

## SUPERCONVERGENCE OF FINITE ELEMENT METHOD FOR PARABOLIC PROBLEM

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**ABSTRACT.** We study superconvergence of a semi-discrete finite element scheme for parabolic problem. Our new scheme is based on introducing different approximation of initial condition. First, we give a superconvergence of  $u_h - R_h u$ , then use a postprocessing to improve the accuracy to higher order.

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**1. Introduction.** We consider the following parabolic problem:

$$\begin{aligned}u_t - \Delta u &= f \quad \text{in } \Omega, \text{ for } t > 0, \\u &= 0 \quad \text{on } \partial\Omega, \text{ for } t \geq 0, \\u(\cdot, 0) &= v \quad \text{in } \Omega,\end{aligned}\tag{1.1}$$

where  $\Omega \subset \mathbb{R}^2$  is a domain with smooth boundary. Suppose we are given a family  $\mathcal{T}_h$  of quasi-uniform triangulation of  $\Omega$ , whose maximum diameter is denoted by  $h$ . Let  $S_h \subset H_0^1(\Omega)$  be a standard finite element space consisting of continuous, piecewise polynomial of degree  $k$ . Define an elliptic projection  $R_h : H_0^1(\Omega) \rightarrow S_h$  by

$$(\nabla(R_h w - w), \nabla \chi) = 0 \quad \forall \chi \in S_h.\tag{1.2}$$

We consider the following map  $u_h(t) : [0, T] \rightarrow S_h$  defined by

$$(u_{h,t}, \chi) + (\nabla u_h, \nabla \chi) = (f, \chi), \quad u_h(0) = v_h,\tag{1.3}$$

where  $v_h$  is determined by

$$(\nabla v_h, \nabla \chi) = (f(0), \chi) - (R_h u_t(0), \chi) \quad \forall \chi \in S_h,\tag{1.4}$$

and  $u_t(0)$  is determined by (1.1). Superconvergence of finite element for parabolic problem has been studied by many authors. For example, Thomeé [8], Chen and Huang [1] studied superconvergence of the gradient in  $L^2$  norm while Thomeé et al. [9] studied maximum norm superconvergence of gradient for linear finite element. Superconvergence of the lumped finite element method for linear and nonlinear parabolic problems were studied in [2] and [6], respectively. In this paper, we introduce a different way of approximating the initial condition, namely (1.4) and investigate the superconvergence of finite element for parabolic problem using any order element.

To do so, we decompose the error as  $u_h - u = u_h - R_h u + R_h u - u = \theta + \rho$  and estimate  $\theta$  in a superconvergent order. Next, a postprocessing technique used in [4, 5] is employed to obtain higher order convergence. The rest of the paper is organized as follows. In Section 2, we show  $\theta$  in  $L^2$  and  $H^1$  norm when  $k > 1$ . For  $\theta$ , the superconvergence in  $L^\infty$  and  $W^{1,\infty}$  norm are also considered. In Section 3, the case  $k = 1$  is considered. The superconvergence of  $\theta_t$  in  $H^1$  and  $\theta$  in  $W^{1,\infty}$  norm are shown. In Section 4,  $W^{l,p}$ ,  $l = 0, 1$ , ( $2 < p < \infty$ ) norm estimates are shown. Finally, in Section 5, we give some applications of the results obtained in Sections 2, 3 and 4. For example, a postprocessing technique is employed to obtain second-order superconvergence for gradient and first-order for the solution when  $k > 1$ . First-order superconvergence is shown when  $k = 1$ .

**2. Superconvergence in  $L^2, H^1, L^\infty$ , and  $W^{1,\infty}$  norm.** We recall  $\rho = R_h u - u$  and  $\theta = u_h - R_h u$ .

**LEMMA 2.1.** *Let  $1 < p < \infty$ ,  $(1/p) + (1/p') = 1$ . Then for any  $g \in W^{1,p'}(\Omega)$ , we have, for  $k > 1$ ,*

$$|(D_t^s \rho, g)| \leq Ch^{k+2} \|D_t^s u\|_{k+1,p} \|g\|_{1,p'}, \tag{2.1}$$

where  $D_t^s = \partial^s / \partial t^s$ .

**PROOF.** It suffices to prove the case for  $s = 0$ . From standard finite element theory,

$$\|\rho\|_{1,p} \leq Ch^k \|u\|_{k+1,p}. \tag{2.2}$$

Consider the dual problem: given  $g \in L^p(\Omega)$ , find  $w \in W^{3,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$  satisfying

$$(\nabla v, \nabla w) = (g, v), \quad \forall v \in H_0^1(\Omega), \tag{2.3}$$

$$\|w\|_{3,p'} \leq C \|g\|_{1,p'}. \tag{2.4}$$

Let  $\Pi_h$  denote the  $S_h$  interpolation operator. Then by (2.3), (1.2), (2.2), (2.4), and the property of interpolation, we have

$$\begin{aligned} (g, \rho) &= (\nabla \rho, \nabla w) = |(\nabla \rho, \nabla (w - \Pi_h w))| \\ &\leq \|\rho\|_{1,p} \|w - \Pi_h w\|_{1,p'} \leq Ch^k \|u\|_{k+1,p} h^2 \|w\|_{3,p'} \\ &\leq Ch^{k+2} \|u\|_{k+1,p} \|g\|_{1,p'}. \end{aligned} \tag{2.5}$$

□

**LEMMA 2.2.** *We have*

- (i)  $\theta_t(0) = 0$ , i.e.,  $u_{h,t}(0) = R_h u_t(0)$ .
- (ii)  $\|\theta(0)\|_1 \leq Ch^{k+2} \|u_t(0)\|_{k+1}$ .

**PROOF.** From (1.4) and (1.3),

$$(R_h u_t(0), \chi) = (f(0), \chi) - (\nabla v_h, \nabla \chi) = (u_{h,t}(0), \chi), \quad \chi \in S_h. \tag{2.6}$$

Hence  $R_h u_t(0) = u_{h,t}(0)$ . For (ii), we see from (1.1),

$$(u_t, v) + (\nabla u, \nabla v) = (f, v). \tag{2.7}$$

Subtraction of (1.3) from (2.7), and noting (1.2), give

$$(\theta_t, \chi) + (\nabla \theta, \nabla \chi) = -(\rho_t, \chi), \quad \chi \in S_h. \tag{2.8}$$

Set  $t = 0$  and noting that  $\theta_t(0) = 0$ , we have

$$(\nabla \theta(0), \nabla \chi) = -(\rho_t(0), \chi). \tag{2.9}$$

Take  $\chi = \theta(0)$  in (2.9). Then we see from Lemma 2.1,

$$\|\nabla \theta(0)\|^2 = |(\rho_t(0), \theta(0))| \leq Ch^{k+2} \|u_t(0)\|_{k+1} \|\theta(0)\|_1. \tag{2.10}$$

Since  $|\cdot|$  and  $\|\cdot\|$  are equivalent in  $H_0^1(\Omega)$ ,

$$\|\theta(0)\|_1 \leq C \|\nabla \theta(0)\| \leq Ch^{k+2} \|u_t\|_{k+1}. \tag{2.11}$$

□

**THEOREM 2.3.** *We have first-order superconvergence for  $\|\theta_t\|$  and second-order superconvergence for  $\|\nabla \theta_t\|$ . In other words,*

$$\|\theta_t(t)\| + \left( \int_0^t \|\nabla \theta_t\|^2 d\tau \right)^{1/2} \leq Ch^{k+2} \left( \int_0^t \|u_{tt}\|_{k+1}^2 d\tau \right)^{1/2} \tag{2.12}$$

holds.

**PROOF.** Differentiating error equation (2.8),

$$(\theta_{tt}, \chi) + (\nabla \theta_t, \nabla \chi) = -(\rho_{tt}, \chi), \quad \chi \in S_h. \tag{2.13}$$

Take  $\chi = \theta_t$ . Then by Lemma 2.1, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\theta_t\|^2 + \|\nabla \theta_t\|^2 &= |(\rho_{tt}, \theta_t)| \leq Ch^{k+2} \|u_{tt}\|_{k+1} \|\theta_t\|_1 \\ &\leq Ch^{2(k+2)} \|u_{tt}\|_{k+1}^2 + \frac{1}{2} \|\nabla \theta_t\|^2, \end{aligned} \tag{2.14}$$

where arithmetic-geometric inequality was used in the last line. Elimination of  $(1/2)\|\nabla \theta_t\|^2$  and integration, give, by Lemma 2.2(i),

$$\begin{aligned} \|\theta_t(t)\|^2 + \int_0^t \|\nabla \theta_t(\tau)\|^2 d\tau &\leq \|\theta_t(0)\|^2 + Ch^{2(k+2)} \int_0^t \|u_{tt}(\tau)\|_{k+1}^2 d\tau \\ &\leq Ch^{2(k+2)} \int_0^t \|u_{tt}(\tau)\|_{k+1}^2 d\tau. \end{aligned} \tag{2.15}$$

□

**THEOREM 2.4.** *We have second-order superconvergence for  $\|\theta_t\|_1$  and first-order for  $\|\theta_{tt}\|$ ,*

$$\left( \int_0^t \|\theta_{tt}\|^2 d\tau \right)^{1/2} + \|\theta_t(t)\|_1 \leq Ch^{k+2} \left[ \|u_{tt}(t)\|_{k+1} + \left( \int_0^t \|u_{ttt}\|_{k+1}^2 d\tau \right)^{1/2} \right]. \tag{2.16}$$

**PROOF.** From (2.13) with  $\chi = \theta_{tt}$ ,

$$\|\theta_{tt}\|^2 + \frac{1}{2} \frac{d}{dt} \|\nabla \theta_t\|^2 = -(\rho_{tt}, \theta_{tt}). \tag{2.17}$$

Integration, and noting that  $\theta_t(0) = 0$ , gives

$$\begin{aligned} \int_0^t \|\theta_{tt}\|^2 d\tau + \frac{1}{2} \|\nabla \theta_t\|^2 &= - \int_0^t (\rho_{tt}, \theta_{tt}) d\tau \\ &= -(\rho_{tt}, \theta_t)|_0^t + \int_0^t (\rho_{ttt}, \theta_t) d\tau \\ &= -(\rho_{tt}, \theta_t) + \int_0^t (\rho_{ttt}, \theta_t) d\tau \end{aligned} \tag{2.18}$$

Using Lemma 2.1, left-hand side of (2.18) is

$$\begin{aligned} &\leq Ch^{k+2} \|u_{tt}\|_{k+1} \|\theta_t\|_1 + Ch^{k+2} \int_0^t \|u_{ttt}\|_{k+1} \|\theta_t\|_1 d\tau \\ &\leq Ch^{2(k+2)} \|u_{tt}\|_{k+1}^2 + \frac{1}{4} \|\theta_t\|_1^2 + Ch^{2(k+2)} \int_0^t \|u_{ttt}\|_{k+1}^2 d\tau + C \int_0^t \|\theta_t\|_1^2 d\tau. \end{aligned} \tag{2.19}$$

Elimination of  $(1/4)\|\theta_t\|_1^2$  and usage of Gronwall inequality give (2.16). □

**THEOREM 2.5.** *We have second-order superconvergence for  $\|\theta\|_1$ .*

$$\|\theta(t)\|_1 \leq Ch^{k+2} \left[ \|u_t(0)\|_{k+1} + \left( \int_0^t \|u_{tt}\|_{k+1}^2 d\tau \right)^{1/2} \right]. \tag{2.20}$$

**PROOF.** By Lemma 2.2 and Theorem 2.3, we have

$$\begin{aligned} \|\theta(t)\|_1 &\leq \|\theta(0)\|_1 + \int_0^t \|\theta_t\|_1 d\tau \\ &\leq \|\theta(0)\|_1 + C \left( \int_0^t \|\theta_t\|_1^2 d\tau \right)^{1/2} \\ &\leq Ch^{k+2} \|u_t(0)\|_{k+1} + Ch^{k+2} \left( \int_0^t \|u_{tt}\|_{k+1}^2 d\tau \right)^{1/2}. \end{aligned} \tag{2.21}$$

□

**THEOREM 2.6.** *We have first-order superconvergence for  $\|\theta\|$ .*

$$\|\theta(t)\| \leq Ch^{k+2} \left[ \|u_t(0)\|_{k+1} + \left( \int_0^t \|u_t\|_{k+1}^2 d\tau \right)^{1/2} \right]. \tag{2.22}$$

**PROOF.** Recall that error equation (2.8)

$$(\theta_t, \chi) + (\nabla \theta, \nabla \chi) = -(\rho_t, \chi). \tag{2.23}$$

Take  $\chi = \theta$  in (2.8). Then we see from Lemma 2.1,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\theta(t)\|^2 + \|\nabla \theta\|^2 &= -(\rho_t, \theta) \leq Ch^{k+2} \|u_t\|_{k+1} \|\theta\|_1 \\ &\leq Ch^{2(k+2)} \|u_t\|_{k+1}^2 + \|\nabla \theta\|^2. \end{aligned} \tag{2.24}$$

Elimination of  $\|\nabla\theta\|^2$  and integration, give, by Lemma 2.2,

$$\begin{aligned} \|\theta(t)\|^2 &\leq \|\theta(0)\|^2 + ch^{2(k+2)} \int_0^t \|u_t\|_{k+1}^2 d\tau \\ &\leq Ch^{2(k+2)} \|u_t(0)\|_{k+1}^2 + ch^{2(k+2)} \int_0^t \|u_t\|_{k+1}^2 d\tau. \end{aligned} \tag{2.25}$$

Now we study  $L^\infty, W^{1,\infty}$  superconvergence. First we need Green's functions. The discrete Green's function  $G_h^z \in S_h$  for  $z \in \Omega$  is defined by

$$(\nabla G_h^z, \nabla \chi) = \chi(z), \quad \chi \in S_h. \tag{2.26}$$

The derivative type Green's function  $g_{h,i}^z \in S_h, (i = 1, 2)$  is defined by

$$(\nabla g_{h,i}^z, \nabla \chi) = \frac{\partial}{\partial x_i} \chi(z), \quad \chi \in S_h. \tag{2.27}$$

Green's functions posses the following properties (see [9, 10]).

**LEMMA 2.7.** *We have*

$$\|G_h^z\| + \|G_h^z\|_{1,p'} \leq C, \quad 1 \leq p' < 2, \tag{2.28}$$

$$\|g_{h,i}^z\|^2 + \|g_{h,i}^z\|_{1,1} \leq C \log \frac{1}{h}. \tag{2.29}$$

**THEOREM 2.8.** *We have the following estimate:*

$$\|\theta(t)\|_{0,\infty} \leq Ch^{k+2} \left[ \|u_t(t)\|_{k+1,p} + \left( \int_0^t \|u_{tt}\|_{k+1}^2 d\tau \right)^{1/2} \right], \quad p > 2. \tag{2.30}$$

**PROOF.** By taking  $\chi = \theta$  in the definition (2.26), we have by (2.8), Lemmas 2.1, 2.7, and Theorem 2.3,

$$\begin{aligned} |\theta(z, t)| &= |(\nabla G_h^z, \nabla \theta)| = |(\rho_t, G_h^z) + (\theta_t, G_h^z)| \\ &\leq Ch^{k+2} \|u_t\|_{k+1,p} \|G_h^z\|_{1,p'} + \|\theta_t\| \|G_h^z\| \\ &\leq Ch^{k+2} \|u_t\|_{k+1,p} + Ch^{k+2} \left( \int_0^t \|u_{tt}\|_{k+1}^2 d\tau \right)^{1/2}. \end{aligned} \tag{2.31}$$

Now take supremum over all  $z \in \Omega$ . □

**THEOREM 2.9.** *We have the following estimate:*

$$\|\theta(t)\|_{1,\infty} \leq Ch^{k+2-\epsilon} \left[ \|u_t\|_{k+1,p} + \left( \int_0^t \|u_{tt}\|_{k+1}^2 d\tau \right)^{1/2} \right], \tag{2.32}$$

for any  $\epsilon > 2/p, p < \infty$  large enough.

**PROOF.** For  $z \in \Omega$ , we see from (2.27), (2.8), Lemma 2.7, and Theorem 2.3,

$$\begin{aligned}
 \left| \frac{\partial}{\partial x_i} \theta(z) \right| &= |(\nabla \mathcal{G}_{h,i}^z, \nabla \theta)| = |(\rho_t, \mathcal{G}_{h,i}^z) + (\theta_t, \mathcal{G}_{h,i}^z)| \\
 &\leq Ch^{k+2} \|u_t\|_{k+1,p} \|\mathcal{G}_{h,i}^z\|_{1,p'} + \|\theta_t\| \|\mathcal{G}_{h,i}^z\| \\
 &\leq Ch^{k+2-2/p} \|u_t\|_{k+1,p} \|\mathcal{G}_{h,i}^z\|_{1,1} + Ch^{k+2} \left( \int_0^t \|u_{tt}\|_{k+1}^2 d\tau \right)^{1/2} \|\mathcal{G}_{h,i}^z\| \quad (2.33) \\
 &\leq Ch^{k+2-\epsilon} \left( \int_0^t \|u_{tt}\|_{k+1}^2 d\tau \right)^{1/2} \|\mathcal{G}_{h,i}^z\|,
 \end{aligned}$$

where inverse estimate

$$\|\mathcal{G}_{h,i}^z\|_{1,p'} \leq Ch^{-2/p} \|\mathcal{G}_{h,i}^z\|_{1,1}, \quad 1 \leq p' < 2, \quad 2 < p \leq \infty \quad (2.34)$$

was used in the second inequality.  $\square$

**3. The case  $k = 1$ .** Here the corresponding finite element space  $S_h$  is a linear finite element space. We make suitable modification of Lemma 2.2 to obtain the following lemma.

**LEMMA 3.1.**

$$\|\theta(0)\|_1 \leq Ch^2 \|u_t(0)\|_2. \quad (3.1)$$

**PROOF.** We recall (2.9)

$$(\nabla \theta(0), \nabla \chi) = -(\rho_t(0), \chi), \quad \chi \in S_h \quad (3.2)$$

Take  $\chi = \theta(0)$ . Then, we see that

$$\|\nabla \theta(0)\|^2 = |(\rho_t(0), \chi(0))| \leq \|\rho_t(0)\| \cdot \|\theta(0)\| \leq Ch^2 \|u_t(0)\|_2 \cdot \|\nabla \theta(0)\|. \quad (3.3)$$

$\square$

**THEOREM 3.2.** *We have*

$$\|\theta_t(t)\| + \left( \int_0^t \|\nabla \theta_t\|^2 d\tau \right)^{1/2} \leq Ch^2 \left( \int_0^t \|u_{tt}\|_2^2 d\tau \right)^{1/2}. \quad (3.4)$$

**PROOF.** We recall (2.13)

$$(\theta_{tt}, \chi) + (\nabla \theta_t, \nabla \chi) = -(\rho_{tt}, \chi), \quad \chi \in S_h. \quad (3.5)$$

Taking  $\chi = \theta_t$ , we see that

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} \|\theta_t\|^2 + \|\nabla \theta_t\|^2 &\leq C \|\rho_{tt}\| \cdot \|\theta_t\| \\
 &\leq Ch^2 \|u_{tt}\|_2 \|\nabla \theta_t\| \\
 &\leq Ch^4 \|u_{tt}\|_2^2 + \frac{1}{2} \|\nabla \theta_t\|^2.
 \end{aligned} \quad (3.6)$$

Elimination of  $(1/2)\|\nabla \theta_t\|^2$  and integration, give the result.  $\square$

**COROLLARY 3.3.** *We have*

$$\|\theta(t)\|_1 \leq Ch^2 \left( \int_0^t \|u_{tt}\|_2^2 d\tau \right)^{1/2}. \tag{3.7}$$

**THEOREM 3.4.** *We have*

$$\left( \int_0^t \|\theta_{tt}\|^2 d\tau \right)^{1/2} + \|\theta_t(t)\|_1 \leq Ch^2 \left[ \|u_{tt}(t)\|_2 + \left( \int_0^t \|u_{ttt}\|_2^2 d\tau \right)^{1/2} \right]. \tag{3.8}$$

**PROOF.** Taking  $\chi = \theta_{tt}$  in (2.13), we see that

$$\|\theta_{tt}\|^2 + \frac{1}{2} \frac{d}{dt} \|\nabla \theta_t\|^2 = -(\rho_{tt}, \theta_{tt}). \tag{3.9}$$

Integrating and noting  $\theta_t(0) = 0$ , we have

$$\begin{aligned} \int_0^t \|\theta_{tt}\|^2 d\tau + \frac{1}{2} \|\nabla \theta_t\|^2 &= - \int_0^t (\rho_{tt}, \theta_{tt}) dt \\ &= -(\rho_{tt}, \theta_t) + \int_0^t (\rho_{ttt}, \theta_t) d\tau \\ &\leq \|\rho_{tt}\| \cdot \|\theta_t\| + \int_0^t \|\rho_{ttt}\| \cdot \|\theta_t\| d\tau \\ &\leq Ch^2 \|u_{tt}\|_2 \cdot \|\theta_t\| + ch^2 \int_0^t \|u_{ttt}\|_2 \cdot \|\theta_t\| d\tau \\ &\leq Ch^4 \|u_{tt}\|_2^2 + \frac{1}{4} \|\nabla \theta_t\|^2 + ch^4 \int_0^t \|u_{ttt}\|_2^2 d\tau + \int_0^t \|\nabla \theta_t\|^2 d\tau. \end{aligned} \tag{3.10}$$

Now Gronwall inequality gives the result. □

**LEMMA 3.5.** *For  $1 < p < 2$ , we have the following estimate:*

$$\|\nabla g_{h,i}^z\|_{0,p} \leq C \quad \text{for } i = 1, 2. \tag{3.11}$$

**PROOF.** Let  $(1/p) + (1/p') = 1$ . For any  $\phi \in L^{p'}(\Omega)$ , let  $\Psi$  be the solution of

$$-\Delta \Psi = \phi \quad \text{in } \Omega, \quad \Psi = 0 \quad \text{on } \partial\Omega. \tag{3.12}$$

Then we have

$$\|\Psi\|_{2,p'} \leq C \|\phi\|_{0,p'}. \tag{3.13}$$

Setting  $g_h = g_{h,i}^z$ , we have, by (3.12), (1.2), and (2.27),

$$(g_h, \phi) = (\nabla g_h, \nabla \Psi) = (\nabla g_h, \nabla R_h \Psi) = \frac{\partial}{\partial x_i} R_h \Psi(z). \tag{3.14}$$

Thus, we see from  $W^{1,\infty}$  stability of  $R_h$ , imbedding theorem and (3.13) that

$$(g_h, \phi) \leq \|R_h \Psi\|_{1,\infty} \leq C \|\Psi\|_{1,\infty} \leq C \|\Psi\|_{2,p'} \leq C \|\phi\|_{0,p'}, \tag{3.15}$$

we have

$$\|g_h\|_{0,p} = \sup_{\phi \in L^{p'}(\Omega)} \frac{(g_h, \phi)}{\|\phi\|_{0,p'}} \leq C. \tag{3.16} \quad \square$$

**THEOREM 3.6.** *We have*

$$\|\theta(t)\|_{1,\infty} \leq Ch^2 \left[ \|u_t(t)\|_{2,p} + \|u_{tt}(t)\|_2 + \left( \int_0^t \|u_{ttt}\|_2^2 d\tau \right)^{1/2} \right], \quad p > 2. \quad (3.17)$$

**PROOF.** Setting  $\chi = g_{h,i}^z$  in (2.8), we obtain by (2.27), (3.11) and imbedding theorem, we have

$$\begin{aligned} \frac{\partial}{\partial x_i} \theta(z, t) &\leq (u_t - u_{h,t}, g_{h,i}^z) \\ &\leq (\|\rho_t\|_{0,p} + \|\theta_t\|_{0,p}) \|g_{h,i}^z\|_{0,p'}, \quad (1/p) + (1/p') = 1 \\ &\leq C (\|\rho_t\|_{0,p} + \|\theta_t\|_1). \end{aligned} \quad (3.18)$$

By standard estimate, we have

$$\|\rho_t\|_{0,p} \leq Ch^2 \|u_t\|_{2,p}. \quad (3.19)$$

Combining (3.8), (3.19) with (3.18), we obtain the desired result. □

**COROLLARY 3.7.** *We have*

$$\|\theta(t)\|_{0,\infty} \leq Ch^2 \left[ \|u_t(t)\|_{2,p} + \|u_{tt}(t)\|_2 + \left( \int_0^t \|u_{ttt}\|_2^2 d\tau \right)^{1/2} \right], \quad p > 2. \quad (3.20)$$

**4. Superconvergence in  $L^p$  and  $W^{1,p}$ , ( $2 < p < \infty$ )**

**THEOREM 4.1.** *We have*

$$\|\theta\|_{0,p} \leq Ch^{k+2} \left[ \|u_t(0)\|_{k+1} + \left( \int_0^t \|u_{tt}\|_{k+1}^2 d\tau \right)^{1/2} \right], \quad k > 1. \quad (4.1)$$

**PROOF.** From Sobolev inequality, we have, for  $2 < p < \infty$ ,

$$\|\chi\|_{0,p} \leq C \|\chi\|_1, \quad \chi \in S_h. \quad (4.2)$$

The conclusion directly follows from Theorem 2.5. □

**THEOREM 4.2.** *We have*

$$\|\theta(t)\|_{1,p} \leq Ch^{k+2} \left[ \|u_t(t)\|_{k+1,p} + \left( \int_0^t \|u_{tt}\|_{k+1}^2 d\tau \right)^{1/2} \right], \quad k > 1, \quad (4.3)$$

$$\|\theta(t)\|_{1,p} \leq Ch^2 \left[ \|u_t(t)\|_{2,p} + \|u_{tt}(t)\|_2 + \left( \int_0^t \|u_{ttt}\|_2^2 d\tau \right)^{1/2} \right], \quad k = 1. \quad (4.4)$$

**PROOF.** Let  $p$  ( $2 < p < \infty$ ) and  $p'$  be conjugate indices, and let  $\phi \in L^{p'}(\Omega)$  with  $\|\phi\|_{0,p'} = 1$  and  $\phi_x$  be any component of  $\nabla\phi$ . If  $\psi$  is the solution of

$$(\nabla v, \nabla \psi) = -(\phi_x, v), \quad \forall v \in H_0^1(\Omega) \quad (4.5)$$



with the regularity property [7]

$$\|\psi\|_{1,p'} \leq C_p \|\phi\|_{0,p'} = C_p. \tag{4.6}$$

Then by Green's formula, equations (4.5), (1.2), (2.8), Lemma 2.1, Theorem 2.3, Sobolev lemma, and (4.6), we have

$$\begin{aligned} (\theta_x, \phi) &= -(\phi_x, \theta) = (\nabla\theta, \nabla\psi) = (\nabla\theta, \nabla R_h\psi) = -(\rho_t, R_h\psi) - (\theta_t, R_h\psi) \\ &\leq Ch^{k+2} \|u_t(t)\|_{k+1,p} \|R_h\psi\|_{1,p'} + \|\theta_t(t)\| \|R_h\psi\| \\ &\leq Ch^{k+2} \left[ \|u_t(t)\|_{k+1,p} + \left( \int_0^t \|u_{tt}\|_{k+1}^2 d\tau \right)^{1/2} \right] \|R_h\psi\|_{1,p'} \\ &\leq Ch^{k+2} \left[ \|u_t(t)\|_{k+1,p} + \left( \int_0^t \|u_{tt}\|_{k+1}^2 d\tau \right)^{1/2} \right]. \end{aligned} \tag{4.7}$$

Now noting that

$$\|\theta_x\|_{0,p} = \sup_{\psi \in L^{p'}(\Omega)} (\theta_x, \phi), \quad \|\phi\|_{0,p'} = 1, \tag{4.8}$$

the conclusion (4.3) is obtained. To prove (4.4), we note that

$$\|\theta\|_{1,p} \leq C \|\theta\|_{1,\infty}. \tag{4.9}$$

This, together with (3.17), proves the theorem. □

**5. Application.** We now give an application of the results derived in Sections 2 and 3.

As an example, let  $T_h$  be a quasi-uniform rectangular partition of  $\Omega \subset \mathbb{R}^2$  and let  $S_h$  be the space of continuous piecewise polynomials

$$S_h = \{v \in H_0^1(\Omega), v \in Q^k(\tau), \tau \in T_h\}, \tag{5.1}$$

where

$$Q^k = \text{span} \{x_1^i x_2^j, 0 \leq i, j \leq k\}. \tag{5.2}$$

Introduce two kinds of operators (see [3, 4]), the vertices-edges-element interpolation  $i_h^k$  and the high-interpolation operator  $I_{2h}^{k+l}$  ( $l = 1, 2$ ). They satisfy the following properties:

$$\|u - I_{2h}^{k+l}u\|_{m,p} \leq Ch^{k+l+1-m} \|u\|_{k+l+1,p}, \quad 1 \leq k, m = 0, 1, (2 \leq p \leq \infty), l = 1, 2, \tag{5.3}$$

$$I_{2h}^{k+l}i_h^k = I_{2h}^{k+l}, \quad k \geq 1, l = 1, 2, \tag{5.4}$$

$$\|I_{2h}^{k+l}\chi\|_{m,p} \leq C \|\chi\|_{m,p}, \quad \forall \chi \in S_k, 1 \leq k, m = 0, 1, (2 \leq p \leq \infty), l = 1, 2. \tag{5.5}$$

Using these properties we can improve global convergence from  $k$ -to  $k + 2$ -order for gradient, and from  $k + 1$ -to  $k + 2$ -order for solution when  $k \geq 2$ . When  $k = 1$ , we get one order gain for the gradient.

**THEOREM 5.1.** *For  $k \geq 2$ , we have the following results:*

$$\|u - I_{2h}^{k+1}u_h\| \leq Ch^{k+2} \left[ \|u_t(0)\|_{k+1} + \left( \int_0^t \|u_t\|_{k+1}^2 d\tau \right)^{1/2} + \|u(t)\|_{k+3} \right], \tag{5.6}$$

$$\begin{aligned} & \|u - I_{2h}^{k+1}u_h\|_{0,p} \\ & \leq Ch^{k+2} \left[ \|u_t(0)\|_{k+1} + \left( \int_0^t \|u_{tt}\|_{k+1}^2 d\tau \right)^{1/2} + \|u(t)\|_{k+3,p} \right], \quad p > 2, \end{aligned} \tag{5.7}$$

$$\|u_t - I_{2h}^{k+1}u_{h,t}\| \leq Ch^{k+2} \left[ \left( \int_0^t \|u_{tt}\|_{k+1}^2 d\tau \right)^{1/2} + \|u_t(t)\|_{k+3} \right], \tag{5.8}$$

$$\begin{aligned} & \|u - I_{2h}^{k+1}u_h\|_{0,\infty} \\ & \leq Ch^{k+2} \left[ \|u_t(t)\|_{k+1,p} + \left( \int_0^t \|u_{tt}\|_{k+1}^2 d\tau \right)^{1/2} + \|u(t)\|_{k+3,\infty} \right], \quad p > 2, \end{aligned} \tag{5.9}$$

$$\|u - I_{2h}^{k+2}u_h\|_1 \leq Ch^{k+2} \left[ \|u_t(0)\|_{k+1} + \left( \int_0^t \|u_{tt}\|_{k+1}^2 d\tau \right)^{1/2} + \|u(t)\|_{k+3} \right], \tag{5.10}$$

$$\begin{aligned} & \|u - I_{2h}^{k+2}u_h\|_{1,p} \\ & \leq Ch^{k+2} \left[ \|u_t(t)\|_{k+1,p} + \left( \int_0^t \|u_{tt}\|_{k+1}^2 d\tau \right)^{1/2} + \|u(t)\|_{k+3,p} \right], \quad p > 2, \end{aligned} \tag{5.11}$$

$$\begin{aligned} & \|u - I_{2h}^{k+2}u_h\|_{1,\infty} \\ & \leq Ch^{k+2-\epsilon} \left[ \|u_t(t)\|_{k+1,p} + \left( \int_0^t \|u_{tt}\|_{k+1}^2 d\tau \right)^{1/2} + \|u(t)\|_{k+3,\infty} \right] \end{aligned} \tag{5.12}$$

for any  $\epsilon > 2/p$ ,  $p$  large enough,

$$\|u_t - I_{2h}^{k+2}u_{h,t}\|_1 \leq Ch^{k+2} \left[ \|u_{tt}(t)\|_{k+1} + \left( \int_0^t \|u_{ttt}\|_{k+1}^2 d\tau \right)^{1/2} + \|u(t)\|_{k+1} \right]. \tag{5.13}$$

**PROOF.** Obviously, by (5.4) and (5.5), we have

$$u - I_{2h}^{k+l}u_h = u - I_{2h}^{k+l}u + I_{2h}^{k+l}(i_h^k u - R_h u) + I_{2h}^{k+1}(R_h u - u_h), \tag{5.14}$$

$$\|u - I_{2h}^{k+l}u_h\|_{m,p} \leq \|u - I_{2h}^{k+l}u\|_{m,p} + C\|i_h^k u - R_h u\|_{m,p} + C\|R_h u - u_h\|_{m,p}, \tag{5.15}$$

for  $l = 1, 2$ . The estimates of first and third terms are shown in (5.3) and Theorems 2.6, 4.1, 2.3, 2.8, 2.5, 4.2, 2.9 and 2.4 (in this order). It remains estimate the second term. By [5, Corollary to Theorem 3.4.2],

$$\|i_h^k u - R_h u\|_{m,p} \leq Ch^{k+2} \|u\|_{k+3,p}, \quad 2 \leq p \leq \infty, \quad m = 0, 1, \tag{5.16}$$

so that

$$\|i_h^k u_t - R_h u_t\|_{m,p} \leq Ch^{k+2} \|u_t\|_{k+3,p}. \tag{5.17}$$

Thus, the proof is complete. □

**THEOREM 5.2.** For  $k = 1$ , we have

$$\|u - I_{2h}^2 u_h\|_1 \leq Ch^2 \left[ \left( \int_0^t \|u_{tt}\|_2^2 d\tau \right)^{1/2} + \|u(t)\|_3 \right], \tag{5.18}$$

$$\begin{aligned} & \|u - I_{2h}^2 u_h\|_{1,p} \\ & \leq Ch^2 \left[ \|u_t(t)\|_{2,p} + \|u_{tt}\|_2 + \left( \int_0^t \|u_{ttt}\|_2^2 d\tau \right)^{1/2} + \|u(t)\|_{3,p} \right], \quad 2 < p \leq \infty, \end{aligned} \tag{5.19}$$

$$\|u_t - I_{2h}^2 u_{h,t}\|_1 \leq Ch^2 \left[ \|u_{tt}(t)\|_2 + \left( \int_0^t \|u_{ttt}\|_2^2 d\tau \right)^{1/2} + \|u_t(t)\|_3 \right]. \tag{5.20}$$

**PROOF.** When  $k = 1$  and  $m = 1$  in (5.15)

$$\|u - I_{2h}^2 u_h\|_{1,p} \leq \|u - I_{2h}^2 u_h\|_{1,p} + C \|i_h^2 u - R_h u\|_{1,p} + C \|R_h u - u_h\|_{1,p}. \tag{5.21}$$

It suffices to estimate the second term. By [3], for any  $\chi \in S_h$

$$(\nabla(i_h^2 u - R_h u), \nabla\chi) = (\nabla(i_h^2 u - u), \nabla\chi) \tag{5.22}$$

$$= O(h^2) \|u\|_{3,p} \|\chi\|_{1,p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad p \geq 2. \tag{5.23}$$

Using the same method as in [4] we have

$$\begin{aligned} & \|i_h^2 u - R_h u\|_{1,p} \leq Ch^2 \|u\|_{3,p}, \\ & \|i_h^2 u_t - R_h u_t\|_{1,p} \leq Ch^2 \|u_t\|_{3,p}. \end{aligned} \tag{5.24}$$

These together with (3.7), (3.17), and (3.8) completes the proof. □

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