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Mixed finite element methods for general quadrilateral grids

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ABSTRACT

We study a new mixed finite element of lowest order for general quadrilateral grids which gives optimal order error in the $\mathbf{H}(\mathrm{div})$-norm. This new element is designed so that the $\mathbf{H}(\mathrm{div})$-projection Π_h satisfies $\nabla \cdot \Pi_h = P_h\mathrm{div}$. A rigorous optimal order error estimate is carried out by proving a modified version of the Bramble–Hilbert lemma for vector variables. We show that a local $\mathbf{H}(\mathrm{div})$-projection reproducing certain polynomials suffices to yield an optimal $\mathbf{L}^2$-error estimate for the velocity and hence our approach also provides an improved error estimate for original Raviart–Thomas element of lowest order. Numerical experiments are presented to verify our theory.

1. Introduction

Since the 1970's much effort has gone into the computation of the velocity variable of a physical state. The mixed finite element method of decoupling the partial differential equations into a system of lower order equations to solve for both the pressure and velocity has been effective for that purpose and has been a very active area of research [3–6,8,13,15,16,18–20].

The importance of obtaining a more accurate velocity occurs in the problem of fluid flow in porous media. For example, it is well known that the transport and diffusion terms in the flow and transport equations are governed by fluid velocities. Further, the mixed finite element method has the important property of conserving mass locally. It is a meaningful method, since the flow equations are based on the mass balance law.

As for the meshes, there are two standard ways; triangular grids and rectangular grids. For these standard grids, the convergence and the accuracy of these mixed methods are well established [4,6,8,18]. One of the advantages of the triangular grids is the ability to fit complex geometry, while that of the rectangular grids lies in nice data structures. Quadrilateral grids have the advantages of both grids: They not only can fit complex geometry well, but also maintain the structures of rectangular grids. While there have been some attempts for the error estimates of mixed methods using quadrilateral grids, they were carried out under the (sometimes implicit) assumption that the grids are almost parallelogram [10–12,14]. When the distortion is given on a large scale, refinements of the grid yield almost parallelograms so that the preceding analysis is applicable. In reality, there are cases when the fine grid itself has to be distorted; the geology may have fine variations due to the rugged rock formulation in the porous media problem. However, when one attempts to use the usual mixed elements to arbitrary quadrilaterals, the existing theory cannot be extended due to the violation of the condition div $\mathbf{V}_h \subset W_h$, and numerical experiments show that the velocity vector does not converge in the divergence norm (see Sections 3, 4 below). Similar phenomena were observed for primal approaches for elliptic problems in [1,7].

In this paper, we study a new mixed finite element for quadrilaterals by modifying the lowest order Raviart–Thomas (RT) element [18]. This modified element is dependent on each quadrilateral and the role of the modified part is to maintain the

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condition, div $\mathbf{V}_h = W_h$. We believe it was first considered by Shen [21] but they use Taylor series expansion under the assumption that $\mathbf{u}$ is at most linear (see page 11 of [21]). We believe this approach cannot handle the case where $\mathbf{u}$ is not linear or $\mathbf{u}$ does not have Taylor series expansion (see remark 3.1 and the paragraph following it). Further, no computational results are given there. The difficult part of the proof is the approximation of vector variables. For this purpose, we provide a Bramble–Hilbert type of lemma for vector variables, which enables us to provide $O(h)$-error estimate for vector variables for Raviart–Thomas element and its modified version. Also, we give an optimal error estimate for the modified element in the divergence norm. Another family of spaces yielding correct vector approximation were created by Arnold, Boffi and Falk in [2] at the expense of additional degrees of freedom (two in the case of the lowest order), whereas our space has the same degrees of freedom as the RT element.

The rest of this paper is organized as follows. In the next section, we consider a mixed formulation for a model second order elliptic problem and its discretization using quadrilateral grids. In Section 3, we study a modified Raviart–Thomas element of lowest order and give an optimal order error estimate (for velocity and its divergence) by proving a modified version of the Bramble–Hilbert lemma. This approach incidentally results in an improved estimate of Raviart–Thomas element. We briefly discuss how our ideas can be applied to modify the Brezzi–Douglas–Marini (BDM) space [4] of order one to obtain a similar result for the divergence error of the velocity. Finally, in Section 4, we present numerical experiments.

2. Mixed methods for elliptic problems

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ with the boundary $\partial\Omega$. We consider the following second-order elliptic boundary value problem

$$\begin{aligned} &-\nabla\cdot(\kappa\nabla p) = f \quad \text{in } \Omega,\\ &p = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\kappa = \kappa(\mathbf{x})$ is a symmetric and uniformly positive definite matrix. Let us introduce a vector variable $\mathbf{u} = -\kappa\nabla p$ and rewrite problem (1) in mixed form

$$\begin{aligned} &\mathbf{u} + \kappa\nabla p = 0, \quad \text{in } \Omega,\\ &\nabla\cdot\mathbf{u} = f, \quad \text{in } \Omega,\\ &p = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2}$$

Denote by $\mathbf{H}^1(\Omega) = (H^1(\Omega))^2$, $\mathbf{V} := \mathbf{H}(\text{div};\Omega)$ and $W = L^2(\Omega)$ the usual Sobolev spaces with obvious norms. Then we have the following variational form for (2):

$$\begin{aligned} &(\kappa^{-1}\mathbf{u},\mathbf{v}) - (p,\nabla\cdot\mathbf{v}) = 0, \quad \forall\mathbf{v}\in\mathbf{V},\\ &(\nabla\cdot\mathbf{u},q) = (f,q), \quad \forall q\in W. \end{aligned} \tag{3}$$

This problem has a unique solution $(\mathbf{u},p)\in\mathbf{V}\times W$. Assume that we have approximating spaces $\mathbf{V}_h\subset\mathbf{V}$ and $W_h\subset W$. Then we have the following finite dimensional problem corresponding to (3).

$$\begin{aligned} &(\kappa^{-1}\mathbf{u}_h,\mathbf{v}_h) - (p_h,\nabla\cdot\mathbf{v}_h) = 0, \quad \forall\mathbf{v}_h\in\mathbf{V}_h,\\ &(\nabla\cdot\mathbf{u}_h,q_h) = (f,q_h), \quad \forall q_h\in W_h. \end{aligned} \tag{4}$$

Let $\{\mathcal{T}_h : h > 0\}$ be a family of partitions of Ω into convex quadrilaterals. The intersection, if any, of any two (closed) quadrilaterals in the partition is either a common edge or a common vertex.

We assume a usual shape-regularity condition [p. 247,8], on the partition $\{\mathcal{T}_h : h > 0\}$.

(**A**): The quadrilaterals $Q\in\{\mathcal{T}_h\}$ are convex and there exist constants $\gamma > 0$ and $0 < \rho < 1$ independent of h such that the ratio of the diameter and the smallest side of each Q is bounded by γ and the absolute maximum cosine of angles of Q are bounded by ρ.

We can replace (**A**) by other shape regularity condition such as the one in [9], where they used the ratio between the diameter of K and the maximum of diameter of triangles formed by three vertices of K.

Let $\hat{\mathbf{x}} = (\hat{x},\hat{y})$ and $\mathbf{x} = (x,y)$. We use the unit square $\hat{Q} = [0,1]\times[0,1]$ as the reference element in the $\hat{x}\hat{y}$-plane with the vertices

$$\hat{\mathbf{x}}_1 = (0,0), \quad \hat{\mathbf{x}}_2 = (1,0), \quad \hat{\mathbf{x}}_3 = (1,1), \quad \hat{\mathbf{x}}_4 = (0,1).$$

Let Q be a convex quadrilateral with vertices $\mathbf{x}_i$ arranged counterclockwise. Then there exists a unique invertible bilinear transformation F_Q which maps $\hat{Q}$ onto Q and satisfies

$$\mathbf{x}_i = F_Q(\hat{\mathbf{x}}_i), \quad i = 1,2,3,4.$$

The Jacobian matrix DF_Q of F_Q is given by

$$DF_Q = \begin{pmatrix} \frac{\partial x}{\partial\hat{x}} & \frac{\partial x}{\partial\hat{y}}\\ \frac{\partial y}{\partial\hat{x}} & \frac{\partial y}{\partial\hat{y}} \end{pmatrix}. \tag{5}$$

Simple calculation shows that the determinant $J_Q = \det DF_Q$ is a linear function of $\hat{x}$ and $\hat{y}$,

$$J_Q(\hat{x},\hat{y}) = \alpha + \beta\hat{x} + \gamma\hat{y}, \tag{6}$$

for some constants α, β and γ depending on F_Q (see Fig. 1). Note that

$$|Q| = \int_Q 1\, d\mathbf{x} = \int_{\hat{Q}} J_Q\, d\hat{\mathbf{x}} = \int_{\hat{Q}} (\alpha + \beta\hat{x} + \gamma\hat{y})\, d\hat{\mathbf{x}} = \alpha + \frac{1}{2}\beta + \frac{1}{2}\gamma.$$

From (**A**), we derive the following estimates for the Jacobian matrix DF_Q and its determinant J_Q: there exists a constant c independent of h and independent of Q such that

$$\|DF_Q\|_\infty \leqslant ch, \tag{7}$$

$$|J_Q^{-1}|_\infty \leqslant ch^{-2}. \tag{8}$$

The vector valued functions on $\hat{Q}$ are transformed into vector valued functions on Q by the so called Piola transformation

$$\mathbf{v} = \mathcal{P}\hat{\mathbf{v}},$$

where

$$(\mathcal{P}\hat{\mathbf{v}})|_Q = \mathcal{P}_Q(\hat{\mathbf{v}}|_{\hat{Q}}) := \frac{DF_Q}{J_Q}\hat{\mathbf{v}}|_{\hat{Q}} \circ F_Q^{-1}. \tag{9}$$

This transformation maps $\mathbf{H}(\mathrm{div};\hat{Q})$ space on the reference element onto $\mathbf{H}(\mathrm{div};Q)$, and has the following well known properties [6]: if we let $\hat{p} = p \circ F$ and $\mathbf{v} = \mathcal{P}\hat{\mathbf{v}}$, then

$$\int_Q \nabla p \cdot \mathbf{v}\, d\mathbf{x} = \int_{\hat{Q}} \hat{\nabla}\hat{p} \cdot \hat{\mathbf{v}}\, d\hat{\mathbf{x}}, \tag{10}$$

$$\nabla \cdot \mathbf{v} = \frac{1}{J_Q}\hat{\nabla} \cdot \hat{\mathbf{v}}. \tag{11}$$

3. A modified Raviart–Thomas (MRT) space of the lowest order

Let W_h be the space of functions which are piecewise constant on $\mathcal{T}_h$. To define $\mathbf{V}_h$, let $\mathbf{V}_h(\hat{Q})$ denote the local space on the reference element $\hat{Q}$ consisting of all functions of the form:

$$\hat{\mathbf{v}} = \left(a + b\hat{x} + (b+d)\frac{\beta}{2|Q|}\hat{x}(\hat{x}-1), c + d\hat{y} + (b+d)\frac{\gamma}{2|Q|}\hat{y}(\hat{y}-1)\right),$$

where $a,\ b,\ c,\ d \in \mathbb{R},\ \beta, \gamma$ are from (6) and the local space $\mathbf{V}_h(Q)$ on each quadrilateral Q is defined by

$$\mathbf{V}_h(Q) = \left\{\mathbf{v} = \mathcal{P}_Q\hat{\mathbf{v}}\,|\,\hat{\mathbf{v}} \in \mathbf{V}_h(\hat{Q})\right\} \tag{12}$$

and the global space $\mathbf{V}_h$ is defined by

$$\mathbf{V}_h = \left\{\mathbf{v} \in \mathbf{V}\,|\,\mathbf{v}|_Q \in \mathbf{V}_h(Q), \quad \forall Q \in \mathcal{T}_h\right\}. \tag{13}$$

Note that the space $\mathbf{V}_h(\hat{Q})$ is dependent on each quadrilateral Q, hence it should be denoted as $\mathbf{V}_h(\hat{Q}, Q)$. But we drop Q for brevity when there is no worry of confusion. When $\beta = \gamma = 0$, the spaces $\mathbf{V}_h$ and $\mathbf{V}_h(\hat{Q})$ correspond to the usual Raviart–Thomas space which we denote by $\mathbf{U}_h$ and $\mathbf{U}_h(\hat{Q})$, respectively.

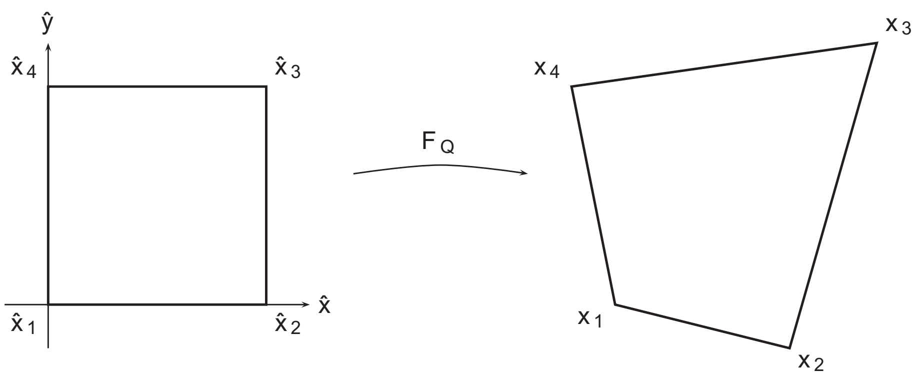

Fig. 1. Bilinear transformation $\mathbf{F}_Q$.

Let $\mathbf{v} \in \mathbf{V}_h$ be arbitrary and let $\hat{\mathbf{v}}$ be the inverse image of the Piola transformation of $\mathbf{v}$. Then for any given $Q \in \mathcal{T}_h$,

$$\hat{\nabla} \cdot \hat{\mathbf{v}}|_Q = b + (b+d)\frac{\beta}{2|Q|}(2\hat{x}-1) + d + (b+d)\frac{\gamma}{2|Q|}(2\hat{y}-1) = (b+d)\frac{2|Q| - \beta - \gamma + 2\beta\hat{x} + 2\gamma\hat{y}}{2|Q|} = (b+d)\frac{J_Q}{|Q|}$$

so that

$$\nabla \cdot \mathbf{v}|_Q = \frac{1}{J_Q}\hat{\nabla} \cdot \hat{\mathbf{v}} = \frac{b+d}{|Q|}$$

is constant. We conclude that the discrete space $\mathbf{V}_h$ has the property that for any $\mathbf{v} \in \mathbf{V}_h$, $\nabla \cdot \mathbf{v}|_Q$ is constant for each $Q \in \mathcal{T}_h$. So div $\mathbf{V}_h \subset W_h$.

3.1. The $\mathbf{H}(\mathrm{div};\Omega)$-projection Π_h

For any domain K with straight edges, we let $\widetilde{\mathbf{H}}(K)$ be a subspace of functions in $\mathbf{H}(\mathrm{div};K)$ having L^2 normal trace on each edge of K. Then the projection $\Pi_h : \mathbf{H}^1(\Omega) \to \mathbf{V}_h$ is defined as follows: First define $\hat{\Pi}_{\hat{Q}}$ from $\widetilde{\mathbf{H}}(\hat{Q})$ onto $\mathbf{V}_h(\hat{Q})$ satisfying

$$\int_{\hat{e}} \hat{\Pi}_{\hat{Q}}\hat{\mathbf{v}} \cdot \hat{\mathbf{n}}\, d\hat{s} = \int_{\hat{e}} \hat{\mathbf{v}} \cdot \hat{\mathbf{n}}\, d\hat{s} \tag{14}$$

for each edge $\hat{e}$ of $\hat{Q}$ and then set

$$\Pi_Q\mathbf{v} = \mathcal{P}_Q(\hat{\Pi}_{\hat{Q}}\hat{\mathbf{v}}), \quad \forall \mathbf{v} \in \mathbf{H}^1(Q),$$

where $\mathcal{P}_Q\hat{\mathbf{v}} = \mathbf{v}$. Finally, we define

$$\Pi_h\mathbf{v}|_Q = \Pi_Q\mathbf{v}.$$

We note that with this modified space, the following crucial commutativity holds:

$$\nabla \cdot \Pi_h = P_h\mathrm{div}, \tag{15}$$

where as usual, we denote by $P_h : W \to W_h$ the L^2 projection defined by

$$(P_h w, v_h) = (w, v_h) \tag{16}$$

for all $v_h \in W_h$.

We also need the standard projection $\hat{\Phi}$ onto the Raviart–Thomas space $\mathbf{U}_h(\hat{Q})$ defined by

$$\int_{\hat{e}} \hat{\Phi}\hat{\mathbf{v}} \cdot \hat{\mathbf{n}}\, ds = \int_{\hat{e}} \hat{\mathbf{v}} \cdot \hat{\mathbf{n}}\, d\hat{s} \tag{17}$$

for each edge $\hat{e}$ of $\hat{Q}$ and let the global projection defined by

$$\Phi_h\mathbf{v}|_Q = \mathcal{P}_Q(\hat{\Phi}\hat{\mathbf{v}}). \tag{18}$$

Lemma 1. *Let $\mathbf{u} = (u, v)^T \in \mathbf{H}^1(Q)$ be arbitrary where Q is an arbitrary quadrilateral in $\mathcal{T}_h$. If $\hat{\mathbf{u}} = (\hat{u}, \hat{v})^T$ satisfies $\mathbf{u} = P_Q\hat{\mathbf{u}}$, then we have*

$$\|\hat{u}_{\hat{x}}\|_{0,\hat{Q}} + \|\hat{v}_{\hat{y}}\|_{0,\hat{Q}} \leqslant C(\|\mathbf{u}\|_{0,Q} + h|\mathbf{u}|_{1,Q}), \tag{19}$$

$$\|\hat{u}_{\hat{y}}\|_{0,\hat{Q}} + \|\hat{v}_{\hat{x}}\|_{0,\hat{Q}} \leqslant Ch|\mathbf{u}|_{1,Q}, \tag{20}$$

$$\|\nabla \cdot \hat{\mathbf{u}}\|_{0,\hat{Q}} \leqslant Ch\|\nabla \cdot \mathbf{u}\|_{0,Q}. \tag{21}$$

Proof. By the definition of the Piola transformation, we have

$$\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = J_Q(DF_Q)^{-1}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{22}$$

Since J_Q is the determinant of DF_Q,

$$J_Q(DF_Q)^{-1} = \begin{pmatrix} y_{\hat{y}} & -x_{\hat{y}} \\ -y_{\hat{x}} & x_{\hat{x}} \end{pmatrix} \tag{23}$$

so that

$$\hat{u} = y_{\hat{y}}u - x_{\hat{y}}v, \tag{24}$$

$$\hat{v} = -y_{\hat{x}}u + x_{\hat{x}}v. \tag{25}$$

Differentiate both side of (24) with respect to $\hat{x}$, then

$$\hat{u}_{\hat{x}} = x_{\hat{x}}y_{\hat{y}}u_x + y_{\hat{x}}y_{\hat{y}}u_y - x_{\hat{x}}x_{\hat{y}}v_x - x_{\hat{y}}y_{\hat{x}}v_y + y_{\hat{x}\hat{y}}u - x_{\hat{x}\hat{y}}v. \tag{26}$$

By (7) and (8) we have

$$\|\hat{u}_{\hat{x}}\|_{0,\hat{Q}}^2 = \int_{\hat{Q}} |\hat{u}_{\hat{x}}|^2 d\hat{\mathbf{x}} \tag{27}$$

$$\leqslant C\int_Q h^4(u_x^2+u_y^2+v_x^2+v_y^2)\frac{1}{J_Q}d\mathbf{x} + C\int_Q h^2(u^2+v^2)\frac{1}{J_Q}d\mathbf{x} \tag{28}$$

$$\leqslant Ch^2|\mathbf{u}|_{1,Q}^2 + C\|\mathbf{u}\|_{0,Q}^2, \tag{29}$$

where C is independent of h and $\mathbf{u}$. Hence

$$\|\hat{u}_{\hat{x}}\|_{0,\hat{Q}} \leqslant C\Big(\|\mathbf{u}\|_{0,Q} + h|\mathbf{u}|_{1,Q}\Big). \tag{30}$$

Similarly, we have

$$\|\hat{v}_{\hat{y}}\|_{0,\hat{Q}} \leqslant C\Big(\|\mathbf{u}\|_{0,Q} + h|\mathbf{u}|_{1,Q}\Big). \tag{31}$$

Now consider the term $\hat{u}_{\hat{y}}$. In this case, we have

$$\hat{u}_{\hat{y}} = x_{\hat{y}}y_{\hat{y}}u_x + (y_{\hat{y}})^2u_y - (x_{\hat{y}})^2v_x - x_{\hat{y}}y_{\hat{y}}v_y, \tag{32}$$

since the term $y_{\hat{y}\hat{y}}u - x_{\hat{y}\hat{y}}v$ vanishes. Proceeding as in (28) and (29) we obtain

$$\|\hat{u}_{\hat{y}}\|_{0,\hat{Q}} \leqslant Ch|\mathbf{u}|_{1,Q}. \tag{33}$$

Similarly, we have

$$\|\hat{v}_{\hat{x}}\|_{0,\hat{Q}} \leqslant Ch|\mathbf{u}|_{1,Q}. \tag{34}$$

Note that $\nabla\cdot\hat{\mathbf{u}} = J_Q\nabla\cdot\mathbf{u}$. Using the estimate of the jacobian $J_Q \leqslant Ch^2$ we can easily obtain the following estimation for the divergence

$$\|\nabla\cdot\hat{\mathbf{u}}\|_{0,\hat{Q}} \leqslant Ch\|\nabla\cdot\mathbf{u}\|_{0,Q}. \quad \square$$

Remark 3.1. Our result is different from the standard result for the affine case;

$$|\hat{\mathbf{u}}|_{1,\hat{Q}} \leqslant Ch|\mathbf{u}|_{1,Q}$$

which is obtained by scaling. If we assume that the quadrilateral is almost parallelogram such that the terms $x_{\hat{x}\hat{y}}$ and $y_{\hat{x}\hat{y}}$ are of order h^2, then (19) reduces to

$$|\hat{\mathbf{u}}|_{1,\hat{Q}} \leqslant Ch\|\mathbf{u}\|_{1,Q}. \tag{35}$$

Now we give a counterexample which shows (35) does not hold for general quadrilaterals. Let Q be a trapezoid with vertices $\mathbf{x}_1=(0,0)$, $\mathbf{x}_2=(h,0)$, $\mathbf{x}_3=(\frac{h}{2},h)$, and $\mathbf{x}_4=(0,h)$. This satisfies (**A**). Let $\mathbf{u}=(u,v)^T=(1,1)^T$ be a constant function. Then

$$\hat{u}_{\hat{x}} = y_{\hat{x}\hat{y}}u - x_{\hat{x}\hat{y}}v = (y_{32}-y_{41}) - (x_{32}-x_{41}) = \frac{h}{2}$$

so that $\|\hat{u}_{\hat{x}}\|_{0,\hat{Q}} = \frac{h}{2}$. But

$$\|\mathbf{u}\|_{1,Q} = \|\mathbf{u}\|_{0,Q} = \left(\int_Q 1\,d\mathbf{x}\right)^{\frac{1}{2}} = |Q|^{\frac{1}{2}} = \frac{\sqrt{3}}{2}h$$

so that

$$\|\hat{u}_{\hat{x}}\|_{0,\hat{Q}} = \frac{1}{\sqrt{3}}\|\mathbf{u}\|_{1,Q}. \tag{36}$$

As we have seen in Remark 3.1, inequalities (19) and (21) are sharp. Hence a simple application of the Bramble–Hilbert lemma cannot lead to obtain h factor by scaling back to original element. To overcome the difficulty, we introduce a special polynomial space and prove a modified Bramble–Hilbert lemma. Let D_0 be the space of polynomials of the form $(a+bx, c-by)$. The following is a key result in proving optimal $O(h)$-error estimates in $\mathbf{L}^2$-norm for RT and MRT elements (see the proof of Lemma 4).

Lemma 2. *Let $\widetilde{\Omega}$ be any Lipschitz domain in $\mathbf{R}^2$. There exists a constant $C_{\widetilde{\Omega}}$ such that for all $\mathbf{v}=(u,v)\in\mathbf{H}^1 := \mathbf{H}^1(\widetilde{\Omega})$,*

$$\inf_{\mathbf{w}\in D_0}\|\mathbf{v}+\mathbf{w}\|_1 \leqslant C_{\widetilde{\Omega}}(\|u_y\|_0 + \|v_x\|_0 + \|\nabla\cdot\mathbf{v}\|_0). \tag{37}$$

Proof. Let f_i, $1 \leqslant i \leqslant 3$, be a dual basis of D_0. By the Hahn–Banach extension theorem, there exist continuous linear forms over the space $\mathbf{H}^1$, again denoted f_i, $1 \leqslant i \leqslant 3$, such that for any $\mathbf{w} \in D_0$, $f_i(\mathbf{w}) = 0$, $1 \leqslant i \leqslant 3$ implies $\mathbf{w} = 0$. We will show that there exists a constant $C_{\widetilde{\Omega}}$ such that for all $\mathbf{v} = (u, v) \in \mathbf{H}^1$,

$$\|\mathbf{v}\|_1 \leqslant C_{\widetilde{\Omega}}\left(\|u_y\|_0 + \|v_x\|_0 + \|\nabla\cdot\mathbf{v}\|_0 + \sum_{i=1}^{3}|f_i(\mathbf{v})|\right). \tag{38}$$

Inequality (37) will follow from inequality (38): Given any function $\mathbf{v} \in \mathbf{H}^1$, let $\mathbf{q} \in D_0$ be such that $f_i(\mathbf{v} + \mathbf{q}) = 0$, $1 \leqslant i \leqslant 3$. Then, we have

$$\inf_{\mathbf{w}\in D_0}\|\mathbf{v}+\mathbf{w}\|_1 \leqslant \|\mathbf{v}+\mathbf{q}\|_1 \leqslant C_{\widetilde{\Omega}}(\|u_y\|_0 + \|v_x\|_0 + \|\nabla\cdot\mathbf{v}\|_0),$$

since $\nabla \cdot \mathbf{q} = 0$. This proves (37).

Assume that (38) is false. There exists a sequence $\{\mathbf{v}^\ell\}_{\ell=1}^{\infty}$ of functions $\mathbf{v}^\ell = (u^\ell, v^\ell) \in \mathbf{H}^1$, such that for all ℓ

$$\|\mathbf{v}^\ell\|_1 = 1 \tag{39}$$

and

$$\lim_{\ell\to\infty}\left(\|u^\ell_y\|_0 + \|v^\ell_x\|_0 + \|\nabla\cdot\mathbf{v}^\ell\|_0 + \sum_{i=1}^{3}|f_i(\mathbf{v}^\ell)|\right) = 0. \tag{40}$$

Since the sequence $\{\mathbf{v}^\ell\}$ is bounded in $\mathbf{H}^1$, there exists a subsequence (Kondrasov Rellich theorem), again denoted $\{\mathbf{v}^\ell\}$, and a function $\mathbf{v} = (u, v) \in \mathbf{L}^2$, such that

$$\lim_{\ell\to\infty}\|\mathbf{v}^\ell - \mathbf{v}\|_0 = 0. \tag{41}$$

Note that, by Banach–Alaoglu theorem, this subsequence converges weakly in $\mathbf{H}^1$. Hence by (40) u_y, v_x, and $\nabla \cdot \mathbf{v}$ exist and are equal to 0. Since any $\mathbf{L}^2$ function $\mathbf{v}$ which satisfies $u_y = v_x = \nabla \cdot \mathbf{v} = 0$ is of the form $(a + bx, c - by)$, $\mathbf{v}$ is in D_0.

Since

$$f_i(\mathbf{v}) = \lim_{\ell\to\infty} f_i(\mathbf{v}^\ell) = 0, \quad 1 \leqslant i \leqslant 3,$$

we have $\mathbf{v} = 0$. But this is a contradiction, since $\|\mathbf{v}^\ell\|_1 = 1$ for all ℓ. □

Theorem 3. *Let $\widetilde{\Omega}$ be an connected open subset of $\mathbb{R}^2$ with a Lipschitz continuous boundary. Let f be a continuous linear form on the space $\mathbf{H}^1(\widetilde{\Omega})$ with the property that for all $\mathbf{w} \in D_0$,*

$$f(\mathbf{w}) = 0.$$

Then there exists a constant $C_{\widetilde{\Omega}}$ such that for all $\mathbf{v} = (u, v) \in \mathbf{H}^1(\widetilde{\Omega})$,

$$|f(\mathbf{v})| \leqslant C_{\widetilde{\Omega}}\|f\|^*(\|u_y\|_0 + \|v_x\|_0 + \|\nabla\cdot\mathbf{v}\|_0), \tag{42}$$

where $\|f\|^*$ is the dual norm of f.

Proof. Since $f(\mathbf{v}) = f(\mathbf{v} + \mathbf{w})$ for all $\mathbf{w} \in D_0$, we have that for all $\mathbf{w} \in D_0$,

$$|f(\mathbf{v})| = |f(\mathbf{v}+\mathbf{w})| \leqslant \|f\|^* \cdot \|\mathbf{v}+\mathbf{w}\|_1$$

and thus

$$|f(\mathbf{v})| \leqslant \|f\|^* \inf_{\mathbf{w}\in D_0}\|\mathbf{v}+\mathbf{w}\|_1. \tag{43}$$

Now the conclusion follows by Lemma 2. □

Notice that, (locally) both the RT-space and the MRT-space contain the space D_0. Hence, using the above result and the estimates in Lemma 1, we can now prove optimal $\mathbf{L}^2$-error estimates for both the projections Π_h and Φ_h.

Lemma 4. *Let π_h denote either Π_h or Φ_h. Then*

$$\|\mathbf{u} - \pi_h\mathbf{u}\|_0 \leqslant Ch|\mathbf{u}|_1, \quad \forall \mathbf{u} \in \mathbf{H}^1(\Omega), \tag{44}$$

Also, the following estimates are valid

$$(\nabla\cdot(\mathbf{u} - \Pi_h\mathbf{u}), w) = 0, \ \forall \mathbf{u} \in \mathbf{H}^1(\Omega), w \in W_h, \tag{45}$$

$$\|\nabla\cdot(\mathbf{u} - \Pi_h\mathbf{u})\|_0 \leqslant Ch|\nabla\cdot\mathbf{u}|_1, \ \textit{if } \mathbf{u} \textit{ satisfies } div\ \mathbf{u} \in H^1(\Omega). \tag{46}$$

Remark 3.2. Note that (45), (46) do not holds if Π_h is replace by the unmodified RT projection Φ_h.

Proof. To prove (44) we take $\pi_h = \Pi_h$. The case $\pi_h = \Phi_h$ is similar. Observe that

$$\|\mathbf{u} - \Pi_h \mathbf{u}\|_0^2 = \sum_{Q \in \mathcal{T}_h} \|\mathbf{u} - \Pi_Q \mathbf{u}\|_{0,Q}^2. \tag{47}$$

By (7), (8), we have

$$\begin{aligned} \|\mathbf{u} - \Pi_Q \mathbf{u}\|_{0,Q}^2 &= \int_Q (\mathbf{u} - \Pi_Q \mathbf{u}) \cdot (\mathbf{u} - \Pi_Q \mathbf{u})\, d\mathbf{x} = \int_{\hat{Q}} \frac{1}{J_Q} \left(\hat{\mathbf{u}} - \hat{\Pi}_{\hat{Q}} \hat{\mathbf{u}}\right)^T DF_Q^T DF_Q \left(\hat{\mathbf{u}} - \hat{\Pi}_{\hat{Q}} \hat{\mathbf{u}}\right) d\hat{\mathbf{x}} \leqslant c_1 \int_{\hat{Q}} |\hat{\mathbf{u}} - \hat{\Pi}_{\hat{Q}} \hat{\mathbf{u}}|^2\, d\hat{\mathbf{x}} \\ &= c_1 \|\hat{\mathbf{u}} - \hat{\Pi}_{\hat{Q}} 3 \hat{\mathbf{u}}\|_{0,\hat{Q}}^2 \leqslant C\left(\|\hat{u}_y\|_{0,\hat{Q}}^2 + \|\hat{v}_x\|_{0,\hat{Q}}^2 + \|\nabla \cdot \hat{\mathbf{u}}\|_{0,\hat{Q}}^2\right) \leqslant Ch^2\left(|\mathbf{u}|_{1,Q}^2 + \|\nabla \cdot \mathbf{u}\|_{0,Q}^2\right), \end{aligned} \tag{48}$$

where Theorem 3, (20) and (21) have been used. Summing over Q, we have (44).

The relation (45) follows from (15) and finally, the estimate (46) is direct from (45) and the property of L^2-projection. □

Now we show the boundedness of Π_h.

Lemma 5. *For any $\mathbf{v} \in \widetilde{\mathbf{H}}(Q)$, there exists a constant c which is independent of h such that*

$$\|\Pi_Q \mathbf{v}\|_{\mathbf{H}(div;Q)} \leqslant c_1 \|\mathbf{v}\|_{\mathbf{H}(div;Q)}. \tag{49}$$

Furthermore, if $\mathbf{v} \in \mathbf{H}(div;\Omega)$ and $\Pi_h \mathbf{v}$ is well defined, the following also holds.

$$\|\Pi_h \mathbf{v}\|_{\mathbf{H}(div;\Omega)} \leqslant c_1 \|\mathbf{v}\|_{\mathbf{H}(div;\Omega)}. \tag{50}$$

Proof. It suffices to show (49). The same argument as (48) shows that

$$\|\Pi_Q \mathbf{v}\|_{0,Q}^2 \leqslant c\|\hat{\Pi}_{\hat{Q}} \hat{\mathbf{v}}\|_{0,\hat{Q}}^2. \tag{51}$$

Observe that

$$\hat{\Pi}_{\hat{Q}} \hat{\mathbf{v}} = \hat{\Pi}_{\hat{Q}} \hat{\mathbf{v}} - \hat{\Phi}\hat{\mathbf{v}} + \hat{\Phi}\hat{\mathbf{v}} = \hat{\Pi}_{\hat{Q}} \hat{\Phi}\hat{\mathbf{v}} - \hat{\Phi}\hat{\mathbf{v}} + \hat{\Phi}\hat{\mathbf{v}}.$$

By direct computation, it is easy to see that

$$\|\hat{\Pi}_{\hat{Q}} \hat{\Phi}\hat{\mathbf{v}} - \hat{\Phi}\hat{\mathbf{v}}\|_{0,\hat{Q}} \leqslant c\|\nabla \cdot \hat{\mathbf{v}}\|_{0,\hat{Q}}. \tag{52}$$

Note that $\hat{\Phi}$ is the standard interpolation of Raviart–Thomas over the reference element $\hat{Q}$. Therefore we have

$$\|\hat{\Phi}\hat{\mathbf{v}}\|_{0,\hat{Q}} \leqslant c\|\hat{\mathbf{v}}\|_{H(\mathrm{div};\hat{Q})}. \tag{53}$$

By (52), (53) and scaling, we have

$$\|\hat{\Pi}_{\hat{Q}} \hat{\mathbf{v}}\|_{0,\hat{Q}} \leqslant c\|\mathbf{v}\|_{H(\mathrm{div};Q)}. \tag{54}$$

Therefore (51) and (54) show that

$$\|\Pi_Q \mathbf{v}\|_{0,Q} \leqslant c\|\mathbf{v}\|_{H(\mathrm{div};Q)}. \tag{55}$$

Now the following is an immediate consequence of (15):

$$\|\nabla \cdot \Pi_Q \mathbf{v}\|_{0,Q} \leqslant \|\nabla \cdot \mathbf{v}\|_{0,Q}. \tag{56}$$

Finally, (55) and (56) implies (49). □

3.2. Error estimates

To prove the existence and uniqueness, we need the inf–sup condition:

Lemma 6. *There exists a positive constant β_0, which is independent of h, such that for all $w_h \in W_h$*

$$\|w_h\|_0 \leqslant \beta_0 \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(w_h, \nabla \cdot \mathbf{v}_h)}{\|\mathbf{v}_h\|_{\mathbf{H}(div)}}. \tag{57}$$

Proof. This follows from a standard method using the boundedness of Π_h, see [6,18]. □

Now the existence and uniqueness follow easily from the standard theory of mixed method since div $\mathbf{V}_h \subset W_h$.

Theorem 7. *Let $(\mathbf{u}, p)$ be the solution pair of (3) and $(\mathbf{u}_h, p_h)$ be the solution pair of (4) using MRT-space. Also, let $(\tilde{\mathbf{u}}_h, \tilde{p}_h)$ be the solution pair of (4) using the unmodified RT-space. Then we have*

$$\|\mathbf{u}-\mathbf{u}_h\|_0 \leqslant Ch|\mathbf{u}|_1,$$
$$\|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_0 \leqslant Ch|\nabla\cdot\mathbf{u}|_1,$$
$$\|p-p_h\|_0 \leqslant Ch\|\mathbf{u}\|_1$$

and

$$\|\mathbf{u}-\tilde{\mathbf{u}}_h\|_0 \leqslant Ch|\mathbf{u}|_1,$$
$$\|p-\tilde{p}_h\|_0 \leqslant Ch\|\mathbf{u}\|_1.$$

Proof. Subtracting (3) from (4), we have

$$(\kappa^{-1}(\mathbf{u}-\mathbf{u}_h), \mathbf{v}_h) - (p-p_h, \nabla\cdot\mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{58}$$
$$(\nabla\cdot(\mathbf{u}-\mathbf{u}_h), q_h) = 0, \quad \forall q_h \in W_h. \tag{59}$$

Hence,

$$\begin{aligned} c\|\mathbf{u}-\mathbf{u}_h\|_0^2 &\leqslant (\kappa^{-1}(\mathbf{u}-\mathbf{u}_h), \mathbf{u}-\mathbf{u}_h) = (\kappa^{-1}(\mathbf{u}-\mathbf{u}_h), \Pi_h\mathbf{u}-\mathbf{u}_h) + (\kappa^{-1}(\mathbf{u}-\mathbf{u}_h), \mathbf{u}-\Pi_h\mathbf{u}) \\ &= (P_hp-p_h, \nabla\cdot(\Pi_h\mathbf{u}-\mathbf{u}_h)) + (\kappa^{-1}(\mathbf{u}-\mathbf{u}_h), \mathbf{u}-\Pi_h\mathbf{u}) = (\kappa^{-1}(\mathbf{u}-\mathbf{u}_h), \mathbf{u}-\Pi_h\mathbf{u}) \\ &\leqslant C\|\mathbf{u}-\mathbf{u}_h\|_0\|\mathbf{u}-\Pi_h\mathbf{u}\|_0, \end{aligned} \tag{60}$$

where c and C are independent of h and $\mathbf{u}$. Here, we note that (60) holds because $\nabla\cdot(\Pi_h\mathbf{u}-\mathbf{u}_h)$ is piecewise constant. Therefore, we have from (44)

$$\|\mathbf{u}-\mathbf{u}_h\|_0 \leqslant c\|\mathbf{u}-\Pi_h\mathbf{u}\|_0 \leqslant Ch|\mathbf{u}|_1. \tag{61}$$

Since $\nabla\cdot(\mathbf{u}-\mathbf{u}_h) = \nabla\cdot(\mathbf{u}-\Pi_h\mathbf{u})$, we have from (46)

$$\|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_0 \leqslant Ch|\nabla\cdot\mathbf{u}|_1. \tag{62}$$

Using the inf–sup condition (57), we have following

$$\|P_hp-p_h\|_0 \leqslant C\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{(P_hp-p_h, \nabla\cdot\mathbf{v}_h)}{\|\mathbf{v}_h\|_{\mathbf{H}(\mathrm{div})}} = C\sup_{\mathbf{v}_h\in\mathbf{V}_h}\frac{(\kappa^{-1}(\mathbf{u}-\mathbf{u}_h), \mathbf{v}_h)}{\|\mathbf{v}_h\|_{\mathbf{H}(\mathrm{div})}} \leqslant C\|\mathbf{u}-\mathbf{u}_h\|_0.$$

Hence we obtain

$$\|p-p_h\|_0 \leqslant \|p-P_hp\|_0 + \|P_hp-p_h\|_0 \leqslant Ch(\|p\|_1 + \|\mathbf{u}\|_1) \leqslant Ch\|\mathbf{u}\|_1.$$

The proof for unmodified case is slightly different. In this case, the equality (60) does not holds. However, if we modify L^2-projection P_h by

$$(P_h^M p, q_h) = (\hat{p}, \hat{q}_h), q_h \in W_h,$$

then (60) holds since

$$(p, \nabla\cdot\mathbf{v}_h) = (\hat{p}, \hat{\nabla}\cdot\hat{\mathbf{v}}_h) = (P_h^M p, \nabla\cdot\mathbf{v}_h)$$

with $\mathbf{v}_h = \Phi_h\mathbf{u} - \mathbf{u}_h$. The rest of the proof is the same as before since the approximation property (44) also holds for Φ_h. □

Remark 3.3. Incidentally, we proved an optimal $\mathbf{L}^2$-error estimate for the original RT space (of lowest order). This is an improvement over [18] in regularity which says

$$\|\mathbf{u}-\tilde{\mathbf{u}}_h\|_0 \leqslant Ch(|\mathbf{u}|_1 + h|\nabla\cdot\mathbf{u}|_1).$$

Arnold, Boffi and Falk [2] have obtained similar result for the RT-space with a different approach. They also obtained an optimal error estimate for the divergence, but at the expense of extra degrees of freedom.

Remark 3.4. The condition that $RT(\hat{Q})$ or $MRT(\hat{Q})$ contains the space D_0 turns out to be sufficient (see Lemma 2, 4 and Theorem 3.7) to show optimal error estimate of velocity in $\mathbf{L}^2$-norm (not the divergence), which is also discovered recently in [2] by a quite different approach.

Remark 3.5. Similar argument can be used to modify the BDM space of lowest order to show $O(h)$-error estimate for divergence. For example, one can modify four basis functions corresponding to the lower weight by the same way as in Section 3. We will not pursue the details here.

Remark 3.6. Our method can be generalized to compute the divergence of velocity accurately for three dimensional problems with distorted grid. But in [17], it is indicated that the standard Raviart–Thomas–Nedelec element of the lowest order does not reproduce a constant vector fields in this case. In our experiment (although not reported), the standard Raviart–Thomas–Nedelec element of the lowest order does not converge even in $\mathbf{L}^2$-norm. It is an interesting future topic to investigate.

4. Numerical experiments

To confirm the theoretical results established in Section 3, several numerical tests are carried out on the unit square $\Omega = (0,1)^2$. First we partition the unit square into uniform squares of size $h := 2^{-J}, J = 1,2,\ldots$ Then, we distort each element by pulling or pushing the vertices alternatively by a factor of α. For example, the quadrilateral in the corner has vertex $(0,0)$, $((1+\alpha)h,0)$, $((1-\alpha)h,h)$, and $(0,h)$ (see Fig. 2). Assuming that the error is of the form Ch^{β}, we report the value β at the end of each table. Tables 1–3 show the results of MRT for $\alpha = 0, 0.4$ and 0.8, while Tables 1, 4, 5 show the results of RT for $\alpha = 0, 0.4$

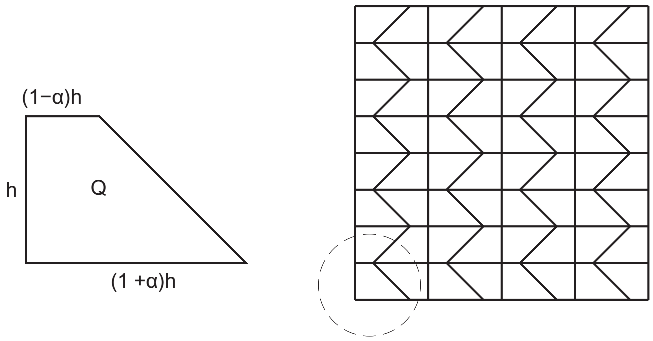

Fig. 2. Partition $\mathcal{T}_h$ on Ω.

Table 1
RT and MRT: $\alpha = 0$.

h	$\|p-p_h\|_0$	$\|P_hp-p_h\|_0$	$\|\mathrm{div}(\mathbf{u}-\mathbf{u}_h)\|_0$	$\|\mathbf{u}-\mathbf{u}_h\|_0$
1/4	3.121(e−2)	2.736(e−3)	5.035(e−1)	1.071(e−1)
1/8	1.640(e−2)	7.400(e−4)	2.610(e−1)	5.538(e−2)
1/16	8.301(e−3)	1.886(e−4)	1.317(e−1)	2.792(e−2)
1/32	4.163(e−3)	4.739(e−5)	6.602(e−2)	1.399(e−2)
1/64	2.083(e−3)	1.188(e−5)	3.304(e−2)	6.998(e−3)
β	0.999	1.996	0.999	0.999

Table 2
MRT: $\alpha = 0.4$.

h	$\|p-p_h\|_0$	$\|P_hp-p_h\|_0$	$\|\mathrm{div}(\mathbf{u}-\mathbf{u}_h)\|_0$	$\|\mathbf{u}-\mathbf{u}_h\|_0$
1/4	3.240(e−2)	3.516(e−3)	5.376(e−1)	1.146(e−1)
1/8	1.730(e−2)	1.098(e−3)	2.839(e−2)	6.487(e−2)
1/16	8.806(e−3)	3.096(e−4)	1.422(e−2)	3.397(e−2)
1/32	4.422(e−3)	8.209(e−5)	7.077(e−2)	1.732(e−2)
1/64	2.214(e−3)	2.114(e−5)	3.526(e−2)	8.738(e−3)
β	0.998	1.957	1.005	0.987

Table 3
MRT: $\alpha = 0.8$.

h	$\|p-p_h\|_0$	$\|P_hp-p_h\|_0$	$\|\mathrm{div}(\mathbf{u}-\mathbf{u}_h)\|_0$	$\|\mathbf{u}-\mathbf{u}_h\|_0$
1/4	3.524(e−2)	6.041(e−3)	6.031(e−1)	1.394(e−1)
1/8	1.952(e−2)	2.217(e−3)	3.240(e−1)	8.622(e−2)
1/16	1.001(e−2)	6.701(e−4)	1.625(e−1)	4.657(e−2)
1/32	5.035(e−3)	1.820(e−5)	8.078(e−2)	2.400(e−2)
1/64	2.521(e−3)	4.731(e−6)	4.021(e−2)	1.216(e−2)
β	0.998	1.944	1.006	0.981

Table 4
RT: $\alpha = 0.4$.

h	$\|p - p_h\|_0$	$\|\mathrm{div}(\mathbf{u} - \mathbf{u}_h)\|_0$	$\|\mathbf{u} - \mathbf{u}_h\|_0$
1/4	3.300(e−2)	8.599(e−1)	1.219(e−1)
1/8	1.757(e−2)	6.953(e−1)	6.858(e−2)
1/16	8.930(e−3)	6.495(e−1)	3.600(e−2)
1/32	4.483(e−3)	6.378(e−1)	1.840(e−2)
1/64	2.244(e−3)	6.349(e−1)	9.298(e−3)
β	0.998	0.007	0.984

Table 5
RT: $\alpha = 0.8$.

h	$\|p - p_h\|_0$	$\|\mathrm{div}(\mathbf{u} - \mathbf{u}_h)\|_0$	$\|\mathbf{u} - \mathbf{u}_h\|_0$
1/4	3.860(e−2)	1.617(e+0)	1.729(e−1)
1/8	2.090(e−2)	1.508(e+0)	1.035(e−1)
1/16	1.063(e−3)	1.483(e+0)	5.595(e−2)
1/32	5.333(e−3)	1.477(e+0)	2.898(e−2)
1/64	2.668(e−3)	1.475(e+0)	1.473(e−2)
β	0.999	0.002	0.976

and 0.8. Tables 1–3 show the $\mathcal{O}(h)$ for all the variables; pressure and divergence and velocity. We observe superconvergence for the pressure. Note that the results for RT and MRT are the same as is seen in Table 1. This is evident because MRT and RT are exactly the same for rectangular meshes. Tables 4 and 5 show that the usual RT-element does not converge in the divergence norm.

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