil of this high-order finite difference scheme is written as follows[16]:

$$\begin{aligned} &\frac{1}{6h^2}\left[\,20u_{i,j} - 4(u_{i-1,j} + u_{i,j-1} + u_{i+1,j} + u_{i,j+1})\right. \\ &\quad \left.-(u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1})\,\right] \\ &\quad = \left[1 + \frac{2h^2}{4!}\Delta + \frac{2h^4}{6!}\left(\Delta^2 + 2\frac{\partial^4}{\partial x^2 \partial y^2}\right)\right] f_{i,j} \end{aligned} \tag{2.3}$$

where $h = h_k$. This nine-point discretization gives a truncation error of $O(h^8)$ over a square mesh(i.e., a convergence rate of $O(h^6)$). We obtain a system of linear equation of the form

$$A_k u = f, \tag{2.4}$$

where $A_k$ is a sparse, $n \times n$, symmetric, positive definite matrix and $u$ is the vector whose entries are $u_{i,j}$, and $f$ is the vector whose entries are $f(x_i, y_j)$.

To describe the multigrid algorithm for this problem, we need certain intergrid transfer operators(called prolongation) between two grids. Assuming we are given a certain prolongation operator $I_{k-1}^k : V_{k-1} \to V_k$, we define the restriction operator $I_k^{k-1} : V_k \to V_{k-1}$ as its adjoint with respect to $(\cdot, \cdot)$ :

$$(I_k^{k-1}u, v)_{k-1} = (u, I_{k-1}^k v)_k \quad \forall u \in V_k, \forall v \in V_{k-1}.$$

Now the multigrid algorithm for solving (2.4) is defined as follows :

**Multigrid Algorithm.** $\mathbf{W(m,m)}$ Set $B_1 = A_1^{-1}$. For $1 < k \le J$, assume that $B_{k-1}$ has been defined and define $B_k f$ for $f \in V_k$ as follows:

1. Set $x^0 = 0$ and $q^0 = 0$.

2. Define $x^l$ for $l = 1, \ldots, m$ by
$$x^l = x^{l-1} + R_k^{(l+m)}(f - A_k x^{l-1}).$$

3. Define $y^m = x^m + I_k q^p$, where $q^i$ is defined by
$$q^i = q^{i-1} + B_{k-1}\left[P_{k-1}^0(f - A_k x^m) - A_{k-1}q^{i-1}\right].\ i = 1, \cdots, p$$

4. Define $y^l$ for $l = m+1, \ldots, 2m$ by
$$y^l = y^{l-1} + R_k^{(l+m)}(f - A_k y^{l-1}).$$

5. Set $B_k f = y^{2m}$.

For comparison, we shall also consider multigrid algorithm with nonsymmetric operators. We will not repeat the algorithm here, instead, they will be considered in later section.

Fix $k$. Let $u_{i,j}, u_{i+1,j}, u_{i,j+1}, u_{i+1,j+1}$ be points of level $k-1$ and $E_{i,j}$ be a cell having them as its vertex. (We borrow the term "cell" from cell-centered method.) Let $u_{i,j}^1, u_{i,j}^2, u_{i,j}^3, u_{i,j}^4$ and $u_{i,j}^5$ be points of level $k$ defined as in Figure 1. Note that $u_{i,j}^1 = u_{i,j-1}^3, u_{i,j}^2 = u_{i+1,j}^4, u_{i,j}^3 = u_{i,j+1}^1$ and $u_{i,j}^4 = u_{i-1,j}^2$. Divide $E_{i,j}$ into four subcells, labeling them counterclockwise as $e_{i,j}^1, e_{i,j}^2, e_{i,j}^3, e_{i,j}^4$ at level $k$. ( See Figure 1.)

Now, we define the prolongation operator $I_{k-1}^k$ be bilinear interpolation of four points $u_{i,j}, u_{i+1,j}, u_{i,j+1}$ and $u_{i+1,j+1}$. First, $u_{i,j}, u_{i+1,j}, u_{i,j+1}$ and $u_{i+1,j+1}$ of level $k$ are the same value of level $k-1$ respectively. The mid points $u_{i,j}^1, u_{i,j}^2, u_{i,j}^3$ and $u_{i,j}^4$ can be written as follows:

$$\begin{aligned} u_{i,j}^1 &= \frac{u_{i,j} + u_{i+1,j}}{2}, \qquad u_{i,j}^2 = \frac{u_{i,j} + u_{i+1,j+1}}{2} \\ u_{i,j}^3 &= \frac{u_{i,j+1} + u_{i+1,j+1}}{2}, \quad u_{i,j}^4 = \frac{u_{i,j} + u_{i,j+1}}{2}. \end{aligned} \tag{2.5}$$

The value of center $u_{i,j}^5$ is the average of $u_{i,j}, u_{i+1,j}, u_{i,j+1}$ and $u_{i+1,j+1}$ :

$$u_{i,j}^5 = \frac{u_{i,j} + u_{i+1,j} + u_{i,j+1} + u_{i+1,j+1}}{4}. \tag{2.6}$$

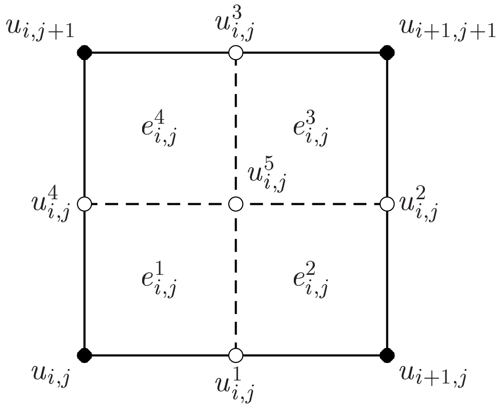


Figure 1: cell $E^k_{i,j}$ and its subcells

**Theorem 2.1** *We have*

$$A_k(I^k_{k-1}u, I^k_{k-1}u) \leq C A_{k-1}(u,u), \quad \forall u \in V_{k-1}. \tag{2.7}$$

**Proof 1** *Using the symmetry, we have*

$$\begin{aligned}
A_{k-1}(u,u) = \frac{1}{6}\sum_{i,j}^{2^{k-1}} & \big[\, 20u_{i,j} - 4(u_{i-1,j} + u_{i,j-1} + u_{i+1,j} + u_{i,j+1}) \\
& -(u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}) \,\big]\, u_{i,j} \\
= \frac{4}{6}\sum_{i,j}^{2^{k-1}} & \big[\, (u_{i,j} - u_{i-1,j})^2 + (u_{i,j} - u_{i,j-1})^2 \\
& +(u_{i,j} - u_{i+1,j})^2 + (u_{i,j} - u_{i,j+1})^2 \big] \\
+\frac{1}{6}\sum_{i,j}^{2^{k-1}} & \big[(u_{i,j} - u_{i-1,j-1})^2 + (u_{i,j} - u_{i+1,j-1})^2 \\
& +(u_{i,j} - u_{i+1,j+1})^2 + (u_{i,j} - u_{i-1,j-1})^2 \,\big]\,.
\end{aligned} \tag{2.8}$$

*Set $v = I_{k-1}^k u$ then, similarly, we have*

$$
\begin{aligned}
A_k(v,v) = \frac{4}{6}\sum_{m,n}^{2^k} & \big[\,(v_{m,n}-v_{m-1,n})^2+(v_{m,n}-v_{m,n-1})^2 \\
& +(v_{m,n}-v_{m+1,n})^2+(v_{m,n}-v_{m,n+1})^2\big] \\
+\frac{1}{6}\sum_{m,n}^{2^k} & \big[(v_{m,n}-v_{m-1,n-1})^2+(v_{m,n}-v_{m+1,n-1})^2 \\
& +(v_{m,n}-v_{m+1,n+1})^2+(v_{m,n}-v_{m-1,n-1})^2\,\big]\,.
\end{aligned} \tag{2.9}
$$

*Each terms in (2.9) is bounded by the terms in (2.8). For example, if $m = 2i$ and $n = 2j$ then $v_{m,n} = u_{i,j}$ and $v_{m,n-1} = (u_{i,j} - u_{i,j-1})/2$ by the definition of our interpolation. So we have*

$$(v_{m,n} - v_{m,n-1})^2 \le (u_{i,j} - u_{i,j-1})^2/4. \tag{2.10}$$

*Note that the difference $(u_{i,j}-u_{i,j-1})^2$ appears finite times(at most 10 times). Thus there exists a positive constant $C$ satisfying the inequality (2.7).*

To prove multigrid convergence theory, we need following property, so-called, "approximation and regularity": There exist a number $0 < \alpha \le 1$ and a constand $C_\alpha$ such that for all $k = 1, \cdots, J$,

$$A_k((I - I_{k-1}^k P_{k-1})u, u) \le C_\alpha \left( \frac{\|A_k u\|_k^2}{\lambda_k} \right)^\alpha A_k(u,u)^{1-\alpha}, \quad \forall u \in V_k. \tag{2.11}$$

Here, $\lambda_k$ is the largest eigenvalue of $A_k$ and $P_{k-1}$ is the elliptic projection defined by

$$A_{k-1}(P_{k-1}u, v) = A_k(u, I_{k-1}^k v), \quad \forall u \in V_k, v \in V_{k-1}. \tag{2.12}$$

The following result can be proved as in [9].

**Lemma 2.1** *Let the operator $P_{k-1}$ be defined by (2.12). Then (2.11) holds for $\alpha = \frac{1}{2}$.*

With these preliminaries, we can prove the W-cycle result by the framework of [9].

**Theorem 2.2** *Let $E_k = I - B_k A_k$ in algorithm W(m,m). Then we have*

$$A_k(E_k u, u) \le \delta_k A_k(u,u), \quad \forall u \in V_k, \tag{2.13}$$

*where $\delta_k < 1$.*

| $h_J$ | $\lambda_{min}$ | $\lambda_{max}$ | $K$ | $\delta$ |
|---|---|---|---|---|
| 1/16 | 0.826 | 0.999 | 1.211 | 0.030 |
| 1/32 | 0.823 | 0.999 | 1.215 | 0.030 |
| 1/64 | 0.823 | 0.999 | 1.215 | 0.031 |
| 1/128 | 0.823 | 0.999 | 1.216 | 0.031 |

Table 1: high-order scheme with Gauss-Seidel smoothing and 9-point interpolation

| $h_J$ | $\lambda_{min}$ | $\lambda_{max}$ | $K$ | $\delta$ |
|---|---|---|---|---|
| 1/16 | 0.776 | 0.999 | 1.290 | 0.050 |
| 1/32 | 0.765 | 0.999 | 1.307 | 0.055 |
| 1/64 | 0.761 | 0.999 | 1.314 | 0.057 |
| 1/128 | 0.756 | 0.999 | 1.321 | 0.059 |

Table 2: 5-point scheme with Gauss-Seidel smoothing and 9-point interpolation

# 3 Numerical Experiments

We consider the following problem on the unit square:

$$\begin{aligned} -\nabla \cdot p\nabla\tilde{u} = f \quad & \text{in } \Omega = (0,1)^2, \\ \tilde{u} = 0 \quad & \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

First, we report the maximum, minimum eigenvalues, condition numbers and contractions of both algorithms with Gauss-Seidel smoothing and 9-point interpolation. Numerical experiment shows that multigrid algorithm of high-order scheme converges faster than that of 5-point scheme. Both algorithms contract independent of the mesh size $h$ and number of levels $J$.

Next, we report the number of iteration and discrete $L^2$ errors of test problem.

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| $h_J$ | high-order scheme Iteration | high-order scheme $L^2$ error | 5-pt scheme Iteration | 5-pt scheme $L^2$ error |
|---|---|---|---|---|
| 1/16 | 8 | 1.47(-06) | 7 | 7.93(-04) |
| 1/32 | 7 | 9.22(-08) | 7 | 1.98(-04) |
| 1/64 | 7 | 5.77(-09) | 7 | 4.95(-05) |
| 1/128 | 7 | 3.65(-10) | 7 | 1.24(-05) |

Table 3: high-order scheme with Gauss-Seidel smoothing and 9-point interpolation


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