Numerical Methods for PDE

D.Y. Kwak

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Chapter 2

Variational Formulation

2.1 Boundary Value problems

Example 2.1.1 (One dim'l problem)**.**

$$-u'' = f \text{ on } I \equiv (0,1), \text{ with B.C. } u(0) = u(1) = 0.$$

Multiply a test function $v \in H_0^1(I)$ and integrate

$$(-u'', v) = -\int_0^1 u'' v dx \tag{2.1}$$

$$= -[u'v]_0^1 + \int_0^1 u'v' dx = \int_0^1 f v dx. \tag{2.2}$$

Thus we have

$$(u', v') = (f, v), \quad v \in V = H_0^1(I).$$

We will replace the space $H_0^1(I)$ by a finite dimensional space $S_h(I)$ of continuous, piecewise linear functions on I.

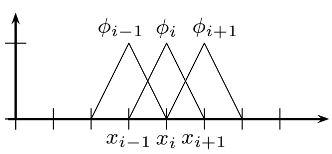


Figure 2.1: Basis in one-dimension

Let $h_i = x_i - x_{i-1},\ I_i = [x_{i-1}, x_i]$. Let $u_h = \sum_{j=1}^{n-1} c_j\phi_j$, where $\phi_i \in S_h(I)$ is the Lagrange basis function associated with the note x_i. Then substituting into (2.1), we obtain

$$\sum_{j=1}^{n-1} \int_0^1 c_j\phi_j'(x)\phi_i'(x)dx = \int_0^1 f(x)\phi_i(x)dx, \quad i = 1, 2, \cdots, n-1.$$

Hence we obtain the matrix equation

$$A_h u_h = f_h,$$

where

$$(A_h)_{ij} = a_{ij} = \int_0^1 \phi_j'(x)\phi_i'(x)dx, \quad f_i = \frac{h_{i-1} + h_i}{2} f(x_i).$$

If we use a uniform spacing, then a typical row of A_h is $[\cdots, 0, -1, 2, -1, 0\cdots]$. This matrix is the same as the one from FDM (up to the factor of h^2).

A typical row of A_h is

$$\left(\cdots, 0, -\frac{1}{h_{i-1}}, \frac{1}{h_{i-1}} + \frac{1}{h_i}, -\frac{1}{h_i}, 0, \cdots\right)$$

2.1.1 The Poisson equation on $\Omega \subset \mathbb{R}^2$

Let Ω be a bounded domain in $\mathbb{R}^2$ and $\partial\Omega$ denote its boundary.

We consider the Poisson equation with homogeneous boundary condition

$$\begin{cases} -\Delta u(x,y) &= f(x,y) \text{ for } (x,y) \in \Omega \\ u(x,y) &= 0 \text{ for } (x,y) \in \Gamma_1, m(\Gamma_1) > 0 \\ \frac{\partial u}{\partial \nu}(x,y) &= 0 \text{ for } (x,y) \in \Gamma_2, \end{cases} \tag{2.3}$$

where Δ is the Laplacian operator and $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ and $\Gamma_1 \cap \Gamma_2$ is a set of measure zero. It is well known that for sufficiently smooth boundary, there exists a unique classical solution $u \in C^2(\Omega), u \in C(\bar{\Omega})$.

We will assume that Ω is a normal domain, i.e., it admits the application of divergence theorem:

$$\int_\Omega \frac{\partial u}{\partial x_i} dxdy = \int_{\partial\Omega} u\nu_i ds, \quad i = 1, 2 \quad u \in C^1(\bar{\Omega}), \tag{2.4}$$

where ν_i are the components of unit outward normal vector to $\partial\Omega$.

Fact: Every polygonal domain or a domain with piecewise smooth boundary is a normal domain.

As a consequence of (2.4) we have the *Green's Formula.*

$$\int_\Omega v\Delta u dxdy = \int_{\partial\Omega} v\frac{\partial u}{\partial\nu}ds - \int_\Omega \nabla u\cdot\nabla v dxdy. \tag{2.5}$$

Suppose that u is a classical solution of (2.3) and that $v \in V = \{\phi \in C^\infty(\Omega) : \phi = 0 \text{ on } \Gamma_1\}$. Then

$$-\int_\Omega v\Delta u dxdy = \int_\Omega \nabla u\cdot\nabla v dxdy := \mathcal{A}(u,v).$$

Hence u satisfy

$$\mathcal{A}(u,v) = (f,v) \quad v \in V, \tag{2.6}$$

where $(f,v) = \int_\Omega f v dxdy$. $\mathcal{A}(u,v)$ is a bilinear form defined on $H^1(\Omega)$ and is called the *Dirichlet* integral associated with the Laplace operator $-\Delta$.

Since the space V is dense in $H^1_{\Gamma_1}(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1\}$, (2.6) actually holds for every $v \in H^1_{\Gamma_1}(\Omega)$, i.e,

$$\mathcal{A}(u,v) = (f,v) \quad v \in H^1_{\Gamma_1}(\Omega). \tag{2.7}$$

We now define a weak solution $u \in H^1_{\Gamma_1}(\Omega)$ of (2.3) as a function u in $H^1_{\Gamma_1}(\Omega)$ which satisfy (2.7). We can show the existence and uniqueness of this weak solution. There exists a unique $u \in H^1_{\Gamma_1}(\Omega)$ such that (2.7) holds. Moreover $\|u\|_1 \le c\|f\|_0$. More can be said about the solution. For example, if $\partial\Omega$ is of class C^r and $f \in H^{r-2}(\Omega)$, then

$$u \in H^r(\Omega) \cap H^1_{\Gamma_1}(\Omega) \tag{2.8}$$

and

$$\|u\|_r \le C\|f\|_{r-2}. \tag{2.9}$$

Results such as (2.8) and (2.9) are known as *elliptic regularity estimates.*

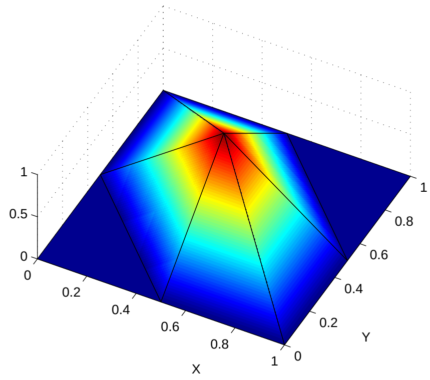


Figure 2.2: Standard nodal Lagrange local basis

An example of FEM

Example 2.1.2 (Poisson problem)**.**

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

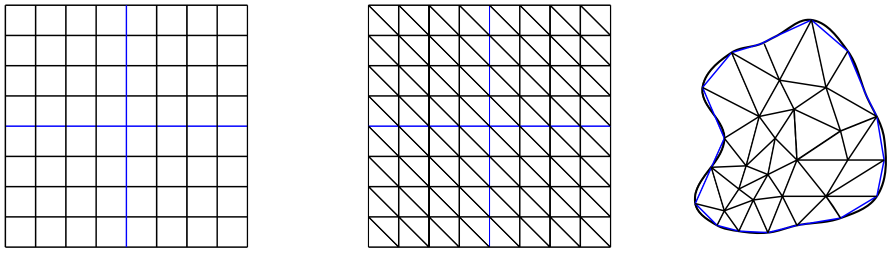


Figure 2.3: Finite element meshes

Assume $\Omega = [0,1]^2$ and that we divide Ω into $2n^2$ right triangles of length h. Let $S_h^0(\Omega)$ be the space of continuous, piecewise linear on each element satisfying zero boundary condition. Let $u_h = \sum_j u_j \phi_j$ where ϕ_j is the nodal(tent shape) basis function satisfying $\phi_j(x_i) = \delta_{ij}$. Then multiply ϕ_i and integrate by part to get

$$\int_\Omega \sum_j u_j \nabla\phi_j \cdot \nabla\phi_i \, dxdy = \int_\Omega f\phi_i \, dx, \text{ for each } i = 1, 2, \cdots .$$

Writing $a(\phi_i, \phi_j) = \int_\Omega \nabla\phi_j \cdot \nabla\phi_i dxdy$, we get

$$\sum_j a(\phi_i, \phi_j)u_j = (f, \phi_i).$$

In matrix form, it is

$$Au = f, \quad A_{ij} = a(\phi_j, \phi_i).$$

A is called the 'stiffness' matrix.

Example 2.1.3 (Neumann problem)**.**

$$\begin{aligned} -\Delta u + u &= f \text{ in } \Omega \\ \frac{\partial u}{\partial n} &= g \text{ on } \Gamma = \partial\Omega. \end{aligned} \tag{2.10}$$

Note in this case u is unknown at the boundary. So we set $V = H^1(\Omega)$(not $H_0^1(\Omega)$!). [1]

$$(-\Delta u, v) + (u, v) = (\nabla u, \nabla v) + (u, v) - < g, v >_\Gamma = (f, v).$$

So the variational problem is: (V) Find $u \in H^1$ such that

$$a(u, v) = (f, v) + < g, v >, \quad v \in H^1,$$

where $a(u, v) = (\nabla u, \nabla v) + (u, v)$.

Show: If $u \in C^2$ is the solution of (V) then it is the solution of (2.10).

Proof. Let u be the solution of (V). Then

$$\begin{aligned} a(u, v) &= (-\Delta u, v) = \int_\Gamma \frac{\partial u}{\partial n} v + (u, v) = (f, v) + < g, v > . \\ \int_\Omega (-\Delta u + u - f)v &= \int_\Gamma (g - \frac{\partial u}{\partial n}) v ds, \quad v \in H^1. \end{aligned}$$

Restrict to $v \in H_0^1$. Then we get

$$-\Delta u + u - f = 0 \text{ in } \Omega.$$

Hence we have

$$\int_\Gamma (g - \frac{\partial u}{\partial n}) v ds = 0, \quad v \in H^1$$

[1] If $g = 0$, we have a physically insulated boundary.

which proves $g = \frac{\partial u}{\partial n}$. The condition $\frac{\partial u}{\partial n} = g$ is called the natural boundary condition. (Look at the space V, we did not impose any condition, but we got B.C naturally from the variational formulation.) □

More general Boundary Conditions

We consider a mixed BC. i.e., on one part, the Dirichlet condition is imposed, while on the other part Neumann condition is imposed. We also consider more general coefficients,

Example 2.1.4. Assume there exists two positive constants p_0, p_1 s.t. $0 < p_0 \le p(x,y) \le p_1$. Consider

$$\begin{aligned} -\nabla \cdot p \nabla u &= f \text{ in } \Omega \\ u &= u_0 \text{ on } \Gamma_1 \\ \frac{\partial u}{\partial n} &= g \text{ on } \Gamma_2 := \partial\Omega \backslash \Gamma_1. \end{aligned} \tag{2.11}$$

Let $V_1 = \{v \in H^1(\Omega), v|_{\Gamma_1} = 0\}$. If $v \in V_1$,

$$\begin{aligned} (-\nabla \cdot p\nabla u, v) &= -\int_\Gamma p \frac{\partial u}{\partial n} v ds + \int_\Omega p \nabla u \cdot \nabla v dxdy \\ &= -\int_{\Gamma_2} p \frac{\partial u}{\partial n} v ds + \int_\Omega p \nabla u \cdot \nabla v \, dxdy = (f, v). \end{aligned}$$

The variational formulation is: Find u satisfying the Dirichlet condition such that

$$a(u, v) = \ell^*(v), \forall v \in V_1, \tag{2.12}$$

where $a(u, v) = (p\nabla u, \nabla v)$ and $\ell^*(v) = (f, v) + < pg, v >_{\Gamma_2}$.

Exercise 2.1.5. A fundamental solution of the PDE is the solution of

$$(L_y G(x,y))(x,y) = \delta_0(y - x), x, y \in \mathbb{R}^2,$$

in the distributional sense. **Greens** function of a PDE is a fundamental solution satisfying the boundary conditions. Greens functions are distributions.

(1) (10pts) In this problem we study Green's function

(a) Find a function $G \in H_0^1(I)$ satisfying

$$(G', v') = v(x_i), \quad \text{for all } v \in H_0^1(I). \tag{2.13}$$

(b) Assuming G'' exists in some sense, interpret

$$-(G'', v) = v(x_i), \text{ for } v \in C(I)$$

This means

$$-G''(x) = \delta(x_i)$$

where $\delta(x_i)$ the Dirac function. So we have a Green's function.

(c) Using this show we show that in one dimensional case, $u_h(x_i) = u(x_i)$ for each node.

(2) (10pts) For this and the next problem, assume $u_0 = g = 0$. Show the solution of (2.12) satisfies (2.11).

(3) (10pts) Show that we have an equivalent minimization problem : Find $u \in V_1$ such that $F(u) \leq F(v)$ for all v where

$$F(v) = \frac{1}{2}a(v,v) - (f,v) - < pg, v >_{\Gamma_2} .$$

(Hint) Take derivative of $F(u + \epsilon v)$ w.r.t ϵ and set it to 0 at $\epsilon = 0$ to obtain (2.12).

Exercise 2.1.6. (1) (trace thm) (10pts) Let Ω be a unit square. Assuming the trace of v exists along the boundary, show that

$$\left(\int_\Gamma v^2 ds \right)^{1/2} \leq C\|v\|_{H^1}, \quad \forall v \in H^1(\Omega).$$

(2) (10pts) Show that

$$\|v\|^2_{L^2(\Omega)} \leq C_1 |v|^2_{1,\Omega} + C_2 \left| \int_\Omega v\, dx \right|^2 . \tag{2.14}$$

(Hint: first assume $v \in C^1(\Omega)$.)

2.2 Variational formulation of BVP

In many cases, second order BVP can be cast into a minimization problem of certain (nonlinear) functional.

Definition 2.2.1. Let V be a set in a Hilbert space. Let $B(u_0, \epsilon) = \{u \in V : \|u - u_0\| < \epsilon\}$ be a neighborhoodof u_0. Let f be a real valued function defined on V. We say $u_0 \in V$ is a *local minimizer* of f if there exists an $\epsilon > 0$ such that $f(u_0) \leq f(u), \quad \forall u \in B(u_0, \epsilon)$. If $f(u_0) < f(u), \quad \forall u \in B(u_0, \epsilon)$ we say $u_0 \in V$ is a *strong local minimizer* of f.

Definition 2.2.2. $u_0 \in V$ is called a *global minimizer* of f if $f(u_0) \leq f(u), \quad \forall u \in V$.

Definition 2.2.3. Let $u, \eta \in V$ with $\|\eta\| = 1$. Suppose there is a $t_0 > 0$ such that the function defined by $g(t) = f(u + t\eta),\ |t| < t_0$ has continuous m-th derivative, then the m-th directional derivative of f at u is

$$f^{(m)}(u; \eta) = g^{(m)}(0) = \frac{d^m f(u + t\eta)}{dt^m}|_{t=0}.$$

Definition 2.2.4. If $f^{(1)}(u_0; \eta) = 0,\ \forall \eta \in V, \|\eta\| = 1$, then f is stationary at u_0.

Theorem 2.2.5. *Suppose there exists a $u_0 \in V$ such that $f^{(1)}(u_0; \eta)$ exist for all direction η. If u_0 is a local minimizer of f, then f is stationary at u_0.*

Proof. By Taylor expansion, $f(u_0 + t\eta) = f(u_0) + tf^{(1)}(u_0; \eta) + o(t),\ \|\eta\| = 1$. Suppose $f^{(1)}(u_0; \eta)$ is nonzero, say positive for some η. then there exists a t_0 such that $tf^{(1)}(u_0; \eta) + o(t) < 0$, for $-t_0 < t < 0$. Hence every nhd of u_0 has a point $u = u_0 + t\eta$ such that $f(u) < f(u_0)$, which is a contradiction. □

Conversely we have

Theorem 2.2.6. *Suppose f is C^2 and u_0 is a stationary point of f. Suppose $f^{(2)}(u_0; \eta) \geq 0$ for all direction η. Then u_0 is a local minimizer of f.*

2.2.1 Euler- Lagrange equation

Green's identities: Let Ω be a domain in $\mathbb{R}^2$ with piecewise smooth boundary. We have for $u, v \in C'(\bar{\Omega})$,

$$\int_\Omega u v_x dxdy = \int_{\partial\Omega} u v \nu_1 ds - \int_\Omega u_x v dxdy \qquad (2.15)$$

$$\int_\Omega u v_y dxdy = \int_{\partial\Omega} u v \nu_2 ds - \int_\Omega u_y v dxdy. \qquad (2.16)$$

Apply this to each component of $\vec{v} = (v_1, v_2)$ and add to get

$$\int_\Omega u\nabla \cdot \vec{v}dxdy = \int_{\partial\Omega} u\vec{v} \cdot \vec{\nu}ds - \int_\Omega \nabla u \cdot \vec{v}dxdy.$$

Now if $\vec{v}$ is replaced by ∇v

$$\int_\Omega u\Delta v dxdy = \int_{\partial\Omega} u\frac{\partial v}{\partial \nu}ds - \int_\Omega \nabla u \cdot \nabla v dxdy.$$

Interchanging u and v and subtracting,

$$\int_\Omega (u\Delta v - v\Delta u)dxdy = \int_{\partial\Omega} (u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu})ds,$$

where $\vec{\nu} = (\nu_1, \nu_2)$ is the outward unit normal vector to $\partial\Omega$.

A minimizer of a functional

Given a continuous function α defined on $\partial\Omega$, we let

$$V_\alpha = \{v \in C^2(\bar{\Omega}) : v = \alpha \text{ on } \partial\Omega\}$$

be the set of admissible functions. Then the corresponding test space is

$$V_0 = \{v \in C^2(\bar{\Omega}) : v = 0 \text{ on } \partial\Omega\}.$$

Consider a functional

$$f(u) = \int_\Omega F(x, y, u, u_x, u_y)dxdy, \ u \in V_\alpha.$$

We get a condition for stationary point for f by letting its first order directional derivative to be zero for all $\eta \in V_0$, i.e,

$$f^{(1)}(u; \eta) = \int_\Omega (\frac{\partial F}{\partial u}\eta + \frac{\partial F}{\partial u_x}\eta_x + \frac{\partial F}{\partial u_y}\eta_y)dxdy = 0. \tag{2.17}$$

Integrating by parts, we have

$$\int_\Omega \left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right]\eta dxdy \tag{2.18}$$

$$+\int_{\partial\Omega} \left(\nu_1\frac{\partial F}{\partial u_x} + \nu_2\frac{\partial F}{\partial u_y}\right)\eta ds = 0, \quad \forall \eta \in V_0. \tag{2.19}$$

Since $\eta = 0$ on $\partial\Omega$, the line integral vanishes and hence get

$$\int_\Omega \left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right] \eta dxdy = 0, \quad \forall \eta \in V_0 \tag{2.20}$$

which in turn implies

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0 \text{ in } \Omega. \tag{2.21}$$

This is called the *Euler Lagrange equation.* The boundary condition $u = \alpha$ is called *essential boundary condition.* To find the *natural boundary condition*, consider $V = \{v \in C^2(\bar{\Omega})\}$. From (2.18) the second term is zero since $V_0 \subset V$. Thus (2.21) still holds. Now from (2.18) again, we have

$$\int_{\partial\Omega} \left(\nu_1 \frac{\partial F}{\partial u_x} + \nu_2 \frac{\partial F}{\partial u_y}\right) \eta ds = 0, \quad \forall \eta \in V.$$

Since $\eta \in V$ can have nonzero boundary conditions, we have

$$\nu_1 \frac{\partial F}{\partial u_x} + \nu_2 \frac{\partial F}{\partial u_y} = 0 \text{ on } \partial\Omega.$$

This is natural boundary condition.

Example 2.2.7. Consider a functional $f(u)$ defined on V_0 by

$$f(u) = \int \left[\frac{1}{2} p(x,y)(u_x^2 + u_y^2) + \frac{1}{2} q(x,y) u^2 - r(x,y) u\right] dxdy. \tag{2.22}$$

Here

$$F(x, y, u, u_x, u_y) = \frac{1}{2} p(x,y)(u_x^2 + u_y^2) + \frac{1}{2} q(x,y) u^2 - r(x,y) u.$$

Thus Its Euler-Lagrange equation is

$$-(pu_x)_x - (pu_y)_y + qu = r. \tag{2.23}$$

The natural boundary condition is

$$p(\nu_1 u_x + \nu_2 u_y) = p \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{2.24}$$

We have shown the solution of the pde (2.23) is a minimizer (in the classical sense) of the functional (2.22) on V_0. It turns out that the minimizer u satisfies natural boundary condition.

In some problems the boundary condition may be imposed on some part of the boundary, say, on $\Gamma_0 \subset \partial\Omega$. Let

$$V_\alpha = \{v \in C^2(\bar{\Omega}) : v = \alpha \text{ on } \Gamma_0\}.$$

In this case the space of test function is

$$V_{\Gamma_0} = \{v \in C^2(\bar{\Omega}) : v = 0 \text{ on } \Gamma_0\}.$$

Then from (2.18), we have

$$\nu_1 \frac{\partial F}{\partial u_x} + \nu_2 \frac{\partial F}{\partial u_y} = 0 \text{ on } \Gamma_1 = \Gamma - \Gamma_0.$$

Boundary conditions of different type on different portion of boundary are called *mixed boundary conditions*.

2.2.2 Existence, Uniqueness of a Weak solution

In this section we rephrase previous discussions in a weaker sense. We deal weak solutions in Sobolev spaces.

Example 2.2.8. Consider

$$\begin{aligned} -\nabla \cdot p\nabla u &= g \text{ in } \Omega \\ u &= 0 \text{ on } \Gamma_1 \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_2 := \partial\Omega \backslash \Gamma_1. \end{aligned} \tag{2.25}$$

Let

$$V = \{v \in H^1(\Omega); v = 0 \text{ on } \Gamma_0\},$$

where $\Gamma_0 \subset \partial\Omega$ is nontrivial and $f \in L^2(\Omega)$. The variational form for this problem is

$$A(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dxdy, \quad v \in V \tag{2.26}$$

$$G(u) = \int_\Omega gu \, dxdy, \quad v \in V. \tag{2.27}$$

For the existence and the uniqueness, we need a theorem. Let H be a Hilbert space equipped with a norm $\|\cdot\|$.

Example 2.2.9. If we choose $H = H^1_{\Gamma_0} = \{v \in H^1, v|_{\Gamma_0} = 0\}$ and consider

$$\begin{aligned} -\Delta u &= g \text{ in } \Omega \quad (2.28)\\ u &= 0 \text{ on } \Gamma_0 \quad (2.29)\\ \frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial\Omega - \Gamma_0. \quad (2.30) \end{aligned}$$

Example 2.2.10. (Robin condition) Next we consider Robin problem.

$$\begin{aligned} -\Delta u &= g \text{ in } \Omega \quad (2.31)\\ \frac{\partial u}{\partial \nu} + \sigma u &= 0 \text{ on } \Gamma = \partial\Omega, \quad (2.32) \end{aligned}$$

where $\sigma \in C(\Gamma), g \in L^2(\Omega),\ 0 < \sigma_0 \le \sigma(x,y) \le \sigma_1,\ (x,y) \in \Gamma$. Then its corresponding variational problem is

$$\begin{aligned} A(u,v) &= \int_\Omega \nabla u \cdot \nabla v dxdy + \int_{\partial\Omega} \sigma uvds, \quad v \in V = H^1(\Omega) \quad (2.33)\\ G(v) &= \int_\Omega gvdxdy, \quad v \in V. \quad (2.34) \end{aligned}$$

2.2.3 More general coefficient

Example 2.2.11. Let $V = \{v \in H^1(\Omega), v = 0 \text{ on } \Gamma_0 \subset \partial\Omega\}$. Consider : find $u \in V$ satisfying

$$A(u,v) = G(v), \forall v \in V, \quad (2.35)$$

where

$$\begin{aligned} A(u,v) &:= \int_\Omega \sum_{i,j=1}^2 a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} dxdy + \int_\Omega cuv\, dxdy, \quad v \in V \quad (2.36)\\ G(v) &= \int_\Omega gv\, dxdy, \quad v \in V, \quad (2.37) \end{aligned}$$

and $a_{ij} = a_{ji} \in C(\bar{\Omega}),\ c \in C(\bar{\Omega}), c \geq 0,\ g \in L^2(\Omega)$. Assume there exists a constant $\lambda > 0$ such that

$$\sum_{i,j} a_{ij}\xi_i\xi_j \geq \lambda \sum_i \xi_i^2, \quad \forall (x,y) \in \Omega, \xi_i \in \mathbb{R}.$$

(This is equivalent to: eigenvalues of $\{a_{ij}\}$ are positive.) Hence we have $A(u,u) \geq \rho \|u\|_1^2, u \in V$. The corresponding boundary value problem is

$$\begin{aligned} \mathcal{L}u &= g \text{ in } \Omega & (2.38)\\ u &= 0 \text{ on } \Gamma_0 & (2.39)\\ \frac{\partial u}{\partial \nu_{\mathcal{L}}} &= 0 \text{ on } \partial\Omega - \Gamma_0, & (2.40) \end{aligned}$$

where $\mathcal{L}u = -\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + cu$ and $\frac{\partial u}{\partial \nu_{\mathcal{L}}}$ is the conormal derivative defined by

$$\frac{\partial u}{\partial \nu_{\mathcal{L}}} = \sum_{i,j} \nu_i a_{ij} \frac{\partial u}{\partial x_j}. \tag{2.41}$$

Exercise 2.2.12. (1) Derive (2.41) from (2.35).

2.2.4 Inhomogeneous boundary condition

Consider

$$\begin{aligned} \mathcal{L}u &= g \text{ in } \Omega & (2.42)\\ u &= \gamma \text{ on } \partial\Omega, & (2.43) \end{aligned}$$

where $\mathcal{L}u = -(pu_x)_x - (pu_y)_y + qu$ on Ω. Take $V = H_0^1(\Omega)$ and let

$$\begin{aligned} A(u,v) &= \int_\Omega (pu_xv_x + pu_yv_y + quv)dxdy,\ v \in V & (2.44)\\ G(v) &= \int_\Omega gv\, dxdy. & (2.45) \end{aligned}$$

Let $H_\gamma^1(\Omega) = \{u \in H^1(\Omega); u = \gamma \text{ on } \partial\Omega\}$.

Question

(1) Given a function $u \in H^1(\Omega)$ how do we define its restriction to $\partial\Omega$?

(2) Given a function $\gamma \in H^1(\Omega)$ does there exist $u \in H^1(\Omega)$ such that the restriction of u to $\partial\Omega$ is γ?

Now the functional to be minimized is

$$f(u) = \frac{1}{2}A(u,u) - G(u), \quad u \in H^1_\gamma(\Omega).$$

Note that $H^1_\gamma(\Omega)$ is not a linear space. For any fixed $u^* \in H^1_\gamma(\Omega)$, we can write

$$H^1_\gamma(\Omega) = \{u \in H^1(\Omega) : u = w + u^*, w \in H^1_0(\Omega)\}$$

and define

$$\begin{aligned} f^*(w) &= f(w+u^*) - f(u^*) && (2.46)\\ &= \frac{1}{2}A(w+u^*, w+u^*) - G(w+u^*) - \frac{1}{2}A(u^*,u^*) - G(u^*)) && (2.47)\\ &= \frac{1}{2}A(w,w) + A(u^*,w) - G(w) && (2.48)\\ &= \frac{1}{2}A(w,w) - G^*(w) && (2.49) \end{aligned}$$

where $G^*(w) := G(w) - A(u^*, w)$. Now Lax-Milgram lemma asserts that there exists a unique element $u_0 \in H^1_0(\Omega)$ (minimizer of $f^*(w)$) such that

$$A(u_0, v) = G^*(v), \quad v \in H^1_0(\Omega)$$

and hence

$$A(u_0 + u^*, v) = G(v), \quad v \in H^1_0(\Omega).$$

Let $u = u_0 + u^*$, then $u \in H^1_\gamma(\Omega)$ and satisfies

$$A(u, v) = G(v), \quad v \in H^1_0(\Omega).$$

u is the generalized solution of (2.42) since

$$A(u,v) - G(v) = (\mathcal{L}u - g, v), \quad v \in H^1_0(\Omega).$$

2.3 Ritz-Galerkn Method

For the simplicity of presentation, we assume Ω is a polygonal domain. Consider

$$-\Delta u = f \text{ in } \Omega \tag{2.50}$$

$$u = 0 \text{ on } \partial\Omega. \tag{2.51}$$

Assume Ω is partitioned by triangles. Let S_h be a finite dimensional subspaces of $H_0^1(\Omega)$. The finite element problem is to find a $u_h \in S_h$ satisfying

$$A(u_h, v_h) = f(v_h), \quad v_h \in S_h. \tag{2.52}$$

Let $u_h = \sum_{j=1}^{N} c_j \phi_j$. Then (2.52) becomes

$$A(\sum_{j=1}^{N} c_j \phi_j, \phi_i) = f(\phi_i), \; i = 1, ..., N \Rightarrow A \cdot \vec{c} = \vec{f}, \tag{2.53}$$

where $A_{ij} = A(\phi_j, \phi_i)$ and $f(\phi_i) = \int f \phi_i dxdy$. Such A is called the *stiffness matrix.*

2.3.1 Choice of S_h and its basis

Thus far we have no assumption on the shape (support, degrees, etc.) of basis functions. Basic idea is to choose a nice basis $\{\phi_i\}$ for S_h so that

(1) Easy to construct A

(2) A is sparse (To save storage and computations)

(3) The condition number of the matrix A is not too large.

Often, we use continuous functions which are piecewise linear on triangular elements. Note that in this case, $A_{ij} \neq 0$ only if the node i, j are adjacent. Thus the matrix is sparse.

2.3.2 Inhomogeneous boundary conditions

Consider solving

$$-\Delta u = f \text{ in } \Omega \tag{2.54}$$
$$u = \gamma \text{ on } \partial\Omega. \tag{2.55}$$

From earlier discussions, the variational formulation is to find a $u_h = \hat{u}_h + u^*$, where $\hat{u}_h \in S_h$ satisfies

$$A(\hat{u}_h, v_h) = f(v_h), \quad v_h \in S_h$$

where u^* is any function in $H^1_\gamma(\Omega) = \{u \in H^1(\Omega), u = \gamma \text{ on } \partial\Omega\}$. Then it amounts to find $\hat{u}_h \in S_h$ such that

$$A(\hat{u}_h, v_h) = f(v_h) - A(u^*, v_h), \quad v_h \in S_h.$$

In matrix form,

$$A \cdot \vec{c} = \vec{f^*}$$

where $f^*_i = f(\phi_i) - A(u^*, \phi_i)$. The Ritz approximation is then

$$u_h = \sum_{j=1}^{N} c_j \phi_j + u^*.$$

Assume the number of unknowns on the boundary is p and let $\{\phi_j\}_{j=N+1}^{N+p}$ are the piecewise linear basis associated with the boundary. One often try to approximate u^* in the form $u^* \doteq \sum_{j=N+1}^{N+p} \alpha_j \phi_j$ so that

$$\sum_{j=N+1}^{N+p} \alpha_j \phi_j(x_j, y_j) = \gamma(x_j, y_j), \ (x_j, y_j) \in \partial\Omega_h(\text{boundary nodes}). \tag{2.56}$$

In particular, if ϕ_j are Lagrange basis so that $\phi_j(x_i, y_i) = \delta_{ij}$, then u^* ca be replaced by $\sum_{j=N+1}^{N+p} \gamma_j \phi_j$.

2.4 Finite Element Method -A Concrete Ritz-Galerkin method

2.4.1 Finite element basis functions

We assume $\bar{\Omega}$ is subdivided by a non-overlapping elements, say triangles or rectangles of certain regular shape. For a mesh generator, see

http://www.cs.cmu.edu/ quake/triangle.html

Notations

- L: Total number of elements
- M: Total number of nodes
- T: Number of nodes in a single element
- e_ℓ: $\ell = 1, 2, \cdots, L$ the elements
- $N_i = (x_i, y_i)$: the nodes
- $\hat{e}$: the standard(reference) element

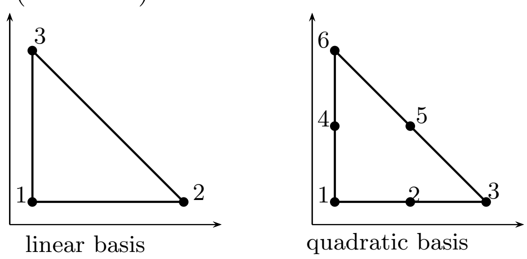


Figure 2.4: Reference element and nodes

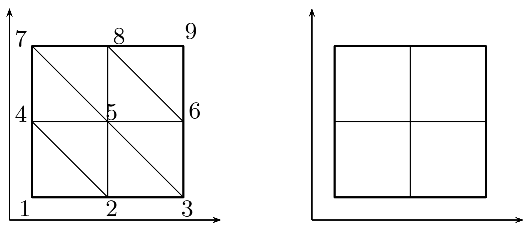


Figure 2.5: Global numbering of Elements and nodes

Since the support of ϕ_i is usually a small subset of $\bar{\Omega}$, we say that ϕ_i has *local support*. A rough geometrical description of ϕ_i is that of a "tent"

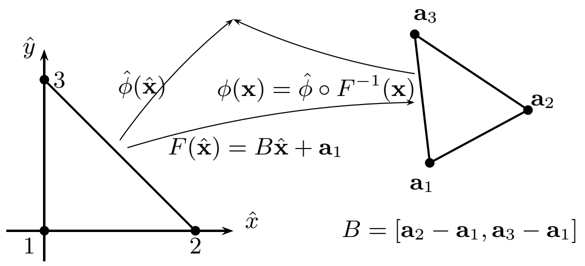


Figure 2.6: Reference triangle and the mapping

centered about N_i. The floor of the tent is the support of ϕ_i. If $N_i \notin \partial\Omega = \Gamma$, ϕ_i vanishes on the boundary of its support.

By definition, a function u belongs to $S_M = S_h$ if and only if it can be expressed as

$$u(x, y) = \sum_{i=1}^{M} c_i \phi_i(x, y), \quad (x, y) \in \bar{\Omega}.$$

Each u is a continuous, piecewise polynomial over $\bar{\Omega}$.

Interpolation

Define the interpolation $I_h : C(K) \to S^h(K)$ by the conditions $(I_h u)(a_i) = u(a_i)$ for all $i = 1, \cdots, T$. For any $u \in C(\bar{\Omega})$ let $u_I \in S_M$ be defined by

$$u_I(x, y) = \sum_{i=1}^{M} u(N_i)\phi_i(x, y), \quad (x, y) \in \bar{\Omega}.$$

The function u_I is the interpolant of u in S_M.

For error analysis we need to view u_I as interpolant of $u \in H^p(\Omega)$. But a function in $H^p(\Omega)$ is actually an equivalence class of functions defined a.e, $u(N_i)$ is not well-defined.

If u is piecewise polynomials then

$$C^{m-1}(\bar{\Omega}) \subset H^m(\bar{\Omega}), \quad H^{m+1}(\bar{\Omega}) \subset C^{m-1}(\bar{\Omega})$$

In general, this does not hold since there exists nowhere differentiable continuous functions. If $m = 1$, then V_M is appropriate for 2nd order elliptic problem, called "conforming". Referring to the figure (2.6), the nodal basis functions are

$$\hat{\phi}_1 = 1 - \hat{x} - \hat{y},\ \hat{\phi}_2 = \hat{x},\ \hat{\phi}_3 = \hat{y}. \tag{2.57}$$

Example 2.4.1. Piecewise linear basis on triangular element. First, on the standard refernce basis element $\hat{e}$,

$$\hat{\phi}_r(\hat{x},\hat{y}) = \hat{c}_r^1 + \hat{c}_r^2\hat{x} + \hat{c}_r^3\hat{y}, r = 1,2,3.$$

Let the local basis function on a general element e_ℓ be give by

$$\phi_r^\ell(x,y) = c_{r,\ell}^1 + c_{r,\ell}^2 x + c_{r,\ell}^3 y, (x,y) \in e_\ell,\ \ell = 1\cdots,L,\ \ r = 1,2,3.$$

It is nothing but the restriction of global basis function $\phi_{i_r^\ell}(x,y), i = 1,\cdots,M$. Since the affine map F_ℓ is of the form

$$\begin{bmatrix} x \\ y \end{bmatrix} = F_\ell \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} a_1\hat{x} + b_1\hat{y} + c_1 \\ a_2\hat{x} + b_2\hat{y} + c_2 \end{bmatrix} \equiv B_\ell \hat{\mathbf{x}} + \hat{d}_\ell, \tag{2.58}$$

we have

$$\phi_r^\ell(x,y) = \hat{\phi}_r(\hat{x},\hat{y}) = \hat{\phi}_r \circ F_\ell^{-1}(x,y).$$

We use conventional counterclockwise ordering as local ordering.

Example 2.4.2. (Piecewise bilinear basis on rectangular elements) Let

$$\phi_r(x,y) = c_1 + c_2 x + c_3 y + c_4 xy.$$

Then the standard basis functions on $\hat{e}$ are

$$\hat{\phi}_1(\hat{x},\hat{y}) = \frac{1}{4}(1-\hat{x})(1-\hat{y}), \qquad \hat{\phi}_2(\hat{x},\hat{y}) = \tfrac{1}{4}(1+\hat{x})(1-\hat{y}) \tag{2.59}$$

$$\hat{\phi}_3(\hat{x},\hat{y}) = \frac{1}{4}(1+\hat{x})(1+\hat{y}), \qquad \hat{\phi}_4(\hat{x},\hat{y}) = \tfrac{1}{4}(1-\hat{x})(1+\hat{y}). \tag{2.60}$$

2.4.2 Assembly of stiffness matrix

Consider

$$-\nabla\cdot p\nabla u + qu = g \text{ in } \Omega \tag{2.61}$$

$$u = \gamma \text{ on } \Gamma_0 \tag{2.62}$$

$$u_\nu + \sigma u = \xi \text{ on } \Gamma_1, \quad \Gamma = \Gamma_0 \cup \Gamma_1. \tag{2.63}$$

Let

$$H^1_\gamma(\Omega) = \{v \in\ H^1(\Omega) : v = \gamma \text{ on } \Gamma_0\} \text{ (affine)}$$

$$H^1_{\Gamma_0}(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_0\} \text{ (linear)}$$

Find $u \in H^1_\gamma(\Omega)$ such that

$$a(u,v) = G(v), \quad \forall v \in H^1_{\Gamma_0}(\Omega), \tag{2.64}$$

where

$$\begin{aligned} a(u,v) &= \iint p u_x v_x + p u_y v_y + q u v dx + \int_{\Gamma_1} p\sigma u v ds & (2.65)\\ &= (p\nabla u, \nabla v) + (qu, v) + < p\sigma u, v >_{\Gamma_1} & (2.66)\\ &= a(u,v)_\Omega + < p\sigma u, v >_{\Gamma_1} & (2.67)\end{aligned}$$

and

$$\begin{aligned} G(v) &= (g,v) + < p\xi, v >_{\Gamma_1} & (2.68)\\ &= G(v)_\Omega + < p\xi, v >_{\Gamma_1} & (2.69)\end{aligned}$$

Then as shown before, the solution u minimizes the functional

$$f(u) = \frac{1}{2}a(u,u) - G(u) \quad \forall u \in H^1_\gamma(\Omega).$$

If u^* is any function in $H^1_\gamma(\Omega)$ one can see

$$H^1_\gamma(\Omega) = H^1_0(\Omega) + u^* = \{u \in H^1(\Omega) : u = u_0 + u^*, u_0 \in H^1_{\Gamma_0}(\Omega)\}$$

and (2.64) is equivalent to finding $u_0 \in H^1_{\Gamma_0}(\Omega)$ such that

$$a(u_0, v) = G(v) - a(u^*, v), \ \forall v \in H^1_{\Gamma_0}(\Omega) \tag{2.70}$$

2.5 Outline of Programming

Let $\mathcal{T}_h = \{K_\ell\}$ be a triangulation of the domain Ω and let

$$V_N = span\{\phi_i \text{ linear on each element and }, \phi_i \in H^1_{\Gamma_0}(\Omega)\}.$$

Let us list some notations:

- L: Total number of elements
- M: Total number of nodes
- Γ_0: the part of boundary where Dirichlet condition is imposed
- M_0: the number of nodes on Γ_0
- $\mathcal{J}_0$: the index set of node numbers of nodes on Γ_0 (Dirichlet condition)
- Γ_1: the part of boundary where Neumann condition is imposed
- L_1: the number of element edges on Γ_1
- N: number of nodes in $\Omega \cup \Gamma_1$ (Total unknowns)
- $\mathcal{J}$: index set of node numbers of nodes in $\Omega \cup \Gamma_1$
- Q: number of integration points in each element

To form a finite element matrix, we need to replace (2.70) by a finite dimensional analog: Find $u_h := u_N + u_N^*$ s.t.

$$a(u_N, v) = G(v) - a(u_N^*, v), \ \forall v \in V_N, \tag{2.71}$$

where

$$u_N = \sum_{j \in \mathcal{J}} \alpha_j \phi_j$$

and u^* is replaced by a P. L. function satisfying the BC(at least weakly.) We usually choose

$$u_N^* = \sum_{i \in \mathcal{J}_0} \gamma_i \phi_i \text{ so that } u_N^*(N_i) = \gamma_i(N_i).$$

Even though $u_N^* \notin H_\gamma^1(\Omega)$, it would suffice our purpose.

A tip for the boundary nodes

When $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 \neq \emptyset$, i is important for application to

(1) place nodes at the end points of $\bar{\Gamma}_0$.

(2) assign nodes to $\bar{\Gamma}_0$ rather than to Γ_1.

Assembly of stiffness matrix

We usually compute so called *element stiffness matrix* and sum them over all elements to assemble the global stiffness matrix, denoted by K here. From (2.71) we have

$$a(u_N,\phi_i)=\sum_{j\in\mathcal{J}}\alpha_j\int_\Omega(p\nabla\phi_j\cdot\nabla\phi_i+q\phi_j\phi_i)+\sum_{j\in\mathcal{J}}\alpha_j\int_{\Gamma_1}p\sigma\phi_j\phi_i ds,\ i\in\mathcal{J},$$

and

$$G_i^*:=G(\phi_i)-a(u^*,\phi_i)=\int_\Omega g\phi_i+\int_{\Gamma_1}p\xi\phi_i\,ds-\sum_{j\in\mathcal{J}_0}\gamma(N_j)a(\phi_j,\phi_i),\ i\in\mathcal{J}.$$

Hence we get

$$K\alpha=G^*,\ \alpha=(\alpha_j)_{j\in\mathcal{J}}$$

where

$$k_{ij}=\int_\Omega(p\nabla\phi_j\cdot\nabla\phi_i+q\phi_j\phi_i)+\int_{\Gamma_1}p\sigma\phi_j\phi_i ds.$$

Some general issues:

(1) Input data : $\Omega,\Gamma_0,\Gamma_1,g,\gamma,\xi$, coefficients, etc.

(2) Construction and representation of $\mathcal{T}_h$

(3) Computation of element stiffness matrix a^K and b^K

(4) Assembly of global stiffness matrix K, G^*

(5) Solver of $K\alpha=G^*$

(6) Presentation of result. Discrete L^2, H^1-error. Numerical Table, order of convergence, graphics.

Remark on (2): quasi uniform—essentially the same size, but it is desirable to vary the size of triangle—adaptive or successive refinement. Conforming: vertex should not lie in the interior of an edge.

We need to compute k_{ij}.

(1) $k_{ij}:=a(\phi_j,\phi_i)_\Omega,\ i,j=1,\cdots,M$

(2) $G_i:=G(\phi_i)_\Omega,\ i=1,\cdots,M$

(3) $k_{ij} := k_{ij} + a(\phi_j, \phi_i)_{\Gamma_1},\ i, j, = 1, \cdots, M$

(4) $G_i := G_i + G(\phi_i)_{\Gamma_1},\ i = 1, \cdots, M$

(5) For $j \in \mathcal{J}_0$:

(a) $G_i := G_i - \gamma(N_j)k_{ij},\ i \in \mathcal{J}$

(b) $k_{ij} := 0 = k_{ji},\ i \in \mathcal{J}$

(c) $k_{ji} := 0,\ i \in \mathcal{J}_0,\ i \neq j$

(d) $G_j := \gamma(N_j); k_{jj} = 1$

Since k_{ij} is symmetric, the computation in (1), (3) and (5) are done for $j = 1, \cdots, i$ only to save time and memory. (For places where k_{ji} appears for k_{ij} we compute for $j > i$ instead. For details see Axellson p. 185)

Remark 2.5.1. (1) We used full matrix notation k_{ij} for simplicity of presentation; However, one need to exploit the sparseness of the matrix to save memory. So it may be nice to provide a single array $KSingle[5 * M]$(for 5 point stencil) for $k[i, j]$ and write subroutines such as

(1) $DoubleToSingle(i, j), \quad 1 \le i, j \le M$

(2) $SingleToDouble(j), \quad j = 1, \cdots, M \times M$

Then step (a) of (5) may look like this:

$$KSingle[DoubleToSingle(i, j)] = a(\phi_j, \phi_i)$$

Here one do not save $a(\phi_j, \phi_i)$. Instead, it is stored into $KSingle[DoubleToSingle(i, j)]$ as soon as it is computed, as shown in the next.

One can use class file to define matrix-function that looks like $A(i, j)$ but has single array structure of length $5M$.

(2) The true number of unknowns are N, not M. The (d) of step (5) means we append the following identity equations to the $N \times N$ equations;

$$u(N_i) = \gamma(N_i), \quad i \in \mathcal{J}_0$$

2.5.1 Computation of $a(\phi_i, \phi_j)$ elementwise.

$$a(\phi_i, \phi_j) = \sum_K \int_K (p\nabla\phi_i \cdot \nabla\phi_j + q\phi_i\phi_j) + \sum_K \int_{\bar{K}\cap\Gamma_1} p\sigma\phi_i\phi_j ds := \sum_K a_{ij}^K$$

where the summation runs through the common support of ϕ_i and ϕ_j. We find this by computing the contribution of $a_K(\phi_i, \phi_j)$ called the *element stiffness matrix.*

Some notations:

L: number of triangles(elements)

M: number of total nodes

K_ℓ, $\ell = 1, \cdots, L$: the elements

Z: $2 \times M$ matrix, $Z(j,i), j = 1,2$, are the coordinates of node i. - vertex coordinates table.

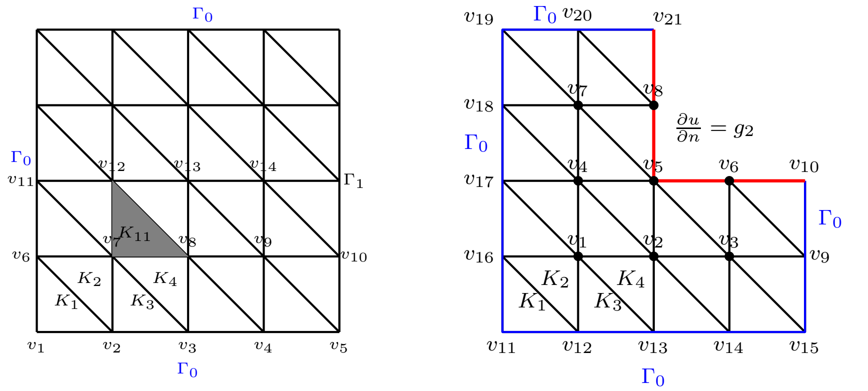
T: $3 \times L$ matrix, $T(\alpha, \ell), \alpha = 1,2,3$, denotes the global node numbering of ℓ-th triangle - element node table.

A triangulation $\mathcal{T}_h$ may be represented by two matrices $Z : 2 \times M$ matrix and $T : 3 \times L$ matrix. Node ordering is important if we want to use Gaussian elimination (For instance, we intend to store A as a banded matrix, hopefully with small band).

Example 2.5.2. Let us divide a unit square by 4×4 uniform meshes where each sub-rectangle is subdivided by the diagonal of slope -1. Label all the vertex nodes $1, 2, 3, \cdots, 25$ lexicographically. Label the elements from lower left corner as $K_1 \backslash K_2$, $K_3 \backslash K_4$, $K_5 \backslash K_6, \cdots$.

Compute the element stiffness matrix for each triangle, and add all the contribution to three vertices as K runs through all element. If $i = j$, K runs through all element having the node i as a vertex. If $i \neq j$, K runs through all element having the line segment $\bar{ij}$ as an edge. In this way, we assemble the global matrix A all together (not for each entry)

Consider $K = K_{11}$. The indices for its vertices are $7, 8$ and 12. For the



(a) Rectangular domain (b) Lshape, mixed BC, unknowns at •

Figure 2.7: Label of elements and vertices

element matrix we need to compute $a_K(\phi_i, \phi_j)$ for $i, j = 7, 8, 12$.

$$
\begin{aligned}
a_K(\phi_7, \phi_7) &= \int_K (-\frac{1}{h}, -\frac{1}{h}) \cdot (-\frac{1}{h}, -\frac{1}{h}) = 1 \\
a_K(\phi_7, \phi_8) &= \int_K (-\frac{1}{h}, -\frac{1}{h}) \cdot (\frac{1}{h}, 0) = -\frac{1}{2} \\
a_K(\phi_8, \phi_8) &= \int_K (\frac{1}{h}, 0) \cdot (\frac{1}{h}, 0) = \frac{1}{2} \\
a_K(\phi_8, \phi_{12}) &= \int_K (\frac{1}{h}, 0) \cdot (0, \frac{1}{h}) = 0
\end{aligned}
$$

The element stiffness matrix $A_{K_{11}}$ (corresponds to the vertices 7,8 and 12) is

$$
\begin{bmatrix} 1, & -\frac{1}{2}, & -\frac{1}{2} \\ -\frac{1}{2}, & \frac{1}{2}, & 0 \\ -\frac{1}{2}, & 0, & \frac{1}{2} \end{bmatrix} = a^{K_{11}}_{\alpha,\beta}
$$

Generate element matrices for all element $K = 1, 2 \cdots$, add its contribution to all pair of vertices (i, j). Let L be the number of elements and let T be the $3 \times L$ matrix whose ℓ-th column denotes the three *global* indices of vertices of ℓ-th element. For example, $T(\cdot, 11) = [7, 8, 12]^t$.

$$Z = \begin{bmatrix} 0 & 0.25 & 0.5 & 0.75 & 1.0 & 0 & 0.25 & \cdots \\ 0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.25 & 0.25 & \cdots \end{bmatrix} \tag{2.72}$$

$$T = \begin{bmatrix} 1 & 2 & 2 & 3 & \cdots \\ 2 & 7 & 3 & 8 & \cdots \\ 6 & 6 & 7 & 7 & \cdots \end{bmatrix} \tag{2.73}$$

(3) Computation of element stiffness matrix.

Let $K_\ell \in \mathcal{T}_h$ be a fixed element. Then $T(\alpha, \ell)$, $\alpha = 1, 2, 3$, are the global numbering of vertices of K_ℓ. The x_i-coordinates of vertices are $Z(i, T(\alpha, \ell)), i = 1, 2$. We note that

$$a^\ell_{\alpha,\beta} := a^{K_\ell}_{\alpha,\beta} = \int_{K_\ell} (p\nabla\phi_\alpha \cdot \nabla\phi_\beta + q\phi_\alpha\phi_\beta)dx + \int_{\bar{K}_\ell \cap \Gamma_1} p\sigma\phi_\alpha\phi_\beta \, ds,$$

$$b^\ell_\alpha = \int_{K_\ell} g\phi_\alpha \, dx - \sum_{j\in\mathcal{J}_0} \gamma(N_j) a_{K_\ell}(\phi_j, \phi_\alpha) + \int_{\bar{K}_\ell \cap \Gamma_1} p\xi\phi_\alpha \, ds, \ \alpha = 1, 2, 3.$$

Then the assembly of global stiffness matrix is as follows:

(4) Assembly of global stiffness matrix: Let $k(M, M), \mathbf{b}(M)$ be arrays. Initially set $k(i, j) = 0, \mathbf{b}(i) = 0, i, j = 1, \cdots, M$.

For each $\ell = 1, 2, \cdots, L$, compute $(a^\ell_{\alpha\beta})$ and $\mathbf{b}^\ell = (b^\ell_\alpha)$ and set

$$\begin{aligned} k(T(\beta, \ell), T(\alpha, \ell))+ = a(T(\alpha, \ell), T(\beta, \ell))+ &= a^\ell_{\alpha\beta}, \quad \alpha, \beta = 1, 2, 3 \\ \mathbf{b}(T(\alpha, \ell))+ &= b^\ell_\alpha, \quad \alpha = 1, 2, 3. \end{aligned}$$

Remark 2.5.3. 1). This process corresponds to step (1) and (2) in previous section.

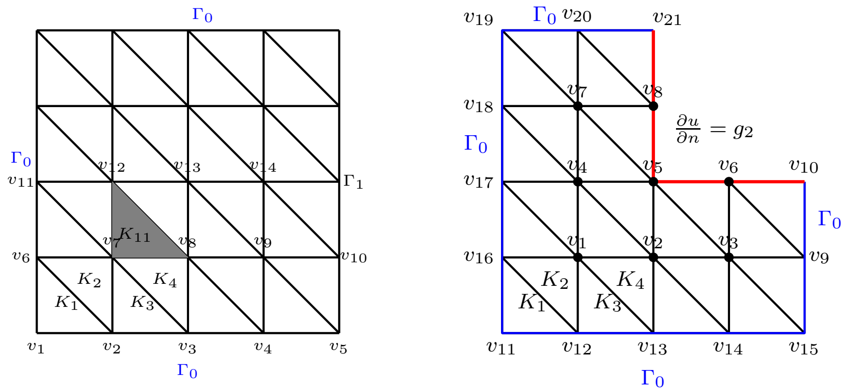
2). In practice we do not use the full array $k(M, M)$. Instead use either banded matrix if Gaussian elimination is used(How big is the band and what happens to the band during the elimination ?) or store only nonzero element if iterative methods are used.

For FEM software; http://www.netlib.org, http://gams.nist.gov( )

Finite element method (FEM) is a powerful and popular numerical method on solving partial differential equations (PDEs), with flexibility in dealing with complex geometric domains and various boundary conditions. MATLAB (Matrix Laboratory) is a powerful and popular software platform using matrix-based script language for scientific and engineering calculations. This project is on the development of an finite element method package in MATLAB based on an innovative programming style: sparse matrixlization. That is to reformulate algorithms in terms of sparse matrix operations to make use of the unique strength of MATLAB on fast matrix operations. iFEM, the resulting package, is a good balance between simplicity, readability, and efficiency. It will benefit not only education but also future research and algorithm development on finite element method.

This package can be downloaded from http://ifem.wordpress.com/

Exercise 2.5.4.



(a) Rectangular domain (b) Lshape, mixed BC, unknowns at •

Figure 2.8: Label of elements and vertices

Mid term take home exam Due Oct 28

Write a FEM code for the Robin problem with the following data. Use the uniform grids of $h = 2^{-k}, k = 2, 3, \cdots, 6$. For all problems, $q = \sigma = \xi = 0$.

(1) Compute the 3×3 element stiffness matrix $A_{K_{11}}$ for the case $p = 1 + x + 2y^2$ when $k = 2$(4×4 grid shown in the note above.)

(2) Print the entries of $G(i), i = 4, 5, 6, 7$ when $k = 2$ for $p = 1 + x + 2y^2$, $g = -(1 + 4y)$ and $\gamma = 1 + x + y$ on the boundary.

(3) Solve the problem for the case $p = 1+x+2y^2$, $\gamma = 0$ (Dirichlet condition on $\partial\Omega$) and $u = x(1 - x)y(1 - y)$.

(4) Solve the problem for the case $p = 1$, $u = 1 + x + 9y$, $g = 0$. $\gamma = u|_{\partial\Omega}$ (Dirichlet condition on $\partial\Omega$). Do you find any special phenomena?

(5) Draw the graph of solution u_h of (3).

To solve the linear system use either Gauss-Seidel method or conjugate gradient method. Report discrete L^2-norm defined by $\|u-u_h\|_h := \sqrt{h^2 \sum_i (u - u_h)^2(c_i)}$. Here c_i is the centroid of each element (triangle). Write the Table in a easily verifiable manner (systematically) for $h = 1/2^k$. Submit the paper report and the coding(by email).

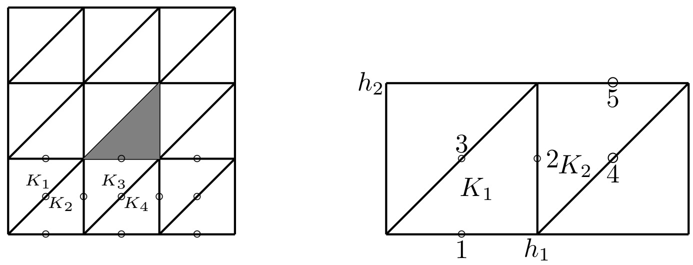


Figure 2.9: Label of elements and vertices

2.5.2 P_1 Nonconforming space of Crouzeix and Raviart

We introduce a P_1 nonconforming finite element method for $-\Delta u = f$. Assume a quasi uniform triangulation of the domain by triangles is given. Consider the space of all piecewise linear functions which is continuous only at mid point of edges. Here the degree of freedom is located at mid point of edges.

Let N_h be the space of all functions which is linear on each triangle and whose degrees of freedoms are determined

$$\begin{cases} u_h(m)|_L = u_h(m)|_R & \text{when } m \text{ is a mid point of interior edges} \\ u_h(m) = 0 & \text{when } m \text{ is a mid point of boundary edges} \end{cases}$$

Since u_h is discontinuous, the $a(u_h, v_h) = \int_\Omega \nabla u_h \cdot \nabla v_h \, dx$- is not well defined. So we define a discrete form a_h as follows:

$$a_h(u_h, v_h) = \sum_{T \in \mathcal{T}_h} \int_T \nabla u_h \cdot \nabla v_h \, dx \tag{2.74}$$

The P_1-nonconforming fem is: Find $u_h \in N_h$ such that

$$a_h(u_h, v_h) = f(v_h), \quad \forall v_h \in N_h.$$

Note that in general

$$a_h(u, v_h) \neq f(v_h).$$

We define a discrete norm on N_h by

$$\|u_h\|_h = a_h(u_h, u_h)^{1/2}.$$

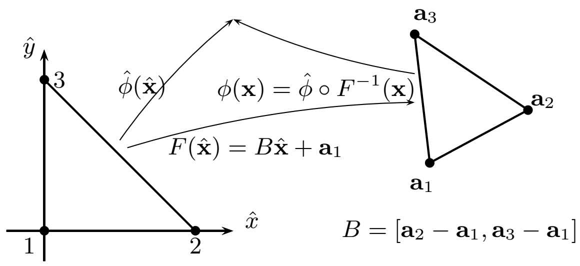


Figure 2.10: Reference triangle and the mapping

Assume the reference element $\hat{K} = K_1$ with $h_1 = h_2$. Then the basis functions are

$$\hat{\phi}_1 = 1 - 2y, \quad \hat{\phi}_2 = 2x - 1, \quad \hat{\phi}_3 = 2y - 2x + 1. \tag{2.75}$$

Consider K_2. By mapping $F_{K_2}(\hat{\mathbf{x}}) = B_K\hat{\mathbf{x}} + \mathbf{b}$ we have

$$\begin{aligned} F_{K_2}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} &= \begin{pmatrix}-h_1 & 0\\ 0 & -h_2\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} + \begin{pmatrix}2h_1\\ h_2\end{pmatrix} \\ \phi_i(\mathbf{x}) &= \hat{\phi}_i \circ (F_K^{-1})(\mathbf{x}) = \hat{\phi}_i \begin{pmatrix}-\frac{1}{h_1} & 0\\ 0 & -\frac{1}{h_2}\end{pmatrix}\begin{pmatrix}x - 2h_1\\ y - h_2\end{pmatrix}, \; i = 2, 4, 5 \end{aligned}$$

Hence (or by direct computation) we can obtain associated with functions

$$\phi_2 = 3 - \frac{2x}{h_1}, \quad \phi_5 = -1 + \frac{2y}{h_2}, \quad \phi_4 = -1 - \frac{2x}{h_1} + \frac{2y}{h_2} \text{ on } K_2$$

2.5.3 Integration using reference element

In practice K is in a general position. Hence we show how to compute the integral

$$a_{ij}^K = \int_K (p\nabla\phi_i \cdot \nabla\phi_j + q\phi_i\phi_j)dxdy$$

through a mapping to a fixed "nice" reference element $\hat{K}$. Let

$$\mathbf{x} = \begin{pmatrix}x\\ y\end{pmatrix} = F(\hat{\mathbf{x}}) = \begin{pmatrix}f_1(\hat{x}, \hat{y})\\ f_2(\hat{x}, \hat{y})\end{pmatrix} \tag{2.76}$$

be a one-to-one invertible map $\hat{K} \to K$, Then any function and $g(\mathbf{x})$ is related to a function $\hat{g}(\hat{\mathbf{x}})$ defined on the reference element $\hat{K}$ by

$$g(\mathbf{x}) = g(F(\hat{\mathbf{x}})) = \hat{g}(\hat{\mathbf{x}}).$$

In particular, if it is affine then $F(\hat{\mathbf{x}}) = B\hat{\mathbf{x}} + \mathbf{b}$. For a scalar function g, we see

$$\int_K g(\mathbf{x})dxdy = \int_{\hat{K}} g(F(\hat{\mathbf{x}}))|J_K|d\hat{x}d\hat{y}, \tag{2.77}$$

where $J_K = det(DF_K) = det(B)$. But for a gradient of a function, it is more complicated: Noting that

$$\hat{\nabla}\hat{g} = B^T \nabla g, \tag{2.78}$$

we see

$$\int_K (p\nabla\phi_i \cdot \nabla\phi_j + q\phi_i\phi_j)dxdy = \int_{\hat{K}} (\hat{p}(B^{-T}\hat{\nabla}\hat{\phi}_i) \cdot (B^{-T}\hat{\nabla}\hat{\phi}_j) + \hat{q}\hat{\phi}_i\hat{\phi}_j)|J|d\hat{x}d\hat{y}, \tag{2.79}$$

where J is the determinant of B. To save computational cost in computing (2.79) we do as follows: A little of algebra shows that (2.79) is

$$a_{ij}^K = \int_{\hat{K}} \left\{ \frac{\hat{p}}{|J|} \left[E_1 \hat{\phi}_{\hat{x}}^i \hat{\phi}_{\hat{x}}^j - E_2(\hat{\phi}_{\hat{y}}^i \hat{\phi}_{\hat{x}}^j + \hat{\phi}_{\hat{x}}^i \hat{\phi}_{\hat{y}}^j) + E_3 \hat{\phi}_{\hat{y}}^i \hat{\phi}_{\hat{y}}^j \right] + |J|\hat{q}\hat{\phi}^i\hat{\phi}^j \right\} d\hat{x}d\hat{y},$$

where

$$J = x_{\hat{x}}y_{\hat{y}} - x_{\hat{y}}y_{\hat{x}}, \quad E_1 = x_{\hat{y}}^2 + y_{\hat{y}}^2 \tag{2.80}$$

$$E_2 = x_{\hat{x}}x_{\hat{y}} + y_{\hat{x}}y_{\hat{y}}, \quad E_3 = x_{\hat{x}}^2 + y_{\hat{x}}^2. \tag{2.81}$$

Things to consider

(1) Use banded storage

(2) Use as many modules as possible.

(3) Iterative method or direct method ?

(4) Output. Check the error by discrete L^2, H^1 inner product. Graphics.

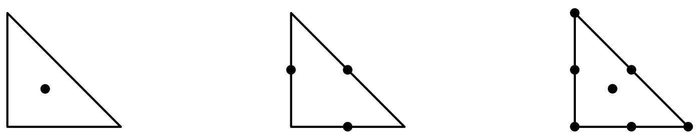


Figure 2.11: Quadrature points for triangle

2.5.4 Numerical Integration

Abramowitz, Stegun. A software package in the public domain by Gautschi. We replace integral by certain weighted sum of function values:

$$I = \int_K g(x,y)dxdy \approx \sum_{i=1}^{Q} w_i g(x^i, y^i)$$

where w_i and (x^i, x^i) are independent of θ.

2.5.5 Quadrature for a triangle

We assume the reference triangle $\hat{K}$ is the right triangle with vertices at $(0,0)$, $(1,0)$ and $(0,1)$.

Example 2.5.5. $Q = 1$. (exact for P_1). Quadrature point for $\hat{K}$ is $(\frac{1}{3}, \frac{1}{3})$, $w = \frac{1}{2}$.

$$\int_K gdxdy \approx |K| g(\frac{1}{3}, \frac{1}{3})$$

Example 2.5.6. $Q = 3$. (exact for P_2). Quadrature points are $(\frac{1}{2}, 0)$, $(\frac{1}{2}, \frac{1}{2})$ and $(0, \frac{1}{2})$ $w = \frac{1}{6}$. Thus

$$\int_K gdxdy \approx \frac{|K|}{3} \left[g(\frac{1}{2}, 0) + g(\frac{1}{2}, \frac{1}{2}) + g(0, \frac{1}{2}) \right]$$

2.5.6 Quadrature for a Rectangle

Example 2.5.7. $\hat{Q} = [-1, 1] \times [-1, 1]$.

(1) $Q = 1$. (Gaussian quadrature) The point is $(0,0)$ and $w = 4$. It is exact for Q_1.

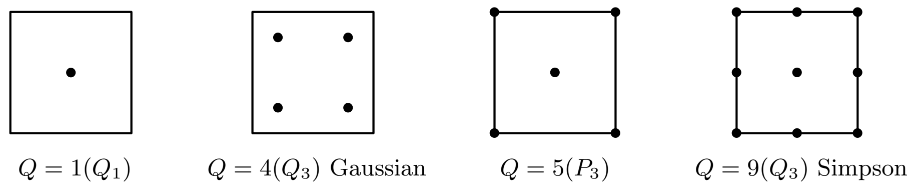


Figure 2.12: Quadrature points for the rectangle

(2) $Q = 4$. (Product of quadrature) The points are $(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}})$ and $w = 1$. Exact for Q_3.

(3) $Q = 5$.

$$\frac{1}{3}\left[g(-1,-1) + g(-1,1) + g(1,-1) + g(1,1) + 8g(0,0)\right]$$

is exact for P_3.

(4) $Q = 7$. (Product of Simpson).

$$\frac{1}{9}\left[\sum g(\pm 1, \pm 1) + 4\sum(g(\pm\frac{1}{2}, 0) + g(0, \pm\frac{1}{2})) + 16g(0,0)\right]$$

is exact for Q_3.

2.6 The Equations of Elasticity

Notations: Consider $\mathbf{H} = (H_0^1(\Omega))^3$. For $\mathbf{u} = (u_1, u_2, u_3) \in \mathbf{H}$, we let

$$\div\mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}, \;\; \|(u_1, u_2, u_3)\|_{\mathbf{H}}^2 = \sum_i \|u_i\|_{H_0^1}^2.$$

Let

$$\boldsymbol{\epsilon}_{ij}(\mathbf{v}) = \frac{1}{2}(\partial_j v_i + \partial_i v_j) = \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T) \tag{2.82}$$

be the linearized strain tensor and the stress tensor be

$$\sigma_{ij}(\mathbf{v}) = \lambda\left(\sum_{k=1}^{3}\boldsymbol{\epsilon}_{kk}(\mathbf{v})\right)\delta_{ij} + 2\mu\boldsymbol{\epsilon}_{ij}(\mathbf{v}), (1 \le i, j \le 3) \tag{2.83}$$

or simply

$$\boldsymbol{\sigma}(\mathbf{v}) = 2\mu\boldsymbol{\epsilon}(\mathbf{v}) + \lambda tr(\boldsymbol{\epsilon}(\mathbf{v}))\boldsymbol{\delta}. \tag{2.84}$$

We also use the following notation (matrix dot product)

$$\boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{v}) = \sum_{i,j=1}^{3} \boldsymbol{\epsilon}_{ij}(\mathbf{u})\boldsymbol{\epsilon}_{ij}(\mathbf{v}).$$

The equation of elasticity in pure displacement problem is

$$\begin{aligned} -\mathbf{div}\,\{2\mu\boldsymbol{\epsilon}(\mathbf{u}) + \lambda\, tr(\boldsymbol{\epsilon}(\mathbf{u}))\boldsymbol{\delta}\} = \mathbf{f}, &\quad \text{in } \Omega, \\ \mathbf{u} = \mathbf{g}, &\quad \text{on } \partial\Omega, \end{aligned} \tag{2.85}$$

where λ and μ are Lamé constants.

Green's formula For any tensor $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_{ij})$ we have

$$-\int_\Omega (\partial_j \boldsymbol{\sigma}_{ij}) v_i \, d\mathbf{x} \;=\; \int_\Omega \boldsymbol{\sigma}_{ij}\partial_j v_i \, d\mathbf{x} - \int_\Gamma \boldsymbol{\sigma}_{ij} v_i \nu_j \, ds. \tag{2.86}$$

In particular, if $\boldsymbol{\sigma} = \operatorname{div}\mathbf{u}\,\boldsymbol{\delta}$, then we have

$$\begin{aligned} -\int_\Omega (\partial_j (\operatorname{div}\mathbf{u})\boldsymbol{\delta}_{ij}) v_i \, d\mathbf{x} &= \int_\Omega \operatorname{div}\mathbf{u}\,\boldsymbol{\delta}_{ij}\partial_j v_i \, d\mathbf{x} - \int_\Gamma \operatorname{div}\mathbf{u}\,\boldsymbol{\delta}_{ij} v_i \nu_j \, ds \\ &= \int_\Omega \operatorname{div}\mathbf{u} \operatorname{div}\mathbf{v}\, d\mathbf{x} - \int_\Gamma \operatorname{div}\mathbf{u}(\mathbf{v}\cdot\mathbf{n})\, ds \\ &= \int_\Omega \operatorname{div}\mathbf{u} \operatorname{div}\mathbf{v}\, d\mathbf{x} - \int_\Gamma tr(\boldsymbol{\epsilon}(\mathbf{u}))\mathbf{v}\cdot\mathbf{n}\, ds. \end{aligned}$$

Since $\mathbf{div}\,(\nabla\mathbf{u})^T = \mathbf{div}\,((\operatorname{div}\mathbf{u})\boldsymbol{\delta})$, the first equation of (2.85) becomes

$$-\mathbf{div}\,\{\mu\nabla\mathbf{u} + (\mu+\lambda)(\operatorname{div}\mathbf{u})\boldsymbol{\delta}\} = \mathbf{f}, \tag{2.87}$$

Hence the weak form for the pure displacement problem is

$$a(\mathbf{u},\mathbf{v}) := \mu\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\, dx + (\mu+\lambda)\int_\Omega \operatorname{div}\mathbf{u}\operatorname{div}\mathbf{v}\, dx = f(\mathbf{v}). \tag{2.88}$$

The pure traction problem is

$$\begin{aligned} -\mathbf{div}\,\{2\mu\boldsymbol{\epsilon}(\mathbf{u}) + \lambda\, tr(\boldsymbol{\epsilon}(\mathbf{u}))\boldsymbol{\delta}\} = \mathbf{f}, &\quad \text{in } \Omega, \\ \boldsymbol{\sigma}(\mathbf{u})\cdot\mathbf{n} = \mathbf{g}, &\quad \text{on } \partial\Omega, \end{aligned} \tag{2.89}$$

with compatibility condition:

$$\int_\Omega \mathbf{f}\cdot\mathbf{v}dx + \int_{\partial\Omega} \mathbf{g}\cdot\mathbf{v}ds = 0, \text{ for } \mathbf{v}\in RM := \{(a+by, c-bx)\}.$$

Multiply $\mathbf{v}\in (H_0^1(\Omega))^2$ and integrate by part of first term in (2.89), we see

$$\begin{aligned} & -2\mu\int_\Omega \sum_{i,j} \frac{\partial \boldsymbol{\epsilon}_{ij}(\mathbf{u})}{\partial x_j} v_i \\ = & -2\mu\int_{\partial\Omega^\pm} \sum_{i,j} \boldsymbol{\epsilon}_{ij}(\mathbf{u}) n_j v_i + 2\mu\int_{\Omega^\pm} \sum_{i,j} \boldsymbol{\epsilon}_{ij}(\mathbf{u})\frac{\partial v_i}{\partial x_j} \end{aligned}$$

Using symmetry of $\boldsymbol{\epsilon}_{ij}(\mathbf{u})$

$$\sum_{i,j} \boldsymbol{\epsilon}_{ij}(\mathbf{u})\frac{\partial v_j}{\partial x_i} = \sum_{i,j} \boldsymbol{\epsilon}_{ji}(\mathbf{u})\frac{\partial v_j}{\partial x_i} = \sum_{i,j} \boldsymbol{\epsilon}_{ij}(\mathbf{u})\frac{\partial v_i}{\partial x_j} \tag{2.90}$$

Hence

$$\begin{aligned} & -2\mu\int_\Omega \sum_{i,j} \frac{\partial \boldsymbol{\epsilon}_{ij}(\mathbf{u})}{\partial x_j} v_i \\ = & -2\mu\int_{\partial\Omega} \sum_{i,j} \boldsymbol{\epsilon}_{ij}(\mathbf{u}) n_j v_i + \mu\int_\Omega \sum_{i,j} \boldsymbol{\epsilon}_{ij}(\mathbf{u})(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}) \\ = & -2\mu\int_{\partial\Omega} \sum_{i,j} \boldsymbol{\epsilon}_{ij}(\mathbf{u}) n_j v_i + 2\mu\int_\Omega \sum_{i,j} \boldsymbol{\epsilon}_{ij}(\mathbf{u})\boldsymbol{\epsilon}_{ij}(\mathbf{v}) \\ = & -2\mu\int_{\partial\Omega} \boldsymbol{\epsilon}(\mathbf{u})\mathbf{n}\cdot\mathbf{v} + 2\mu\int_\Omega \boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{v}). \end{aligned}$$

Meanwhile the second term of (2.89), gives

$$\lambda\int_\Omega \operatorname{div}\mathbf{u}\operatorname{div}\mathbf{v}\,dx - \lambda\int_{\partial\Omega} tr(\boldsymbol{\epsilon}(\mathbf{u}))\boldsymbol{\delta}\mathbf{n}\cdot\mathbf{v}\,ds = \lambda\int_\Omega \operatorname{div}\mathbf{u}\operatorname{div}\mathbf{v}\,dx - \lambda\int_{\partial\Omega} tr(\boldsymbol{\epsilon}(\mathbf{u}))\mathbf{v}\cdot\mathbf{n}\,ds.$$

Hence we get

$$a(\mathbf{u},\mathbf{v}) = f(\mathbf{v}), \tag{2.91}$$

where

$$a(\mathbf{u},\mathbf{v}) = 2\mu\int_\Omega \boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{v})\,dx + \lambda\int_\Omega \operatorname{div}\mathbf{u}\operatorname{div}\mathbf{v}\,dx, \tag{2.92}$$

and

$$f(\mathbf{v}) \quad = \quad \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dx + \int_\Gamma \mathbf{g} \cdot \mathbf{v} \, ds. \tag{2.93}$$

Thus formally we are solving

$$-\sum_{j=1}^{3} \int_\Omega (\partial_j \boldsymbol{\sigma}_{ij}(\mathbf{u})) v_i = \int_\Omega f_i v_i \, d\mathbf{x}, \; i = 1, 2, 3. \tag{2.94}$$

This together with the BC, we can check the weak form is equivalent to the pure traction case. Compatibility condition: If $\mathbf{v} \in RM$ then $\boldsymbol{\epsilon}(\mathbf{v}) = \operatorname{div} \mathbf{v} = 0$. Hence $\int_\Omega \mathbf{f} \cdot \mathbf{v} dx + \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{v} ds = 0$. Conversely, Let $\boldsymbol{\epsilon}(\mathbf{v}) = \operatorname{div} \mathbf{v} = 0$. Then

$$\begin{pmatrix} 2v_{11} & v_{12} + v_{21} \\ v_{12} + v_{21} & 2v_{22} \end{pmatrix} = v_{11} + v_{22} = 0.$$

Hence

$$v_1 = g_1(y) + c_1, \; v_2 = g_2(x) + c_2.$$

Further, $v_{12} + v_{21} = 0$ implies

$$g_1'(y) + g_2'(x) = 0$$

and hence

$$g_1'(y) = b - g_2'(x).$$

This shows the compatibility condition holds precisely when $\mathbf{v} \in RM := \{(a + by, c - bx)\}$.

Relation to Stokes equation

Meanwhile if we introduce $p = -\lambda tr(\boldsymbol{\epsilon}(\mathbf{u}))$ then

$$\begin{aligned} -\mathbf{div}\,\{2\mu\boldsymbol{\epsilon}(\mathbf{u})\} + \nabla p = \mathbf{f}, &\quad \text{in } \Omega, \\ \mathbf{div}\,\mathbf{u} = -\frac{p}{\lambda}, &\quad \text{in } \Omega, \\ \mathbf{u} = \mathbf{g}, &\quad \text{on } \partial\Omega. \end{aligned} \tag{2.95}$$

This is a mixed form. If $\lambda \to \infty$ then $\mathbf{div}\,\mathbf{u} = 0$ hence we get Stoke problem.