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Research Article

Enriched P_1-Conforming Methods for Elliptic Interface Problems with Implicit Jump Conditions

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We develop a numerical method for elliptic interface problems with implicit jumps. To handle the discontinuity, we enrich usual P_1-conforming finite element space by adding extra degrees of freedom on one side of the interface. Next, we define a new bilinear form, which incorporates the implicit jump conditions. We show that the bilinear form is coercive and bounded if the penalty term is sufficiently large. We prove the optimal error estimates in both energy-like norm and L^2-norm. We provide numerical experiments. We observe that our scheme converges with optimal rates, which coincides with our error analysis.

1. Introduction

Interface problems arise in various disciplines including mechanical, material, and medical image and petroleum engineering [1–9]. There are several difficulties to solve for the governing equation of such problems.

Firstly, partial differential equations may have different coefficients along the interface due to the change of material properties. When the geometry of interface is complex, one needs to generate grids that align with the interfaces. Once a fitted grid is generated, one uses finite element method (FEM) or finite volume method (FVM) based on this grid. Secondly, the problem may have nonhomogeneous jump conditions along the interface. When the jumps along the interface are known explicitly, (say $[u] = g_1$, $[\partial u/\partial \mathbf{n}] = g_2$, with known g_1 and g_2), these jumps may be handled effectively by discontinuous Galerkin (DG) [10, 11] by incorporating jumps into the bilinear form with proper penalty terms. For example, an effective DG scheme was developed to describe discontinuous phenomena arising from porous media with discontinuous capillary pressure [12]. The interface problems with known jumps can be solved with immersed interface methods [13–15] or discontinuous bubble-immersed finite element methods [16].

However, when the jumps are implicit along the interface problems, numerically solving the governing equations becomes more challenging. Let us consider some problems with interface conditions, where the jumps of primary variables are related to the normal fluxes. Firstly, these problems arise in the medical imaging of cancer cells using MREIT [3, 4] or electrochemotherapy [17], where the jumps of an electric voltage across the cell membrane appear. Next, an elastic body has spring-type jumps that are related to stress [18, 19]. The heat in the material interface may have implicit jump conditions along the interface [20, 21]. Also, a generalized jump condition for Laplace equation or Helmholtz equation has been considered in [22–24].

The first attempt to solve the elliptic interface problems having implicit jump conditions seems to be introduced in [25], where the iterative method was used. Recently, some XFEM-based nonfitted methods were developed in [26, 27] for the elliptic problems and elasticity problems, respectively, where the extra degrees of freedom are introduced on elements cut by the interface. On the other hand, an immersed finite element type method was developed in [28].

In this work, we introduce a new numerical method to solve elliptic interface problems, where the jumps are related to the normal fluxes and some known functions. A main idea of our work is to include the jump conditions implicitly on the bilinear form so that the numerical solutions for the weak problems satisfy the implicit jump conditions. We enrich the usual P_1 FEM space near the interface. We show that our bilinear form is coercive and bounded and prove the optimal error estimates. In numerical section, we provide several numerical examples supporting our analysis.

Let Ω be a convex domain in $\mathbb{R}^n$ $(n = 2, 3)$, which is divided into Ω^+ and Ω^- by a C^2 closed interface Γ. The governing equations on Ω are given by

$$-\nabla \cdot \beta \nabla u = f, \quad \text{in } \Omega := \Omega^- \cup \Omega^+, \tag{1}$$

$$u = g_0, \quad \text{on } \partial\Omega, \tag{2}$$

$$[u]_\Gamma = \alpha \frac{\partial u^+}{\partial \mathbf{n}^+} + g_1, \quad \text{on } \Gamma, \tag{3}$$

$$\left[\beta \frac{\partial u}{\partial \mathbf{n}_\Gamma}\right]_\Gamma = g_2, \quad \text{on } \Gamma, \tag{4}$$

where $f \in L^2(\Omega)$ and $g_0 \in H^{3/2}(\partial\Omega)$, $g_1 \in H^{3/2}(\Gamma)$, $g_2 \in H^{1/2}(\Gamma)$, and β is a positive piecewise constant; that is, $\beta = \beta^+$ in Ω^+ and $\beta = \beta^-$ in Ω^-, where β^+ and β^- are some positive constants. Here, $\mathbf{n}^s$ is the outer unit normal vector to Ω^s $(s = -, +)$ and $[\cdot]_\Gamma$ is the jump along the interface; that is, $[u]_\Gamma = u|_{\Omega^-} - u|_{\Omega^+}$. Also, we define $\mathbf{n}_\Gamma$ to be an outer normal vector to Ω^-. The jump of normal derivatives of u is defined as

$$\left[\beta \frac{\partial u}{\partial \mathbf{n}_\Gamma}\right]_\Gamma = \beta^- \frac{\partial u|_{\Omega^-}}{\partial \mathbf{n}_\Gamma} - \beta^+ \frac{\partial u|_{\Omega^+}}{\partial \mathbf{n}_\Gamma}. \tag{5}$$

We assume that α is a positive constant.

We introduce some notations. Let O be any domain and let $H^m(O)$, $m = 1, 2$, be a usual Sobolev space with norm $\|u\|_{m,O}$. We define $H_0^1(O)$ as the set of functions in $H^1(O)$ with vanishing trace on ∂O. We define the subspaces of $H^m(O)$,

$$\widetilde{H}^m(O) := H^m(\Omega^- \cap O) \cap H^m(\Omega^+ \cap O), \tag{6}$$

equipped with the (semi)norms:

$$\begin{aligned} |u|_{\widetilde{H}^m(O)} &:= |u|_{H^m(\Omega^- \cap O)} + |u|_{H^m(\Omega^+ \cap O)}, \\ \|u\|_{\widetilde{H}^m(O)} &:= \|u\|_{H^m(\Omega^- \cap O)} + \|u\|_{H^m(\Omega^+ \cap O)}. \end{aligned} \tag{7}$$

Finally, we define subspace of $\widetilde{H}^1(\Omega)$:

$$\widetilde{H}_0^1(\Omega) := \left\{u : u \in \widetilde{H}^1(O) \mid u|_{\partial\Omega} = 0\right\}. \tag{8}$$

We state a theorem regarding the existence and regularity of the problem [29, 30].

Theorem 1. *Problem (1)–(4) has a unique solution $u \in \widetilde{H}^2(\Omega)$ such that, for some constant $C > 0$,*

$$\begin{aligned} \|u\|_{\widetilde{H}^2(\Omega)} \le C \Big(&\|f\|_{L^2(\Omega)} + \|g_0\|_{H^{3/2}(\partial\Omega)} + \|g_1\|_{H^{3/2}(\Gamma)} \\ &+ \|g_2\|_{H^{1/2}(\Gamma)} \Big). \end{aligned} \tag{9}$$

The rest of the paper is organized as follows. In Section 2, we derive the variational forms for the problems with implicit jump conditions. We introduce new numerical methods in Section 3 and in Section 4 we prove the error estimates. In Section 5, we give numerical results that support our analysis. The conclusion follows in Section 6.

2. Variational Form

In this section, we derive a variational formulation of the model problem. Without loss of generality, we may assume that $g_0 = 0$. First, we multiply $v \in \widetilde{H}_0^1(\Omega)$ to (1) and apply integration by parts on each subdomain to get

$$\int_{\Omega^s} \beta \nabla u \cdot \nabla v \, d\mathbf{x} - \int_{\partial\Omega^s} \beta^s \frac{\partial u^s}{\partial \mathbf{n}^s} v \, ds = \int_{\Omega^s} f v \, d\mathbf{x}, \quad s = +, -. \tag{10}$$

By summation, we have

$$\begin{aligned} \sum_{s=+,-} \int_{\Omega^s} \beta \nabla u \cdot \nabla v \, d\mathbf{x} - \sum_{s=+,-} \int_{\partial\Omega^s} \beta^s \frac{\partial u^s}{\partial \mathbf{n}^s} v \, ds \\ = \sum_{s=+,-} \int_{\Omega^s} f v \, d\mathbf{x}. \end{aligned} \tag{11}$$

Using the jump conditions (3) and (4), we see the second terms become

$$\begin{aligned} -\sum_{s=+,-} \int_{\partial\Omega^s} \beta^s \frac{\partial u^s}{\partial \mathbf{n}^s} v \, ds &= -\int_{\partial\Omega^-} \beta^- \frac{\partial u^-}{\partial \mathbf{n}_\Gamma} v^- ds \\ &\quad + \int_{\partial\Omega^+} \beta^+ \frac{\partial u^+}{\partial \mathbf{n}_\Gamma} v^+ ds, \\ &= -\int_\Gamma \beta^+ \frac{\partial u^+}{\partial \mathbf{n}_\Gamma} [v]_\Gamma \, ds \\ &\quad - \int_\Gamma g_2 v^- ds, \quad \text{by (4)} \\ &= -\int_\Gamma \frac{\beta^+}{\alpha} \left(g_1 - [u]_\Gamma\right) [v]_\Gamma \, ds \\ &\quad - \int_\Gamma g_2 v^- ds, \quad \text{by (3)} \\ &= \frac{\beta^+}{\alpha} \int_\Gamma [u]_\Gamma [v]_\Gamma \, ds \\ &\quad - \frac{\beta^+}{\alpha} \int_\Gamma g_1 [v]_\Gamma \, ds \\ &\quad - \int_\Gamma g_2 v^- ds. \end{aligned} \tag{12}$$

We define a bilinear form and a functional on $\widetilde{H^1}(\Omega)$:

$$\begin{aligned} a(u, v) := \int_{\Omega^-} \beta \nabla u \cdot \nabla v \, d\mathbf{x} + \int_{\Omega^+} \beta \nabla u \cdot \nabla v \, d\mathbf{x} \\ + \frac{\beta^+}{\alpha} \left\langle [u]_\Gamma, [v]_\Gamma \right\rangle_\Gamma, \end{aligned} \tag{13}$$

$$\widetilde{F}(v) := (f, v) + \frac{\beta^+}{\alpha} \left\langle g_1, [v]_\Gamma \right\rangle_\Gamma + \left\langle g_2, v^- \right\rangle_\Gamma, \tag{14}$$

where $(\cdot,\cdot)$ denotes $L^2(\Omega)$ inner product on Ω and $\langle\cdot,\cdot\rangle_\Gamma$ denotes the $L^2(\Gamma)$ inner product. By (12) and (13), we have the weak problem: find $u \in \widetilde{H}^2(\Omega) \cap \widetilde{H}^1_0(\Omega)$ satisfying

$$a(u,v) = \widetilde{F}(v), \tag{15}$$

for all $v \in \widetilde{H}^1_0(\Omega)$.

Now let us show that the weak problem (15) is equivalent to (1)–(4). Suppose that $u \in \widetilde{H}^2(\Omega) \cap \widetilde{H}^1_0(\Omega)$ satisfies (15). First, let v be any function $H^1_0(\Omega^-)$ (or $v \in H^1_0(\Omega^+)$). Then, we have

$$\begin{gathered}\int_{\Omega^-} \beta\nabla u\cdot\nabla v\,dx = \int_{\Omega^-} f v\,d\mathbf{x},\\ \left(\text{or, } \int_{\Omega^+} \beta\nabla u\cdot\nabla v\,d\mathbf{x} = \int_{\Omega^+} f v\,d\mathbf{x}\right).\end{gathered} \tag{16}$$

By integration by parts, we see that u satisfies

$$-\nabla\cdot\beta\nabla u = f, \quad \text{on } \Omega^- \cup \Omega^+. \tag{17}$$

Now, assume that $v \in H^1_0(\Omega)$ in (15). By Green's theorem and the fact that $[v]_\Gamma = 0$, the left side of (15) becomes

$$\begin{aligned}&\sum_{s=+,-}\left(\int_{\partial\Omega^s} \beta^s \frac{\partial u^s}{\partial \mathbf{n}^s} v\,ds - \int_{\Omega^s} \nabla\cdot(\beta\nabla u)\, v\,d\mathbf{x}\right)\\ &\quad = \int_\Gamma \left[\beta\frac{\partial u}{\partial \mathbf{n}_\Gamma}\right]_\Gamma v\,ds + \int_\Omega f v\,d\mathbf{x}.\end{aligned} \tag{18}$$

Comparing with the right side of (15), we have

$$\left[\beta\frac{\partial u}{\partial \mathbf{n}_\Gamma}\right]_\Gamma = g_2. \tag{19}$$

Finally, assume that $v \in \widetilde{H}^1_0(\Omega)$ in (15). By integration by parts, the left hand side of (15) becomes

$$\begin{aligned}&\sum_{s=+,-}\left(\int_{\partial\Omega^s} \beta^s \frac{\partial u^s}{\partial \mathbf{n}^s} v\,ds - \int_{\Omega^s} \nabla\cdot(\beta\nabla u)\, v\,dx\right)\\ &\quad + \frac{\beta^+}{\alpha}\int_\Gamma [u]_\Gamma [v]_\Gamma\,ds\\ &= \int_\Gamma \beta^+ \frac{\partial u^+}{\partial \mathbf{n}_\Gamma}[v]_\Gamma\,ds + \int_\Gamma g_2 v^- + \frac{\beta^+}{\alpha}\int_\Gamma [u]_\Gamma [v]_\Gamma\,ds\\ &\quad + \int_\Omega f v\,d\mathbf{x}.\end{aligned} \tag{20}$$

Comparing with the right side of (15), we see that u satisfies

$$[u]_\Gamma = \alpha\frac{\partial u^+}{\partial \mathbf{n}^+} + g_1, \quad \text{on } \Gamma. \tag{21}$$

3. Numerical Methods

In this section, we develop a numerical method for (1)–(4). Our method is obtained by adding extra degrees of freedom to P_1-conforming space on one side of the interface. For simplicity, we assume that $\Omega \subset \mathbb{R}^2$. However, similar constructions are possible for the case of $\Omega \subset \mathbb{R}^3$ as well.

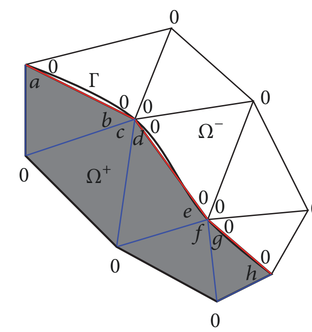

Figure 1: The support and the degrees of freedom of a function ϕ_h in $B_h(\Omega)$.

Let $\mathcal{T}_h$ be a given regular triangulation of Ω fitted with the interface. We let $\mathcal{T}_h^+$ and $\mathcal{T}_h^-$ be set of elements in $\mathcal{T}_h$ which belong to Ω^+ and Ω^-, respectively. We let $S_h(\Omega)$ be the usual P_1-conforming space; that is, any function in $S_h(\Omega)$ is continuous and piecewise linear and is vanishing on the boundary. We use notation $S_h(T)$ for the set of linear functions on T.

We let $\mathcal{I}_h$ be the set of all neighboring elements of interface Γ in $\mathcal{T}_h^+$; that is, $T \in \mathcal{T}_h$ belongs to $\mathcal{I}_h$ if and only if $T \in \mathcal{T}_h^+$ and at least one node of T is located on Γ. We let $B_h(T)$ be the space of functions in $S_h(T)$ vanishing on nodes not lying at the interface. For example, suppose that T has three nodes A_1, A_2, and A_3, where A_1 and A_2 are located on Γ. Then, a function in $B_h(T)$ is linear on T vanishing at A_3. In this case, $B_h(T)$ has dimension two. On the other hand, if T have only one node located on Γ, the dimension of $B_h(T)$ is one. A function in $B_h(T)$ is extended to Ω as follows:

$$B_h(\Omega) = \begin{cases}\phi|_T = 0, & \text{on } T \in \mathcal{T}_h - \mathcal{I}_h,\\ \phi|_T \in B_h(T), & \text{on } T \in \mathcal{I}_h.\end{cases} \tag{22}$$

For example, suppose that there are seven elements aligning with interface (see Figure 1). Then function ϕ_h in $B_h(\Omega)$ has a support on grey region. Moreover, ϕ_h has vanishing values on outside nodes on Ω^+. Thus, ϕ_h has seven degrees of freedom, that is, a, b, c, d, e, f, g, h.

We decompose u as

$$u \coloneqq u^0 + u^*, \tag{23}$$

where u^0 belongs to $H^1_0(\Omega)$ and u^* belongs to $\widetilde{H}^1_0(\Omega)$. We approximate u^0 from $S_h(\Omega)$ and approximate u^* from $B_h(\Omega)$. Thus, $u = u^0 + u^*$ is approximated in $\widetilde{S}_h(\Omega) \coloneqq S_h(\Omega) + B_h(\Omega)$.

Now, we discretize (1)–(4). Let $\mathcal{E}_h$ be edges of elements in $\mathcal{T}_h$ and let us define subspaces of $\mathcal{E}_h$ (see Figure 1).

(i) $\mathcal{E}_h^I$ is the set of edges of $\mathcal{E}_h$ whose two endpoints are located on Γ.

(ii) $\mathcal{E}_h^N$ is the set of edges of $\mathcal{E}_h$ whose one endpoint is located on Γ and the other is located in the interior Ω^-.

(iii) $\mathscr{E}_h^O := \mathscr{E}_h - \mathscr{E}_h^I - \mathscr{E}_h^N$.

We note that $\mathscr{E}_h = \mathscr{E}_h^I \cup \mathscr{E}_h^N \cup \mathscr{E}_h^O$. For all edges e in $\mathscr{E}_h$, we fix a normal vector $\mathbf{n}_e$ once and for all. We define jumps and averages across the edges:

$$\begin{aligned} [u]_e &:= u|_{T_e^1} - u|_{T_e^2}, \\ \{u\}_e &:= 0.5u|_{T_e^1} + 0.5u|_{T_e^2}, \end{aligned} \tag{24}$$

where T_e^1 and T_e^2 are two neighboring elements of e.

We multiply $v_h \in \widetilde{S}_h(\Omega)$ to (1) and use integration by parts to obtain the following:

$$\sum_{T\in\mathscr{T}_h} \int_T \beta\nabla u \cdot \nabla v_h \, d\mathbf{x} + \sum_{e\in\mathscr{E}_h} D_e = \sum_{T\in\mathscr{T}_h} (f, v_h)_T, \tag{25}$$

where

$$\begin{aligned} D_e := -\int_e \Big((\beta\nabla u)|_{T_e^1} \cdot \mathbf{n}_e \, v_h|_{T_e^1} - (\beta\nabla u)|_{T_e^2} \\ \cdot \mathbf{n}_e \, v_h|_{T_e^2} \Big) ds. \end{aligned} \tag{26}$$

Let us classify D_e into three categories. Firstly, if $e \in \mathscr{E}_h^I$, then using the similar method used in deriving (12), we have

$$\begin{aligned} D_e = \frac{\beta^+}{\alpha} \int_e [u]_e [v_h]_e \, ds - \frac{\beta^+}{\alpha} \int_e g_1 [v_h]_e \, ds \\ - \int_e g_2 v_h^- ds. \end{aligned} \tag{27}$$

Secondly, if $e \in \mathscr{E}_h^N$, then by using the identity

$$\begin{aligned} a^- b^- - a^+ b^+ = (a^- + a^+)\left(\frac{b^- - b^+}{2}\right) \\ + \left(\frac{a^- - a^+}{2}\right)(b^- + b^+) \end{aligned} \tag{28}$$

and the fact that

$$[\beta\nabla u \cdot \mathbf{n}_e]_e = 0, \tag{29}$$

we have

$$\begin{aligned} D_e &= -\int_e \{\beta\nabla u \cdot \mathbf{n}_e\}_e [v_h]_e \, ds \\ &\quad - \int_e [\beta\nabla u \cdot \mathbf{n}_e]_e \{v_h\}_e \, ds, \\ &= -\int_e \{\beta\nabla u \cdot \mathbf{n}_e\}_e [v_h]_e \, ds. \end{aligned} \tag{30}$$

Finally, if $e \in \mathscr{E}_h^O$, D_e vanishes, since both $\beta\nabla u$ and v_h are continuous across e. Thus, (25) becomes

$$\begin{aligned} &\sum_{T\in\mathscr{T}_h} \int_T \beta\nabla u \cdot \nabla v_h d\mathbf{x} - \sum_{e\in\mathscr{E}_h^N} \int_e \{\beta\nabla u \cdot \mathbf{n}_e\}_e [v_h]_e \, ds \\ &\quad + \sum_{e\in\mathscr{E}_h^I} \frac{\beta^+}{\alpha} \int_e [u]_e [v_h]_e \, ds \\ &= \sum_{T\in\mathscr{T}_h} (f, v_h)_T + \sum_{e\in\mathscr{E}_h^I} \frac{\beta^+}{\alpha} \int_e g_1 [v_h]_e \, ds \\ &\quad + \sum_{e\in\mathscr{E}_h^I} \int_e g_2 v_h^- ds. \end{aligned} \tag{31}$$

Now we propose our method based on enriched P_1-conforming space: find $u_h \in \widetilde{S}_h(\Omega)$ satisfying

$$a_h^\epsilon(u_h, v_h) = \widetilde{F}_h(v_h), \tag{32}$$

for all $v_h \in \widetilde{S}_h(\Omega)$, where

$$\begin{aligned} a_h^\epsilon(u, v) &:= \sum_{T\in\mathscr{T}_h} \int_T \beta\nabla u \cdot \nabla v \, d\mathbf{x} \\ &\quad - \sum_{e\in\mathscr{E}_h^N} \int_e \{\beta\nabla u \cdot \mathbf{n}_e\}_e [v]_e \, ds \\ &\quad + \epsilon \sum_{e\in\mathscr{E}_h^N} \int_e \{\beta\nabla v \cdot \mathbf{n}_e\}_e [u]_e \, ds \\ &\quad + \sum_{e\in\mathscr{E}_h^N} \int_e \frac{\sigma}{|e|} [u]_e [v]_e \, ds \\ &\quad + \sum_{e\in\mathscr{E}_h^I} \frac{\beta^+}{\alpha} \int_e [u]_e [v]_e \, ds, \end{aligned} \tag{33}$$

$$\begin{aligned} \widetilde{F}_h(v_h) &= (f, v_h) + \sum_{e\in\mathscr{E}_h^I} \frac{\beta^+}{\alpha} \int_e g_1 [v_h]_e \, ds \\ &\quad + \sum_{e\in\mathscr{E}_h^I} \int_e g_2 v_h^- ds. \end{aligned} \tag{34}$$

In (33), the parameter σ is positive and the parameter ϵ is 1, 0, or -1, which is motivated by NIPG, IIPG, and SIPG of DG scheme [11].

We show that our scheme is consistent.

Lemma 2. *Suppose that u is the solution of (1)–(4). Then, for all $v_h \in \widetilde{S}_h(\Omega)$, the following holds:*

$$a_h^\epsilon(u, v_h) = \widetilde{F}_h(v_h). \tag{35}$$

In other words, we have

$$a_h^\epsilon(u - u_h, v_h) = 0, \tag{36}$$

for all $v_h \in \widetilde{S}_h(\Omega)$.

Proof. Since $[u]_e = 0$ for all $e \in \mathscr{E}_h^N$, we have

$$a_h^\epsilon(u, v_h) := \sum_{T\in\mathscr{T}_h} \int_T \beta\nabla u \cdot \nabla v_h d\mathbf{x} - \sum_{e\in\mathscr{E}_h^N} \int_e \{\beta\nabla u \cdot \mathbf{n}_e\}_e [v_h]_e \, ds + \sum_{e\in\mathscr{E}_h^I} \frac{\beta^+}{\alpha} \int_e [u]_e [v_h]_e \, ds, \tag{37}$$

for all $v_h \in \widetilde{S}_h(\Omega)$. By (31), we have the conclusion. □

4. Error Estimates

We define energy-like norm $|||\cdot|||_h$ on $H_h(\Omega) := \widetilde{H}^2(\Omega)+\widetilde{S}_h(\Omega)$.

$$|||\phi|||^2 = \sum_{T\in\mathscr{T}_h} \int_T \beta\nabla\phi \cdot \nabla\phi \, d\mathbf{x} + \sum_{e\in\mathscr{E}_h^N} \frac{1}{|e|} \int_e [\phi]_e^2 \, ds + \sum_{e\in\mathscr{E}_h^I} \int_e [\phi]_e^2 \, ds. \tag{38}$$

Let $\gamma_0 : H^1(T) \to H^1(\partial T)$ be the usual trace operator. Then we have the following theorem [31, 32].

Lemma 3 (trace theorem). *There exists a constant C such that*

$$\|\gamma_0 v\|_{L^2(\partial T)} \le C\left(h^{-1/2}\|v\|_{L^2(T)} + h^{1/2}\|\nabla v\|_{L^2(T)}\right), \tag{39}$$

for all $v \in H^1(T)$.

We define a local interpolation operator $I_h : u \in H^2(T) \to I_h u \in \widetilde{S}_h(T)$ by

$$(I_h u)(X_i) = u(X_i), \quad i = 1, 2, 3, \tag{40}$$

where X_i's are nodes of T. The operator I_h is extended to $u \in \widetilde{H}^2(\Omega)$ by $(I_h u)|_T = I_h(u|_T)$ for each element T. We note that if e belongs to $\mathscr{E}_h^I$, $I_h u$ is discontinuous across e; that is, $[I_h u]_e \ne 0$. However, $I_h u$ is continuous on each subdomain Ω^- and Ω^+. Then, by the standard interpolation theory, there exists a constant C such that

$$\begin{aligned} &\sum_{T\in\mathscr{T}_h^+} \|u - I_h u\|_{0,T} + h\sum_{T\in\mathscr{T}_h^+} \|\nabla(u - I_h u)\|_{0,T} \le Ch^2\|u\|_{\widetilde{H}^2(\Omega^+)}, \\ &\sum_{T\in\mathscr{T}_h^-} \|u - I_h u\|_{0,T} + h\sum_{T\in\mathscr{T}_h^-} \|\nabla(u - I_h u)\|_{0,T} \le Ch^2\|u\|_{\widetilde{H}^2(\Omega^-)}. \end{aligned} \tag{41}$$

From this, we can obtain interpolation estimate in $|||\cdot|||$ norm.

Corollary 4. *There exists a constant C_I such that, for all $u \in \widetilde{H}^2(\Omega)$,*

$$|||u - I_h u||| \le C_I h\|u\|_{\widetilde{H}^2(\Omega)}. \tag{42}$$

Proof. Since $u - u_h$ is continuous across $e \in \mathscr{E}_h^N$, we have

$$\begin{aligned} |||u - I_h u|||^2 &= \sum_{T\in\mathscr{T}_h} \left\|\beta^{1/2}\nabla(u - I_h u)\right\|_{0,T}^2 + \sum_{e\in\mathscr{E}_h^I} \int_e [u - I_h u]^2 \, ds \\ &\le Ch^2\|u\|_{\widetilde{H}^2(\Omega)}^2 + \sum_{e\in\mathscr{E}_h^I} \int_e [u - I_h u]^2 \, ds. \end{aligned} \tag{43}$$

It suffices to show that

$$\sum_{e\in\mathscr{E}_h^I} \int_e [u - I_h u]^2 \, ds \le Ch^2\|u\|_{\widetilde{H}^2(\Omega)}, \tag{44}$$

for some constant $C > 0$. Let $\phi = u - I_h u$. By the trace inequality (41), we have

$$\begin{aligned} \sum_{e\in\mathscr{E}_h^I} \int_e [\phi]^2 \, ds &\le \sum_{e\in\mathscr{E}_h^I} \sum_{i=1,2} \left\|\phi|_{T_e^i}\right\|_{0,e}^2 \\ &\le C\sum_{e\in\mathscr{E}_h^I} \sum_{i=1,2} \left(h^{-1}\|\phi\|_{0,T_e^i}^2 + h^1\|\nabla\phi\|_{0,T_e^i}^2\right) \\ &\le \sum_{e\in\mathscr{E}_h^I} \sum_{i=1,2} Ch^3\|u\|_{H^2(T_e^i)}^2 \le Ch^2\sum_{T\in\mathscr{T}_h} \|u\|_{H^2(T)}^2. \end{aligned} \tag{45}$$

□

We have the following coercivity property.

Theorem 5. *If we choose σ so that*

$$\sigma > \frac{3(1-\epsilon)^2 C_1^2}{2}, \tag{46}$$

then the following holds:

$$a_h^\epsilon(v_h, v_h) \ge C_1|||v_h|||^2, \tag{47}$$

for all $v_h \in \widetilde{S}_h(\Omega)$, where

$$C_1 = \min\left\{\frac{1}{2}, \sigma, \frac{\beta^+}{\alpha}\right\}. \tag{48}$$

Proof. If $\epsilon = 1$, then by definition of $a_h(\cdot,\cdot)$ and $|||\cdot|||$ norm, we have

$$a_h^\epsilon(v_h, v_h) \ge \min\left\{1, \sigma, \frac{\beta^+}{\alpha}\right\}\|v_h\|_h^2. \tag{49}$$

Now, suppose that $\epsilon = -1$ or 0. We have

$$\begin{aligned} a_h^\epsilon(v_h, v_h) &= \sum_{T\in\mathscr{T}_h} \left\|\beta^{1/2}\nabla v_h\right\|_{0,T}^2 - (1-\epsilon)\sum_{e\in\mathscr{E}_h^N} \int_e \{\beta\nabla v_h \cdot \mathbf{n}_e\}_e [v_h]_e \, ds \\ &\quad + \sum_{e\in\mathscr{E}_h^N} \frac{\sigma}{|e|}\left\|[v_h]_e\right\|_{0,e}^2 + \sum_{e\in\mathscr{E}_h^I} \frac{\beta^+}{\alpha}\left\|[v_h]_e\right\|_{0,e}^2 \\ &:= A_1 + A_2 + A_3 + A_4. \end{aligned} \tag{50}$$

We bound A_2 by Cauchy-Schwarz inequality, (39), and Young's inequality.

$$\begin{aligned}
&(1-\epsilon)\sum_{e\in\mathscr{E}_h^N}\int_e\{\beta\nabla v_h\cdot\mathbf{n}_e\}_e[v_h]_e\,ds \le (1-\epsilon)\\
&\cdot\sum_{e\in\mathscr{E}_h^N}\left\|\{\beta^{1/2}v_h\}_e\right\|_{0,e}\left\|[v_h]_e\right\|_{0,e} \le \sum_{e\in\mathscr{E}_h^N}(1-\epsilon)\\
&\cdot C_1h^{-1/2}\left(\frac{1}{2}\left\|\beta^{1/2}\nabla v_h\right\|_{0,T_e^1}+\frac{1}{2}\left\|\beta^{1/2}\nabla v_h\right\|_{0,T_e^2}\right)\\
&\cdot\left\|[v_h]_e\right\|_{0,e} \le (1-\epsilon)\\
&\cdot C_1h^{-1/2}\left(\sum_{e\in\mathscr{E}_h^N}\left\|\beta^{1/2}\nabla v_h\right\|_{0,T_e^1}^2\right)^{1/2}\\
&\cdot\left(\sum_{e\in\mathscr{E}_h^N}\left\|[v_h]_e\right\|_{0,e}^2\right)^{1/2} \le (1-\epsilon)\\
&\cdot C_1h^{-1/2}\left(3\sum_{T\in\mathscr{T}_h}\left\|\beta^{1/2}\nabla v_h\right\|_{0,T}^2\right)^{1/2}\\
&\cdot\left(\sum_{e\in\mathscr{E}_h^N}\left\|[v_h]_e\right\|_{0,e}^2\right)^{1/2} \le \frac{\epsilon_1}{2}\left(\sum_{T\in\mathscr{T}_h}\left\|\beta^{1/2}\nabla v_h\right\|_{0,T}^2\right)\\
&+\frac{3(1-\epsilon)^2C_1^2}{2\epsilon_1}\left(\sum_{e\in\mathscr{E}_h^N}\frac{1}{|e|}\left\|[v_h]_e\right\|_{0,e}^2\right),
\end{aligned}\tag{51}$$

for all $\epsilon_1 > 0$. Thus, we have

$$\begin{aligned}
&a_h^\epsilon(v_h,v_h)\\
&\quad\ge\left(1-\frac{\epsilon_1}{2}\right)\sum_{T\in\mathscr{T}_h}\left\|\beta^{1/2}\nabla v_h\right\|_{0,T}^2\\
&\qquad+\left(\sigma-\frac{3(1-\epsilon)^2C_1^2}{2\epsilon_1}\right)\sum_{e\in\mathscr{E}_h^N}\frac{1}{|e|}\left\|[v_h]_e\right\|_{0,e}^2\\
&\qquad+\frac{\beta^+}{\alpha}\sum_{e\in\mathscr{E}_h^I}\left\|[v_h]_e\right\|_{0,e}^2.
\end{aligned}\tag{52}$$

Then (47) is obtained by taking $\epsilon = 1$. □

By a similar technique, we can show that a_h is bounded.

Theorem 6. *There exists a constant $C_B > 0$ such that following holds:*

$$a_h^\epsilon(u_h,v_h) \le C_B|||u_h|||\cdot|||v_h|||_h,\tag{53}$$

for all $u_h, v_h \in \widetilde{S}_h(\Omega)$.

Finally, we prove the error estimate in the energy-like norm.

Theorem 7. *Assume that $\sigma > 0$ satisfies (46). There exists a constant $C > 0$ such that*

$$|||u-u_h||| \le Ch\left(\|f\|_{L^2(\Omega)}+\|g_1\|_{H^{3/2}(\Gamma)}+\|g_2\|_{H^{1/2}(\Gamma)}\right).\tag{54}$$

Proof. By (36), we have

$$a_h^\epsilon(u-u_h,u-u_h) = a_h^\epsilon(u-u_h,u-\phi_h),\tag{55}$$

for all $\phi_h \in \widetilde{S}_h(\Omega)$. By (55), (47), and (53) we have

$$\begin{aligned}
C_1|||u-u_h|||^2 &\le a_h^\epsilon(u-u_h,u-\phi_h)\\
&\le C_B|||u-u_h|||\cdot|||u-\phi_h|||.
\end{aligned}\tag{56}$$

If we take $\phi_h = I_hu$, then, by (42) and (9), we have

$$\begin{aligned}
|||u-u_h||| &\le \frac{C_B}{C_1}C_Ih\|u\|_{\widetilde{H}^2(\Omega)}\\
&\le \frac{C_B}{C_1}C_Ih\left(\|f\|_{L^2(\Omega)}+\|g_1\|_{H^{3/2}(\Gamma)}+\|g_2\|_{H^{1/2}(\Gamma)}\right).
\end{aligned}\tag{57}$$

□

Next, we prove L^2 estimates using duality argument when $\epsilon = -1$.

Theorem 8. *If $\sigma > 0$ satisfies (46) and $\epsilon = -1$, there exists a constant $C > 0$ such that*

$$\begin{aligned}
&\|u-u_h\|_{L^2(\Omega)}\\
&\quad\le Ch^2\left(\|f\|_{L^2(\Omega)}+\|g_1\|_{H^{3/2}(\Gamma)}+\|g_2\|_{H^{1/2}(\Gamma)}\right).
\end{aligned}\tag{58}$$

Proof. We define an auxiliary problem. Let $\psi \in \widetilde{H}^2(\Omega)$ be solution of

$$-\nabla\cdot\beta\nabla\psi = e_h,\quad \text{on } \Omega^+\cup\Omega^-,\tag{59}$$

$$[\psi]_\Gamma = -\alpha\frac{\partial\psi^+}{\partial\mathbf{n}^+},\quad \text{on } \Gamma,\tag{60}$$

$$\left[\beta\frac{\partial\psi}{\partial\mathbf{n}}\right]_\Gamma = 0,\quad \text{on } \Gamma,\tag{61}$$

$$\psi = 0,\quad \text{on } \partial\Omega,\tag{62}$$

where $e_h := u - u_h \in L^2(\Omega)$. We multiply e_h to (59) and we use integration by parts to have

$$\begin{aligned}
\|e_h\|_{L^2(\Omega)}^2 &= -\sum_{T\in\mathscr{T}_h}\int_T(\nabla\cdot\beta\nabla\psi)\,e_h d\mathbf{x} = \sum_{T\in\mathscr{T}_h}\int_T\beta\nabla\psi\\
&\cdot\nabla e_h d\mathbf{x} - \sum_{e\in\mathscr{E}_h}\int_e\left((\beta\nabla\psi)|_{T_e^1}\cdot\mathbf{n}_e\,e_h|_{T_e^1}-(\beta\nabla\psi)|_{T_e^2}\right.\\
&\left.\cdot\mathbf{n}_e\,e_h|_{T_e^2}\right)ds.
\end{aligned}\tag{63}$$

Table 1: L^2 and H^1 errors of Example 1.

Elements	DoF	$\|u-u_h\|_{L_2(\Omega)}$	Order	$\|u-u_h\|_{1,h}$	Order
70	80	3.934×10^{-1}		4.629×10^{-1}	
264	221	1.019×10^{-1}	1.949	1.793×10^{-1}	1.368
1062	709	2.568×10^{-2}	1.988	8.009×10^{-2}	1.163
4171	2439	6.432×10^{-3}	1.998	3.940×10^{-2}	1.023
16557	8986	1.610×10^{-3}	1.998	1.967×10^{-2}	1.002
65521	34174	4.023×10^{-4}	2.001	9.832×10^{-3}	1.000
260961	133311	1.006×10^{-4}	2.000	4.911×10^{-3}	1.002

Table 2: L^2 and H^1 errors of Example 2.

Elements	DoF	$\|u-u_h\|_{L_2(\Omega)}$	Order	$\|u-u_h\|_{1,h}$	Order
110	118	5.964×10^{-3}		1.008×10^{-1}	
382	313	2.720×10^{-3}	1.133	7.348×10^{-2}	0.456
1355	920	6.077×10^{-4}	2.162	3.589×10^{-2}	1.034
5005	2983	1.714×10^{-4}	1.826	1.924×10^{-2}	0.900
18993	10461	4.321×10^{-5}	1.988	9.900×10^{-3}	0.959
74225	39037	1.090×10^{-5}	1.987	4.989×10^{-3}	0.989
292404	150041	2.769×10^{-6}	1.976	2.519×10^{-3}	0.986

We use similar techniques in the classification of D_e of (25) to derive

$$-\sum_{e\in\mathscr{E}_h}\int_e\Big(\left(\beta\nabla\psi\right)|_{T_e^1}\cdot\mathbf{n}_e\, e_h|_{T_e^1}-\left(\beta\nabla\psi\right)|_{T_e^2}\cdot\mathbf{n}_e\, e_h|_{T_e^2}\Big)\,ds=-\sum_{e\in\mathscr{E}_h^N}\int_e\{\beta\nabla\psi\cdot\mathbf{n}_e\}_e\,[e_h]_e\,ds+\sum_{e\in\mathscr{E}_h^I}\frac{\beta^+}{\alpha}\int_e[\psi]_e\,[e_h]_e\,ds. \quad (64)$$

Combining with (63) and (64) and the fact that ψ is continuous on each subdomain Ω^- and Ω^+, we have

$$\|e_h\|^2_{L^2(\Omega)}=a_h^\epsilon(\psi,e_h). \quad (65)$$

By definition of e_h and the fact that $a_h^\epsilon(\cdot,\cdot)$ is symmetric and by (36), (54), (42), and (9), we have

$$\begin{aligned}\|u-u_h\|^2_{L^2(\Omega)}&=a_h^\epsilon(u-u_h,\psi)=a_h^\epsilon(u-u_h,\psi-I_h\psi)\\&\le Ch\|u\|_{\widetilde{H}^2(\Omega)}Ch\|\psi\|_{\widetilde{H}^2(\Omega)}\\&\le Ch^2\left(\|f\|_{L^2(\Omega)}+\|g_1\|_{H^{3/2}(\Gamma)}+\|g_2\|_{H^{1/2}(\Gamma)}\right)\\&\quad\cdot\|u-u_h\|_{L^2(\Omega)}.\end{aligned} \quad (66)$$

Thus, we have the conclusion. □

5. Numerical Results

In this section, we provide some numerical experiments of elliptic interface problems with implicit jump conditions. We consider circle- and ellipse-type interface shapes.

We let the domain $\Omega=[-1,1]^2$ and we let $\mathscr{T}_h$ be a triangulation of Ω by regular triangles, which aligns with interfaces. We set $\epsilon=-1$ in bilinear form (33). We take σ in (33) as a multiple of β. We report the number of elements, degrees of freedom, L^2-errors, and H^1-errors for $h_k=2^{-k}$, $k=1,2,\ldots$, in Tables 1 and 2. For both examples, we observe optimal error convergence, which supports our analysis in Section 4.

Example 1 (circular interface). The interface is given by $\Gamma=\{(x,y): x^2+y^2=r_0^2\}$ and Ω^- and Ω^+ are inside and outside of Γ, respectively. The coefficients are $(\beta^-,\beta^+)=(10,1)$, $\alpha=1$, and $r_0=0.5$. The exact solution is

$$u=\begin{cases}\dfrac{r^2}{2\beta^-}-\alpha r_0+\left(\dfrac{1}{\beta^+}-\dfrac{1}{\beta^-}\right)\dfrac{r_0^2}{2}+1.5-r^2, & \text{on } \Omega^-,\\ \dfrac{r^2}{2\beta^+}, & \text{on } \Omega^+,\end{cases} \quad (67)$$

where $r=\sqrt{x^2+y^2}$. We remark that u satisfies jump conditions (3) and (4), where g_1 and g_2 are given as

$$g_1=1.5-{r_0}^2,\qquad g_2=-20r_0. \quad (68)$$

Table 1 shows the number of elements and L^2 and piecewise H^1 errors. Figure 2 shows the numerical solution. We observe that our scheme has optimal convergence in L^2 and piecewise H^1-norms.

Example 2 (elliptical shape interface). The interface is given by $\Gamma=\{(x,y): x^2+y^2/2=r_0^2\}$ and Ω^- and Ω^+ are inside and

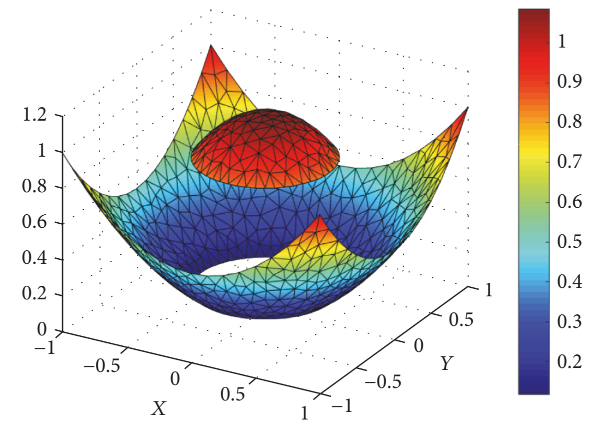

Figure 2: Numerical solution of Example 1.

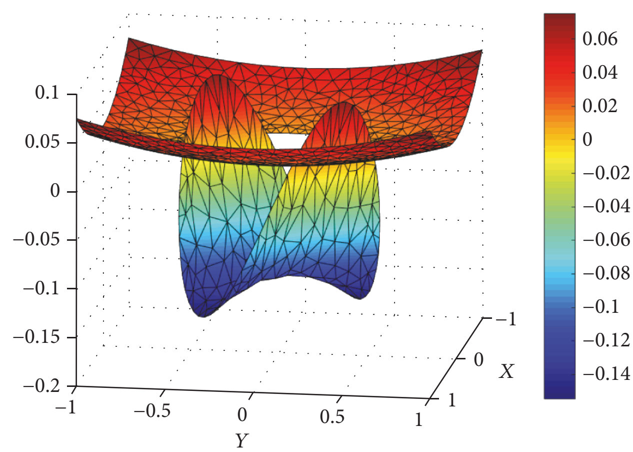

Figure 3: Numerical solution of Example 2.

outside of Γ, respectively. The coefficients are $(\beta^-, \beta^+) = (1, 10)$ and $r_0 = 0.4$. In the previous example, α was constant. However, in this example, we set α as a function of (x, y):

$$\alpha = \frac{1}{(x^2 + y^2/4)^{1/2}}. \tag{69}$$

The exact solution is

$$u = \begin{cases} \dfrac{q}{2\beta^-} - \dfrac{1}{2\beta^+} + \left(\dfrac{1}{\beta^+} - \dfrac{1}{\beta^-}\right)\dfrac{r_0^2}{2} + xy, & \text{on } \Omega^-, \\ \dfrac{q}{2\beta^+} & \text{on } \Omega^+, \end{cases} \tag{70}$$

where $q = x^2 + y^2/4$. We remark that g_1 and g_2 in (3) and (4) are given as

$$g_1 = xy,$$
$$g_2 = \frac{3xy}{\sqrt{4x^2 + y^2}}. \tag{71}$$

Table 2 shows the errors and Figure 3 shows the numerical solution. Again, we observe the optimal convergence.

6. Conclusion

In this work, we introduce a numerical method for elliptic interface problems, where the jumps are related to the normal fluxes. We enrich usual P_1 space by extra degrees of freedom on one side of the interface. We define bilinear form that includes the jump conditions implicitly. We prove that the bilinear form is coercive and bounded. Using Cea's Lemma, we prove the error estimates in energy-like norm. Next, we prove L^2 error estimate using the duality arguments. We provide numerical experiments that support our analysis.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] K. W. Garrett and H. M. Rosenberg, "The thermal conductivity of epoxy-resin / powder composite materials," *Journal of Physics D: Applied Physics*, vol. 7, no. 9, article no. 311, pp. 1247–1258, 1974.

[2] T. Belytschko, N. Moës, S. Usui, and C. Parimi, "Arbitrary discontinuities in finite elements," *International Journal for Numerical Methods in Engineering*, vol. 50, no. 4, pp. 993–1013, 2001.

[3] H. Ammari, J. Garnier, L. Giovangigli, W. Jing, and J.-K. Seo, "Spectroscopic imaging of a dilute cell suspension," *Journal de Mathématiques Pures et Appliquées*, vol. 105, no. 5, pp. 603–661, 2016.

[4] H. Ammari, J. Garnier, H. Kang, M. Lim, and S. Yu, "Generalized polarization tensors for shape description," *Numerische Mathematik*, vol. 126, no. 2, pp. 199–224, 2014.

[5] Z. Hashin, "Thin interphase/imperfect interface in elasticity with application to coated fiber composites," *Journal of the Mechanics and Physics of Solids*, vol. 50, no. 12, pp. 2509–2537, 2002.

[6] A. Buffa, "Remarks on the discretization of some noncoercive operator with applications to heterogeneous Maxwell equations," *SIAM Journal on Numerical Analysis*, vol. 43, no. 1, pp. 1–18, 2005.

[7] Z. Chen, *Reservoir simulation: mathematical techniques in oil recovery*, vol. 77 of *CBMS-NSF Regional Conference Series in Applied Mathematics*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2007.

[8] H. L. Duan and B. L. Karihaloo, "Effective thermal conductivities of heterogeneous media containing multiple imperfectly bonded inclusions," *Physical Review B: Condensed Matter and Materials Physics*, vol. 75, no. 6, Article ID 064206, 2007.

[9] F. Pavanello, F. Manca, P. Luca Palla, and S. Giordano, "Generalized interface models for transport phenomena: Unusual scale effects in composite nanomaterials," *Journal of Applied Physics*, vol. 112, no. 8, Article ID 084306, 2012.

[10] M. F. Wheeler, "An elliptic collocation-finite element method with interior penalties," *SIAM Journal on Numerical Analysis*, vol. 15, no. 1, pp. 152–161, 1978.

[11] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini, "Unified analysis of discontinuous Galerkin methods for elliptic problems," *SIAM Journal on Numerical Analysis*, vol. 39, no. 5, pp. 1749–1779, 2001/02.

[12] A. Ern, I. Mozolevski, and L. Schuh, "Discontinuous Galerkin approximation of two-phase flows in heterogeneous porous media with discontinuous capillary pressures," *Computer Methods Applied Mechanics and Engineering*, vol. 199, no. 23-24, pp. 1491–1501, 2010.

[13] R. J. LeVeque and Z. L. Li, "The immersed interface method for elliptic equations with discontinuous coefficients and singular sources," *SIAM Journal on Numerical Analysis*, vol. 31, no. 4, pp. 1019–1044, 1994.

[14] T. Y. Hou, Z. Li, S. Osher, and H. Zhao, "A hybrid method for moving interface problems with application to the Hele-Shaw flow," *Journal of Computational Physics*, vol. 134, no. 2, pp. 236–252, 1997.

[15] C. Zhang and R. J. LeVeque, "The immersed interface method for acoustic wave equations with discontinuous coefficients," *Wave Motion. An International Journal Reporting Research on Wave Phenomena*, vol. 25, no. 3, pp. 237–263, 1997.

[16] K. S. Chang and D. Y. Kwak, "Discontinuous bubble scheme for elliptic problems with jumps in the solution," *Computer Methods Applied Mechanics and Engineering*, vol. 200, no. 5-8, pp. 494–508, 2011.

[17] I. Lackovic, R. Magjarevic, and D. Miklavcic, "Analysis of tissue heating during electroporation based therapy: A 3D FEM model for a pair of needle electrodes," in *Proceedings of the 11th Mediterranean Conference on Medical and Biological Engineering and Computing, MEDICON '07*, pp. 631–634, Springer, June 2007.

[18] S. Kim, J. K. Seo, and T. Ha, "A nondestructive evaluation method for concrete voids: frequency differential electrical impedance scanning," *SIAM Journal on Applied Mathematics*, vol. 69, no. 6, pp. 1759–1771, 2009.

[19] D. Kyeong and D. Y. Kwak, "An immersed finite element method for the elasticity problems with displacement jump," *Advances in Applied Mathematics and Mechanics*, vol. 9, no. 2, pp. 407–428, 2017.

[20] J. Kacur and R. Van Keer, "Numerical method for a class of parabolic problems in composite media," *Numerical Methods for Partial Differential Equations*, vol. 9, no. 6, pp. 711–731, 1993.

[21] D. W. Hahn and M. N. Ã-zisik, *Heat conduction*, John Wiley & Sons, 2012.

[22] P. A. Krutitskii, "The jump problem for the Helmholtz equation and singularities at the edges," *Applied Mathematics Letters*, vol. 13, no. 4, pp. 71–76, 2000.

[23] P. A. Krutitskii, "The modified jump problem for the Laplace equation and singularities at the tips," *Journal of Computational and Applied Mathematics*, vol. 183, no. 1, pp. 232–240, 2005.

[24] P. A. Krutitskii, "Generalization of the Neumann problem to harmonic functions outside cuts on the plane," *Journal of Differential Equations*, vol. 49, no. 9, pp. 1062–1074, 2013.

[25] J. Weisz, "On an iterative method for the solution of discretized elliptic problems with imperfect contact condition," *Journal of Computational and Applied Mathematics*, vol. 72, no. 2, pp. 319–333, 1996.

[26] J. Yvonnet, Q.-C. He, Q.-Z. Zhu, and J.-F. Shao, "A general and efficient computational procedure for modelling the Kapitza thermal resistance based on XFEM," *Computational Materials Science*, vol. 50, no. 4, pp. 1220–1224, 2011.

[27] A. Hansbo and P. Hansbo, "A finite element method for the simulation of strong and weak discontinuities in solid mechanics," *Computer Methods Applied Mechanics and Engineering*, vol. 193, no. 33-35, pp. 3523–3540, 2004.

[28] D. Y. Kwak, S. Lee, and Y. Hyon, "A new finite element for interface problems having Robin type jump," *International Journal of Numerical Analysis & Modeling*, vol. 14, no. 4-5, pp. 532–549, 2017.

[29] J. A. Rotberg and Z. G. Šeftel, *A theorem on homeomorphisms for elliptic systems and its applications, Mathematics of the USSR-Sbornik*, vol. 7, 1969.

[30] J. H. Bramble and J. T. King, "A finite element method for interface problems in domains with smooth boundaries and interfaces," *Advances in Computational Mathematics*, vol. 6, no. 2, pp. 109–138 (1997), 1996.

[31] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, The Netherlands, 1978.

[32] D. Braess, *Finite elements: Theory, fast solvers, and applications in solid mechanics*, Cambridge University Press, 3rd edition, 2007.