Mixed Method

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Chapter 1

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1.1 Introduction

Consider an elliptic problem:

$$-\Delta p = f \quad \text{in } \Omega \tag{1.1a}$$
$$p = 0 \quad \text{on } \Gamma. \tag{1.1b}$$

Let us introduce some notations: Given $m \geq 0$ a nonnegative integer,

$$H^m(\Omega) = \{p \in L^2(\Omega) : \partial^\alpha p \in L^2(\Omega), |\alpha| \leq m\}$$

is the usual Sobolev space of order m with the semi norm and norm

$$|p|_{m,\Omega} = \left(\sum_{|\alpha|=m} \int_\Omega |\partial^\alpha p|^2 dx \right)^{1/2}, \quad \|p\|_{m,\Omega} = \left(\sum_{|\alpha|\leq m} \int_\Omega |\partial^\alpha p|^2 dx \right)^{1/2}.$$

Given a vector valued functions $\mathbf{v} \in (H^m(\Omega))^n$ set

$$|\mathbf{v}|_{m,\Omega} = \left(\sum_{i=1}^n |v_i|^2_{m,\Omega} \right)^{1/2}, \quad \|\mathbf{v}\|_{m,\Omega} = \left(\sum_{i=1}^n \|v_i\|^2_{m,\Omega} \right)^{1/2}.$$

Let $P_k(K)$ denote the space of polynomials of total degree k and $\tilde{P}_k$ be the homogeneous polynomials of degree k. Let $Q_{i,j}(K)$ denote the space of polynomials of degree $\leq i$ and $\leq j$ in each variable and let $Q_k = Q_{k,k}$. Let

$$R_k(\partial K) = \{\phi \in L^2(\partial K), \phi_{e_i} \in P_k(e_i), \forall e_i\}. \tag{1.2}$$

The dimension of $P_k(K)$ is number of different terms in the expansion of $(1+x+y)^k$(or $(1+x+y+z)^k$) which is

$$_{n+1}\Pi_k =_{n+k} C_k = \frac{(n+k)!}{k!n!} = \frac{(k+n)\cdots(k+1)}{n!}, \; n = 2, 3.$$

Hence

$$\text{dimension of } P_k(K) = \begin{cases} \frac{1}{2}(k+1)(k+2) \text{ for } n = 2 \\ \frac{1}{6}(k+1)(k+2)(k+3) \text{ for } n = 3. \end{cases}$$

Let

$$\operatorname{curl} \mathbf{u} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \quad \mathbf{curl}\, p = (\frac{\partial p}{\partial y}, -\frac{\partial p}{\partial x})$$

Theorem 1.1.1. *(Stokes Theorem 2D and 3D)*

$$\int_\Omega \operatorname{curl} \mathbf{u} \cdot \xi \, d\mathbf{x} = \int_\Omega \mathbf{u}\mathbf{curl}\, \xi \, d\mathbf{x} + \int_{\partial\Omega} \mathbf{u} \cdot \tau\xi \, ds \tag{1.3}$$

$$\int_\Omega (\nabla \times \mathbf{u}) \cdot \mathbf{v} \, d\mathbf{x} = \int_\Omega \mathbf{u} \cdot (\nabla \times \mathbf{v}) d\mathbf{x} + \int_{\partial\Omega} \mathbf{n} \times \mathbf{u} \cdot \mathbf{v} \, dA \tag{1.4}$$

1.2 Mixed Formulation

Introduce the space

$$\mathbf{H}(\operatorname{div}; \Omega) = \{\mathbf{v} \in (L^2(\Omega))^n : \operatorname{div} \mathbf{v} \in L^2(\Omega)\} \tag{1.5}$$

with the norm equipped with

$$\|\mathbf{v}\|_{H(\operatorname{div};\Omega)} = \{\|\mathbf{v}\|^2_{0,\Omega} + \|\operatorname{div} \mathbf{v}\|^2_{0,\Omega}\}^{1/2}. \tag{1.6}$$

Given $\mathbf{v} \in \mathbf{H}(\operatorname{div}; \Omega)$ we can define its normal components $\mathbf{v} \cdot \mathbf{n} \in H^{-1/2}(\Gamma)$ where $H^{-1/2}(\Gamma)$ is the dual space of $H^{1/2}(\Gamma)$ and $\mathbf{n}$ is the unit outward normal along Γ. Indeed by Green's formula, we see

$$\int_\Omega (\nabla q \cdot \mathbf{v} + q \operatorname{div} \mathbf{v}) \, dx = \int_\Gamma q\mathbf{v} \cdot \mathbf{n} \, d\gamma, \qquad q \in H^1(\Omega). \tag{1.7}$$

Then the line integral $\int_\Gamma$ represent the duality between the spaces $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, sometimes written as $< \mathbf{v} \cdot \mathbf{n}, q >_\Gamma$.

Define the dual problem $(\mathbf{P})$

Definition 1.2.1. Find $(\mathbf{u}, p)$ in $\mathbf{H}(\text{div}\,; \Omega) \times L^2(\Omega)$ such that

$$\int_\Omega \mathbf{u} \cdot \mathbf{v}\, dx - \int_\Omega p \text{div}\, \mathbf{v}\, dx = 0, \ \forall \mathbf{v} \in \mathbf{H}(\text{div}\,; \Omega) \tag{1.8}$$

$$\int_\Omega q(\text{div}\, \mathbf{u} - f)\, dx = 0, \ \forall q \in L^2(\Omega). \tag{1.9}$$

Theorem 1.2.2. *The problem (P) ha a unique solution* $(\mathbf{u}, p)$ *in* $\mathbf{H}(div; \Omega) \times L^2(\Omega)$*. In addition* p *is the solution of (1.1) and we have*

$$\mathbf{u} = -\nabla p. \tag{1.10}$$

Proof. Uniqueness: Assume $f = 0$ in (1.9). Then we see $\text{div}\, \mathbf{u} = 0$. Taking $\mathbf{v} = \mathbf{u}$, in (1.8) we obtain $\mathbf{u} = 0$. Therefore

$$\int_\Omega p \text{div}\, \mathbf{v}\, dx = 0, \forall \mathbf{v} \in \mathbf{H}(\text{div}\,; \Omega). \tag{1.11}$$

Let $w \in H^1(\Omega)$ be such that

$$\Delta w = p \text{ in } \Omega.$$

Then by choosing $\mathbf{v} = \nabla w$ in (1.11), we get $p = 0$. To show existence, we use the solution p of (1.1). In fact we show that the pair $(\mathbf{u}, p) = (-\nabla p, p)$ is a solution of (P). We first check

$$\text{div}\, \mathbf{u} - f = -\Delta p - f = 0.$$

Since $p = 0$ on boundary, we have by Green's formula

$$-\int_\Omega (\mathbf{u} \cdot \mathbf{v} - p \text{div}\, \mathbf{v})\, dx = \int_\Gamma p \mathbf{v} \cdot \mathbf{n}\, d\gamma = 0$$

which is (1.8). Note by uniqueness, p automatically satisfies homog. B.C. □

Remark 1.2.3. One can check that the solution of the problem (P) may be characterized as the unique saddle point of the functional

$$L(\mathbf{v}, q) = \frac{1}{2} \int_\Omega \mathbf{v} \cdot \mathbf{v}\, dx + \int_\Omega q(\text{div}\, \mathbf{v} - f)\, dx$$

over the space $\mathbf{H}(\text{div}\,; \Omega) \times L^2(\Omega)$, i.e.,

$$L(\mathbf{u}, q) \le L(\mathbf{u}, p) \le L(\mathbf{v}, p), \forall \mathbf{v} \in \mathbf{H}(\text{div}\,; \Omega), q \in L^2(\Omega).$$

Hence p is the Lagrange multiplier associated with the constraint $\text{div}\, \mathbf{u} - f = 0$.

First inequality implies

$$\int_\Omega (\operatorname{div}\mathbf{v} - f)(q-p)\,dx \le 0, \quad \forall q \Rightarrow \int_\Omega (\operatorname{div}\mathbf{v} - f)q\,dx = 0, \quad \forall q.$$

Second inequality implies:

$$DL_u(\mathbf{u},p)\cdot\mathbf{v} = \int \mathbf{u}\cdot\mathbf{v} - p\operatorname{div}\mathbf{v}\,dx = 0, \quad \forall \mathbf{v} \Rightarrow \int_\Omega (\mathbf{u}+\nabla p)\cdot\mathbf{v}\,dx = 0 \Rightarrow \mathbf{u} = -\nabla p.$$

1.2.1 More general coefficients, nonhomog. B.C.

$$\left\{\begin{array}{rcl} \mathbf{u} &=& -\mathcal{K}\nabla p \text{ in } \Omega \\ \operatorname{div}\mathbf{u} &=& f \text{ in } \Omega \\ p &=& g \text{ on } \partial\Omega. \end{array}\right. \tag{1.12}$$

Its weak form is

$$\begin{aligned} (\mathcal{K}^{-1}\mathbf{u},\mathbf{v}) - (p,\operatorname{div}\mathbf{v}) &= -< g, \mathbf{v}\cdot\mathbf{n} >_{\partial\Omega}, \quad \mathbf{v}\in H(\operatorname{div};\Omega) && (1.13)\\ (\operatorname{div}\mathbf{u}, q) &= (f,q), \quad q \in L^2(\Omega). && (1.14)\end{aligned}$$

Define

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) &= (\mathcal{K}^{-1}\mathbf{u},\mathbf{v}), \quad \mathbf{v}\in H(\operatorname{div};\Omega) && (1.15)\\ b(\mathbf{v},q) &= -(\operatorname{div}\mathbf{u}, q), \quad q\in L^2(\Omega). && (1.16)\end{aligned}$$

Then the problem has the following abstract form

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) &= -< g, \mathbf{v}\cdot\mathbf{n} >_{\partial\Omega}, \quad \mathbf{v}\in H(\operatorname{div};\Omega) && (1.17)\\ b(\mathbf{u},q) &= -(f,q), \quad q\in L^2(\Omega). && (1.18)\end{aligned}$$

1.2.2 FEM Discretization

Given two finite dimensional spaces $\mathbf{V}_h \subset \mathbf{H}(\operatorname{div};\Omega)$ and $W_h \subset L^2(\Omega)$, consider the problem (P_h): Find the pair $(\mathbf{u}_h, p_h)$ satisfying

$$\begin{aligned} \int_\Omega \mathbf{u}_h\cdot\mathbf{v}_h\,dx - \int_\Omega p_h \operatorname{div}\mathbf{v}_h\,dx &= 0, \quad \mathbf{v}_h\in\mathbf{V}_h, && (1.19)\\ \int_\Omega \operatorname{div}\mathbf{u}_h q_h\,dx &= \int_\Omega f q_h\,dx, \quad q_h\in W_h. && (1.20)\end{aligned}$$

The following result is from Brezzi[3].

Theorem 1.2.4. *Assume*

$$\begin{cases} \mathbf{v}_h \in \mathbf{V}_h \\ \int_\Omega q_h \mathrm{div}\, \mathbf{v}_\mathrm{h} \,\mathrm{dx} = 0, \quad \forall q_h \in W_h \end{cases} \Rightarrow \mathrm{div}\, \mathbf{v}_\mathrm{h} = 0. \tag{1.21}$$

(This holds when $\mathrm{div}\, \mathbf{V}_\mathrm{h} \subset \mathrm{W}_\mathrm{h}$*) and that there exists a constant* $c > 0$ *such that*

$$\inf_{q_h \in W_h} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{\int_\Omega q_h \, div \, \mathbf{v}_h \, dx}{\|\mathbf{v}_h\|_{\mathbf{H}(div\,;\Omega)} \|q_h\|_{0,\Omega}} \geq c. \tag{1.22}$$

Then the problem (P_h) *has a unique solution* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ *and there exists a constant* $\tau > 0$ *which depends only on* c *such that*

$$\begin{aligned} &\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}(div\,;\Omega)} + \|p - p_h\|_{0,\Omega} \leq \\ &\tau \left\{ \inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{H}(div\,;\Omega)} + \inf_{q_h \in W_h} \|p - q_h\|_{0,\Omega} \right\} \end{aligned} \tag{1.23}$$

(These are approximation properties of $\mathbf{V}_h$ *and* W_h*)*

Construction of $\mathbf{V}_h$ and W_h

Now we construct finite dimensional spaces $\mathbf{V}_h$ and W_h so that they satisfy a good approximation property and stability conditions (1.21) and (1.22). Also we need to ensure $P_0 \mathrm{div} = \mathrm{div}_h \Pi$. So we assume

$$\mathrm{div}\, V_h \subset W_h.$$

From here and thereafter, we shall assume, for convenience, that a bounded polygon and triangulation $\mathcal{T}_h$ consists of triangles and parallelograms whose diameters are $\leq h$.

Remark 1.2.5. Define $\nabla_h \in L(W_h, \mathbf{V}_h)$ by

$$\int_\Omega \nabla_h q_h \cdot \mathbf{v}_h \, dx = -\int_\Omega q_h \mathrm{div}\, \mathbf{v}_h \, dx, \quad \forall q_h \in W_h, \mathbf{v}_h \in \mathbf{V}_h. \tag{1.24}$$

This operator ∇_h is clearly an approximation of ∇. Now the function p_h may be characterized as the unique solution of the following problem:

$$\int_\Omega \nabla_h p_h \cdot \nabla_h q_h dx = \int_\Omega f q_h dx, \forall q_h \in W_h. \tag{1.25}$$

Clearly (1.25) has a unique solution p_h and the pair $(-\nabla p_h, p_h)$ is the solution of problem (P_h). In general, $W_h \not\subset H_0^1(\Omega)$.

Exer. Construct explicit form of ∇_h in case of rectangular(and triangular) grid for $k = 0$. i.e, Let $q_h = 1$ on one element, zero at the other. Construct $\nabla_h q_h \in \mathbf{V}_h$.

Lemma 1.2.6. *A function* $\mathbf{v} \in (L^2(\Omega))^2$ *belongs to* $\mathbf{H}(div;\Omega)$ *if and only if the following conditions hold:*

(i) for all $K \in \mathcal{T}_h$*, the restriction* $\mathbf{v}|_K$ *of* $\mathbf{v}$ *to the set* K *belongs to* $\mathbf{H}(div;K)$*.*

(ii) for any pair of adjacent elements $K_1, K_2 \in \mathcal{T}_h$*, we have the following reciprocity condition*

$$\mathbf{v}\cdot\mathbf{n}_{K_1} + \mathbf{v}\cdot\mathbf{n}_{K_2} = 0 \quad \text{on } e = K_1 \cap K_2, \tag{1.26}$$

where $\mathbf{n}_{K_i}$ *is the unit outward normal vector along the boundary of* K_i*,* $i = 1,2$*.*

Proof. Without loss of generality, we may assume $\Omega = K_1 \cup K_2$. Necessity is trivial. For sufficiency, let $\mathbf{v} \in (L^2(\Omega))^2$ satisfy two conditions (i) and (ii). Then for any $q \in C_0^\infty(\Omega)$,

$$\begin{aligned} & \int_{K_1} \operatorname{div}\mathbf{v} q\,dx + \int_{K_2} \operatorname{div}\mathbf{v} q\,dx \\ = & \int_{\partial K_1} \mathbf{v}\cdot\mathbf{n} q\,ds + \int_{\partial K_2} \mathbf{v}\cdot\mathbf{n} q\,ds - \int_{K_1} \mathbf{v}\cdot\nabla q\,dx - \int_{K_2} \mathbf{v}\cdot\nabla q\,dx \\ = & -\int_{K_1} \mathbf{v}\cdot\nabla q\,dx - \int_{K_2} \mathbf{v}\cdot\nabla q\,dx = -\int_{\Omega} \mathbf{v}\cdot\nabla q\,dx. \end{aligned}$$

This relation defines $\operatorname{div}\mathbf{v}$ weakly(in the distribution sense). i.e, if we define

$$(\operatorname{div}\mathbf{v})|_{K_i} = \operatorname{div}(\mathbf{v}|_{K_i}), \quad i = 1,2$$

then $\operatorname{div}\mathbf{v}$ is defined a.e. and by taking a sequence $q_n \to \operatorname{div}(\mathbf{v}|_{K_i})$ on each i, we see that $\operatorname{div}\mathbf{v}$ is in $L^2(\Omega)$.

□

Furthermore, we have by Green's identity:

$$\iint_\Omega (\operatorname{div}\mathbf{u}) q\,dx = \sum_K \iint_K (\operatorname{div}\mathbf{u})\, q\,dx \tag{1.27}$$

$$= \sum_K \int_{\partial K} \mathbf{u}\cdot\nu q\,ds - \sum_K \iint_K \mathbf{u}\cdot\nabla q\,dx, \; q \in W_h. \tag{1.28}$$

From this, it is easy to guess the degrees of freedom of the mixed finite element space involves certain moments of normal component along edges and possibly certain moments of $\mathbf{v}_h$. In particular, the basis functions must have continuous normal component on edges.

Also, it is advised to have $\mathbf{u}_h$ contains at least a polynomial of degree $k \geq 0$. The simplest possible element is obtained if we choose $\mathbf{V}_h = \{\mathbf{u}_h \cdot \nu \text{ constant on edges}\}$ and take $W_h(Q) = P_0(Q)$.

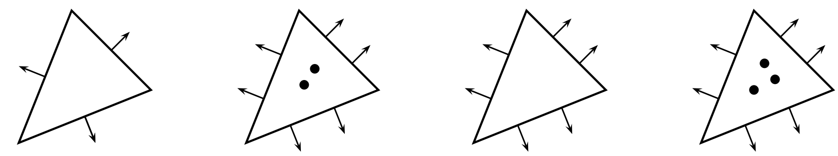

Figure 1.1: Degrees of freedom for RT_0, RT_1, BDM_1 and $BDFM_2$

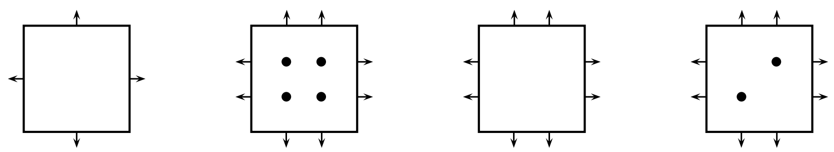

Figure 1.2: Degrees of freedom for RT_0, RT_1, BDM_1 and $BDFM_2$

1.2.3 Rectangular element

Assume the partition along x, y–axis,

$$0 = x_0 < x_1 < \cdots < x_n = 1, \quad 0 = y_0 < y_1 < \cdots < y_n = 1.$$

We let

$$\mathbf{V}_h(K) = Q_{k+1,k}(K) \times Q_{k,k+1}(K), \quad W_h|_K = Q_k(K)$$

on each rectangle K. The functions are determined by the following conditions:

$$\begin{cases} \displaystyle\int_e \mathbf{v}_h \cdot \mathbf{n} q_k \, ds, & \forall q_k \in P_k(e), \text{ for each edge } e \text{ of } K \\ \displaystyle\int_K \mathbf{v}_h \cdot \boldsymbol{\phi}_k \, dx & \forall \boldsymbol{\phi}_k \in Q_{k-1,k}(K) \times Q_{k,k-1}(K), (k \geq 1). \end{cases} \tag{1.29}$$

For $k = 0$, we have simplest element:

$$\begin{cases} \mathbf{u}_h &= (a + bx, c + dy) \text{ on } K \\ p_h &= \text{constant on } K. \end{cases} \tag{1.30}$$

Of course, we have to impose the continuity condition of $\mathbf{u}_h \cdot \mathbf{n}$ on each edge e.

Explicit Form of B in Case of Rectangular grid, $k = 0$

Let B be the matrix representation of $b(\cdot, \cdot)$ form by

$$(B\mathbf{v}_h, q) = b(\mathbf{v}_h, q), \quad \mathbf{v}_h \in \mathbf{V}_h, q \in W_h.$$

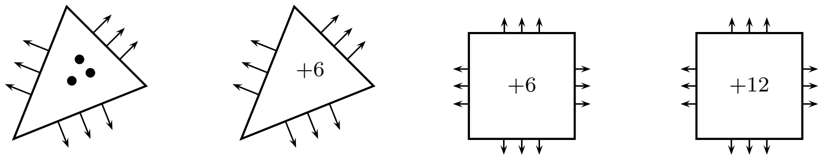

Figure 1.3: Degrees of freedom for BDM_2, RT_2, for triangle/rectangle

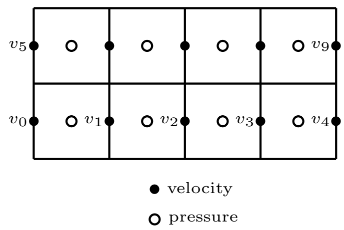

Figure 1.4: Nodes for velocity and pressure

For rectangular grid, we see

$$\begin{aligned}
(B\mathbf{v}_h, p) &= -(\text{ div } \mathbf{v}_h, p_h) = -\int_\Omega p \text{ div } \mathbf{v}_h \\
&= -h_2[p_1(v_1 - v_0) + p_2(v_2 - v_1) + \cdots + p_{n-1}(v_{n-1} - v_{n-2}) + p_n(v_n - v_{n-1})] \\
&\quad (+y \text{ direction }) \\
&= -h_2[v_0(-p_1) + v_1(p_1 - p_2) + \cdots + v_{n-1}(p_{n-1} - p_n) + v_n p_n](+y \text{ direction }) \\
&= h_2[v_1(p_2 - p_1) + \cdots + v_{n-1}(p_n - p_{n-1})] \\
&= (\mathbf{v}_h, B^t p)
\end{aligned}$$

since $v_0 = v_n = 0$. So the action of B and B^t are given by above.(In fact, B is discrete $-$div and B^t is discrete gradient).

1.2.4 Iterative Methods

Eliminate u and get a SPD system for p

In matrix form, we have

$$\left\{ \begin{aligned} AU + B^t P &= -G \\ BU &= -F \end{aligned} \right. \tag{1.31}$$

$$\begin{bmatrix} A_x & 0 & B_x^t \\ 0 & A_y & B_y^t \\ B_x & B_y & 0 \end{bmatrix} \begin{bmatrix} U_x \\ U_y \\ P \end{bmatrix} = \begin{bmatrix} -G_x \\ -G_y \\ -F \end{bmatrix}$$

This system is symmetric, but indefinite. So cg cannot be used directly here. Instead, eliminate U from the first equation: $U = -A^{-1}(G + B^t P)$ and substitute in the second equation to get

$$BA^{-1}B^t P = F - BA^{-1}G.$$

Use conjugate gradient to solve for P and then get $U = -A^{-1}(G + B^t P)$.

Standard Uzawa

Let p_h^0 given. With small $\epsilon > 0$, Solve

$$\begin{aligned} a(\mathbf{u}^{m+1}, \mathbf{v}) + b(\mathbf{v}, p_h^m) &= -< g, \mathbf{v}\cdot\mathbf{n} >_{\partial\Omega} \quad \mathbf{v} \in \mathbf{V}_h \\ (p^{m+1} - p^m, q) &= \epsilon[b(\mathbf{u}^{m+1}, q) + (f, q)], \quad q \in W_h. \end{aligned}$$

Stop if $\|p^{m+1} - p^m\|$ is sufficiently small. In CFD note by Verfuth, they take $\epsilon = 1.5$ for Stokes equation.

1.2.5 Triangular Element

Assume K is a triangle. Define $\mathbf{V}(K) \subset \mathbf{H}(\mathrm{div}\,; K)$ by

$$\mathbf{V}(K) = \mathbf{P}_k(K) \oplus Span(\mathbf{x}\tilde{P}_k(K)),$$

where $\tilde{P}_k(K)$ is the homogeneous polynomial of degree k. The dimension is

$$dim\, RT_k(K) = \begin{cases} (k+1)(k+3) & \text{for } n = 2 \\ \frac{1}{2}(k+1)(k+2)(k+4) & \text{for } n = 3. \end{cases} \tag{1.32}$$

Proposition 1.2.7. *For $k \geq 0$ and for any $\mathbf{v} \in RT_k(K)$ the following relations imply $\mathbf{v} = 0$.*

$$\int_{\partial K} \mathbf{v}\cdot\mathbf{n} p_k\, ds = 0, \forall p_k \in R_k(\partial K), \text{ moments up to } k \text{ on } \partial K \tag{1.33}$$

$$\int_K \mathbf{v}\cdot p_{k-1}\, dx = 0, \forall p_{k-1} \in (P_{k-1}(K))^n \text{ moments up to } k-1 \text{ on } K \tag{1.34}$$

Also

$$\mathbf{V}_0 := RT_k^0 = \{\mathbf{v} \in \mathbf{V}(K), div\,\mathbf{v} = 0\} \subset (P_k(K))^2. \tag{1.35}$$

Proof. The number conditions for $n = 2$ is

$$dim(\mathbf{V}(K)) = 3(k+1) + 2 \times_3 \Pi_{k+1} = 2 \times_{k+1} C_{k-1} = (k+1)(k+3). \quad (1.36)$$

The number of conditions is equal to the dimension of $\mathbf{V}(K)$. Let $\mathbf{v}_h \in \mathbf{V}(K)$ satisfy (1.33), (1.34). We shall show that $\mathbf{v}_h = 0$. Since

$$\mathbf{v} \cdot \mathbf{n} = \mathbf{v}_0 \cdot \mathbf{n} + \tilde{p}_k \mathbf{x} \cdot \mathbf{n}, \quad \mathbf{v}_0 \in (P_k(K))^2$$

and an edge is determined by a linear equation $\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$. Hence $\mathbf{v} \cdot \mathbf{n}$ belongs to $P_k(e)$ on that side. So $\mathbf{v} \cdot \mathbf{n}|_{\partial K} \in R_k(\partial K)$ and (1.33) implies $\mathbf{v} \cdot \mathbf{n}|_{\partial K} = 0$. So

$$\int_K div\mathbf{v} \cdot p_k = \int_{\partial K} \mathbf{v} \cdot \mathbf{n} p_k ds - \int_K \mathbf{v} \cdot \mathrm{grad}\, p_k \, dx = 0, \forall p_k \in P_k(K).$$

Hence $\mathrm{div}\, \mathbf{v} = 0$ and since $\mathbf{v} = \mathbf{v}_0 + \mathbf{x}\tilde{p}_k$,

$$0 = \mathrm{div}\, \mathbf{v}_0 + \mathrm{div}\, \mathbf{x}\tilde{p}_k + \mathbf{x} \cdot \mathrm{grad}\, \tilde{p}_k = \mathrm{div}\, \mathbf{v}_0 + (n+k)\tilde{p}_k \quad (1.37)$$

which implies $\mathrm{div}\, \mathbf{v}_0 = 0$ and $(n+k)\tilde{p}_k = 0$. Hence there exists $w \in P_{k+1}$(unique up to an additive constant) such that

$$\mathbf{v}_h = \mathrm{curl}\, w = (\frac{\partial w}{\partial y}, -\frac{\partial w}{\partial x}).$$

Since $\frac{\partial w}{\partial \tau} = \mathbf{v}_h \cdot \mathbf{n} = 0$ on ∂K, w is constant on ∂K, which we may assume 0 and we can set

$$w = \lambda_1 \lambda_2 \lambda_3 z, \quad z \in P_{k-2} (z = 0, \text{ if } k \le 1).$$

Again by (1.34), we have for any $\mathbf{r} \in (P_{k-1})^2$

$$0 = \int_K \mathbf{v}_h \cdot \mathbf{r} \, dx = \int_K \mathrm{curl}\, w \cdot \mathbf{r} \, dx = \int_K w \mathrm{curl}\, \mathbf{r} \, dx = \int_K \lambda_1 \lambda_2 \lambda_3 z \mathrm{curl}\, \mathbf{r} \, dx,$$

where $\mathrm{curl}\, \mathbf{r} = \frac{\partial r_2}{\partial x} - \frac{\partial r_1}{\partial y} \in P_{k-2}$. We can choose $\mathbf{r}$ so that $z = \mathrm{curl}\, \mathbf{r}$ and then

$$\int_K \lambda_1 \lambda_2 \lambda_3 z^2 \, dx = 0.$$

Therefore $z = 0$ and $w = 0$, $\mathbf{v}_h = \mathrm{curl}\, w = 0$. Finally using (1.37), one can show that (1.35) holds. □

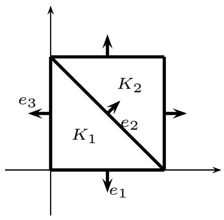

Figure 1.5: Fix normal vectors once and for all

Basis functions on reference element for $k = 0$

For computation, one need to fix the unit normal vector $\mathbf{n}_e$ for each edge once and for all. We compute $\boldsymbol{\phi}_1$ on K_1, half of $[0,1]^2$. Let

$$\boldsymbol{\phi}_1 = (a_1 + b_1 x, c_1 + b_1 y).$$

We need solve

$$\int_{e_i} \boldsymbol{\phi}_j \cdot \mathbf{n} ds \quad = \quad \delta_{ij}, \quad i = 1, 2, 3.$$

Hence for $j = 1$, we have

$$\begin{aligned}
\int_{e_1} \boldsymbol{\phi}_1 \cdot \mathbf{n} ds &= \int_{e_1} (a + bx, c + by) \cdot (0, -1) ds = -c = 1 \\
\int_{e_2} \boldsymbol{\phi}_1 \cdot \mathbf{n} ds &= \int_{e_2} (a + bx, c + by) \cdot \frac{1}{\sqrt{2}}(1, 1) ds = (a + c + b(x + y))|_{(\frac{1}{2}, \frac{1}{2})} = 0 \\
\int_{e_3} \boldsymbol{\phi}_1 \cdot \mathbf{n} ds &= \int_{e_3} (a + bx, c + by) \cdot (-1, 0) ds = -a = 0.
\end{aligned}$$

From this, we get

$$\boldsymbol{\phi}_1 = (x, -1 + y)$$

Triangle element by affine mapping

Let $\hat{\mathbf{V}}$ be any mixed fem space defined on a reference element $\hat{K}$. Consider any triangle K in the plane whose vertices are denoted by $\mathbf{a}_i, i = 1, \cdots, 3$. Set

$$h_K = \text{diam } K \tag{1.38}$$

$$\rho_K = \text{diameter of inscribed circle in } K. \tag{1.39}$$

Let $F_K : \hat{\mathbf{x}} \to F_K(\hat{\mathbf{x}}) = B_K\hat{\mathbf{x}} + \mathbf{b}_K, B_K \in L(\mathbb{R}^2), \mathbf{b}_K \in \mathbb{R}^2$ be the unique affine invertible mapping such that

$$F_K(\hat{\mathbf{a}}_i) = \mathbf{a}_i, i = 1, 2, 3.$$

If $\hat{\phi}$ is any scalar function defined over $\hat{K}(\partial\hat{K})$, we associate a function on K(pull back) by

$$\phi = \hat{\phi} \circ F_K^{-1}. \tag{1.40}$$

On the other hand, for any vector valued function $\hat{\mathbf{v}} = (\hat{q}_1, \hat{q}_2)$ we associate $\mathbf{v}$ by

$$\mathbf{v} = \frac{1}{J_K} B_K \hat{\mathbf{v}} \circ F_K^{-1}, \tag{1.41}$$

where $J_K = \det(B_K)$. Now for each K, we associate the space

$$\mathbf{V}(K) = \{\mathbf{v} \in H(\mathrm{div}\,; K); \hat{\mathbf{v}} \in \hat{\mathbf{V}}\}. \tag{1.42}$$

Lemma 1.2.8. *For any $\hat{\mathbf{v}} \in (H^1(\hat{K}))^2$, we have*

$$\int_{\hat{K}} \hat{\phi}\, div\, \hat{\mathbf{v}}\, d\hat{x} = \int_K \phi\, div\, \mathbf{v}\, dx, \quad \hat{\phi} \in L^2(\hat{K}) \tag{1.43}$$

$$\int_{\partial\hat{K}} \hat{\phi}\hat{\mathbf{v}} \cdot \hat{\mathbf{n}}\, d\hat{s} = \int_{\partial K} \phi\mathbf{v} \cdot \mathbf{n}\, ds, \quad \hat{\phi} \in L^2(\partial\hat{K}). \tag{1.44}$$

Lemma 1.2.9. *For any nonnegative integer ℓ*

$$|\hat{\phi}|_{\ell,\hat{K}} \le \|B_K\|^\ell |J_K|^{-1/2} |\phi|_{\ell,K}, \quad \hat{\phi} \in H^\ell(\hat{K}) \tag{1.45}$$

$$|\hat{\mathbf{v}}|_{\ell,\hat{K}} \le \|B_K\|^\ell \|B_K\|^{-1} |J_K|^{1/2} |\mathbf{v}|_{\ell,K}, \hat{\mathbf{v}} \in (H^\ell(\hat{K}))^2. \tag{1.46}$$

1.3 Error Estimates

Let $k \ge 0$. The (local) Raviart-Thomas spaces for for triangles are

$$RT_{[k]} = \{\mathbf{P}_k(K) \oplus Span(\mathbf{x}\tilde{P}_k(K)\},\ dim = \begin{cases} (k+1)(k+3), & n = 2 \\ \frac{1}{2}(k+1)(k+2)(k+4), & n = 3. \end{cases} \tag{1.47}$$

For rectangles we have $\mathbf{V}(K) = RT_{[k]} = \{\mathbf{Q}_k(K) \oplus Span(\mathbf{x}\tilde{Q}_{k,k}(K)\}$, i.e.,

$$RT_{[k]} = \begin{cases} Q_{k+1,k} \times Q_{k,k+1}, dim = 2(k+1)(k+2) & n = 2 \\ Q_{k+1,k,k} \times Q_{k,k+1,k} \times Q_{k,k+1,k}, dim\ = 3(k+1)^2(k+2) & n = 3, \end{cases} \tag{1.48}$$

These spaces have been defined in order to have $\mathrm{div}\,\mathbf{v}|_K \in Q_k(K)$ and

$$\begin{cases} \mathbf{v} \cdot \mathbf{n}|_{e_i} \in P_k(e_i) & \text{for } n = 2 \\ \mathbf{v} \cdot \mathbf{n}|_{f_i} \in Q_k(f_i) & \text{for } n = 3. \end{cases} \tag{1.49}$$

Lemma 1.3.1. *For $n = 2$ if $\mathbf{q} \in RT^0_{[k]}(\hat{K})$(div free), there exists $\psi \in Q_{k+1}(\hat{K})$ such that $\mathbf{q} = curl\psi$. Its dimension is $(k+1)(k+3)$.*

Let us define

$$\Psi_k(K) = \begin{cases} Q_{k-1,k}(K) \times Q_{k,k-1}(K) & \text{for } n = 2 \\ Q_{k-1,k,k}(K) \times Q_{k,k-1,k}(K) \times Q_{k,k,k-1}(K) & \text{for } n = 3. \end{cases} \tag{1.50}$$

Proposition 1.3.2. *For any $\mathbf{v} \in RT_{[k]}(\hat{K})$, the relations(when $n = 2$)*

$$\int_{e_i} \mathbf{v} \cdot \mathbf{n}\phi \, ds = 0, \quad \forall \phi \in P_k(e_i) \tag{1.51}$$

$$\int_{\hat{K}} \mathbf{v} \cdot \boldsymbol{\phi} \, dx = 0, \quad \forall \boldsymbol{\phi} \in \Psi_k(\hat{K}) \tag{1.52}$$

imply $\mathbf{v} = 0$. For $n = 3$, e_i must be replaced by a face f_i and $P_k(e_i)$ is replaced by $Q(f_i)$.

1.3.1 $H(\mathbf{div})$ interpolation

Let

$$\begin{aligned} \mathbf{V}_h &= \{\mathbf{v} \in \mathbf{H}(\mathrm{div}\,;\Omega), \mathbf{v}|_K \in \mathbf{V}(K), K \in \mathcal{T}_h\} \\ W_h &= \{w \in L^2(\Omega) : w_K \in W(K), K \in \mathcal{T}_h\} \end{aligned}$$

be any stable pair of mixed spaces so that $\mathrm{div}\,\mathbf{V}_h(K) \subset W_h(K)$ holds. Then there exists a mapping $\Pi_h = \Pi_h^k : \mathbf{H}^1(K) \to \mathbf{V}(K)$[1] such that

$$\begin{cases} \int_{\partial K} (\mathbf{v} - \Pi_h \mathbf{v}) \cdot \mathbf{n} p_k \, ds = 0, & \forall p_k \in R_k(\partial K), \\ \int_K (\mathbf{v} - \Pi_h \mathbf{v}) \cdot \boldsymbol{\phi}_{k-1} \, dx = 0, & \forall \boldsymbol{\phi}_{k-1} \in (P_{k-1}(K))^2, (k \geq 1) \end{cases} \tag{1.53}$$

for triangles and for rectangles

$$\begin{cases} \int_{\partial K} (\mathbf{v} - \Pi_h \mathbf{v}) \cdot \mathbf{n} p_k \, ds = 0, & \forall p_k \in R_k(\partial K), \\ \int_K (\mathbf{v} - \Pi_h \mathbf{v}) \cdot \boldsymbol{\phi}_k \, dx = 0, & \forall \boldsymbol{\phi}_k \in Q_{k-1,k}(K) \times Q_{k,k-1}(K), (k \geq 1). \end{cases} \tag{1.54}$$

The proofs are standard as long as the following commuting diagram holds:

[1]This requires more regularity than $\mathbf{H}(\mathrm{div}\,;K)$, i.e., $\mathbf{W} = \{\mathbf{v} : \mathrm{div}\,\mathbf{v} \in L^s(K),\ s > 2\}$ is enough, see p.125 Brezzi book.

$$\begin{array}{ccc} \mathbf{V}(K) & \xrightarrow{\text{div}} & L^2(K) \\ \Pi_h^k \downarrow & & \downarrow P_h^{k-1} \\ \mathbf{V}_h^k(K) & \xrightarrow{\text{div}} & W_h^{k-1}(K) \end{array}$$

In fact, for $q \in \text{div}\, \mathbf{V}_h(K) \subset W_h(K)$ we have

$$\begin{aligned} \int_K \text{div}\, \Pi_h \mathbf{u}\, q\, dx &= \int_{\partial K} \Pi_h \mathbf{u} \cdot \mathbf{n}\, q\, ds - \int_K \Pi_h \mathbf{u} \cdot \nabla q\, dx \\ &= \int_{\partial K} \mathbf{u} \cdot \mathbf{n}\, q\, ds - \int_K \mathbf{u} \cdot \nabla q\, dx \qquad (1.55) \\ &= \int_K \text{div}\, \mathbf{u}\, q\, dx = \int_K P_h \text{div}\, \mathbf{u}\, q\, dx. \end{aligned}$$

Thus

$$\text{div}\, \Pi_h^k = P_h^{k-1} \text{div}\, .$$

Lemma 1.3.3. *This projection Π_h has the following properties:*

$$\|\mathbf{v} - \Pi_h \mathbf{v}\|_{L^2(K)} \le C h_K^{k+1} |\mathbf{v}|_{k+1,K} \qquad (1.56)$$

$$\|\text{div}\, (\mathbf{v} - \Pi_\mathrm{h} \mathbf{v})\|_{\mathrm{L^2(K)}} \le \mathrm{C h_K^{k+1}} |\text{div}\, \mathbf{v}|_{\mathrm{k+1,K}}. \qquad (1.57)$$

Proof. By Bramble Hilbert lemma(vector form) [13]

$$\|\hat{\mathbf{v}} - \hat{\Pi}_h \hat{\mathbf{v}}\|_{0,\hat{K}} \le C |\hat{\mathbf{v}}|_{k+1,\hat{K}}. \qquad (1.58)$$

On the other hand, by Green's formula one can verify

$$(\text{div}\, (\hat{\Pi}_h \hat{\mathbf{v}}), \hat{\phi}) = (P_0 \text{div}\, \hat{\mathbf{v}}, \hat{\phi}), \quad \hat{\phi} \in P_k.$$

Hence

$$(\text{div}\, \hat{\mathbf{v}} - \text{div}\, (\hat{\Pi}_h \hat{\mathbf{v}}), \hat{\phi}) = ((I - P_0) \text{div}\, \hat{\mathbf{v}}, \hat{\phi}), \quad \hat{\phi} \in P_k.$$

Hence again by BH lemma

$$\|\text{div}\, (\hat{\mathbf{v}} - \hat{\Pi}_h \hat{\mathbf{v}})\|_{0,\hat{K}} \le C |\text{div}\, \mathbf{v}|_{k+1,\hat{K}}. \qquad (1.59)$$

Define Π_h by

$$\widehat{\Pi_h \mathbf{v}} = \hat{\Pi}_h \hat{\mathbf{v}}, \quad \mathbf{v} \in (\mathbf{H}^1(K))^2.$$

Now use scaling argument and the fact that

$$\operatorname{div}\hat{\mathbf{v}} = J_K \widehat{\operatorname{div}\mathbf{v}}$$

and

$$\|B_K\| \leq \frac{h_K}{\rho_{\hat{K}}}, \quad \|B_K^{-1}\| \leq \frac{h_{\hat{K}}}{\rho_K}. \tag{1.60}$$

This will give the desired result (1.56).(fill in gaps) □

Theorem 1.3.4. *Let* $(\mathbf{u}, p)$ *be the solution pair of (1.13) and* $(\mathbf{u}_h, p_h)$ *be the solution pair of (1.19). Then we have*

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leq Ch^{k+1}|\mathbf{u}|_{k+1},$$
$$\|\operatorname{div}(\mathbf{u} - \mathbf{u}_\mathrm{h})\|_0 \leq Ch^{k+1}|\operatorname{div}\mathbf{u}|_{\mathrm{k+1}},$$
$$\|p - p_h\|_0 \leq Ch^{k+1}\|\mathbf{u}\|_{k+1}.$$

Proof. Subtracting (1.13) from (1.19), we have

$$(\mathcal{K}^{-1}(\mathbf{u} - \mathbf{u}_h), \mathbf{v}_h) - (p - p_h, \operatorname{div}\mathbf{v}_h) = 0, \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{1.61}$$
$$(\operatorname{div}(\mathbf{u} - \mathbf{u}_h), q_h) = 0, \forall q_h \in W_h. \tag{1.62}$$

Hence,

$$\begin{aligned}
c\|\mathbf{u} - \mathbf{u}_h\|_0^2 &\leq (\mathcal{K}^{-1}(\mathbf{u} - \mathbf{u}_h), \mathbf{u} - \mathbf{u}_h) \\
&= (\mathcal{K}^{-1}(\mathbf{u} - \mathbf{u}_h), \Pi_h\mathbf{u} - \mathbf{u}_h) + (\mathcal{K}^{-1}(\mathbf{u} - \mathbf{u}_h), \mathbf{u} - \Pi_h\mathbf{u}) \\
&= (P_h p - p_h, \operatorname{div}(\Pi_h\mathbf{u} - \mathbf{u}_h)) + (\mathcal{K}^{-1}(\mathbf{u} - \mathbf{u}_h), \mathbf{u} - \Pi_h\mathbf{u}) \qquad (1.63) \\
&= (\mathcal{K}^{-1}(\mathbf{u} - \mathbf{u}_h), \mathbf{u} - \Pi_h\mathbf{u}) \\
&\leq C\|\mathbf{u} - \mathbf{u}_h\|_0\|\mathbf{u} - \Pi_h\mathbf{u}\|_0,
\end{aligned}$$

where c and C are independent of h and $\mathbf{u}$. Therefore, we have from (1.56)

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leq c\|\mathbf{u} - \Pi_h\mathbf{u}\|_0 \leq Ch^{k+1}|\mathbf{u}|_{k+1}. \tag{1.64}$$

Since $\operatorname{div}(\mathbf{u} - \mathbf{u}_h) = \operatorname{div}(\mathbf{u} - \Pi_h\mathbf{u})$, we have from (1.56)

$$\|\operatorname{div}(\mathbf{u} - \mathbf{u}_h)\|_0 \leq Ch^{k+1}|\operatorname{div}\mathbf{u}|_{k+1}. \tag{1.65}$$

Using the inf-sup condition (1.22), we have following

$$\|P_h p - p_h\|_0 \leq C \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(P_h p - p_h, \operatorname{div}\mathbf{v}_h)}{\|\mathbf{v}_h\|_{\mathbf{H}(\operatorname{div})}}$$

$$= C \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(\kappa^{-1}(\mathbf{u}-\mathbf{u}_h), \mathbf{v}_h)}{\|\mathbf{v}_h\|_{\mathbf{H}(\mathrm{div}\,)}}$$
$$\leq C\|\mathbf{u}-\mathbf{u}_h\|_0.$$

Hence we obtain

$$\begin{aligned}\|p-p_h\|_0 &\leq \|p-P_hp\|_0 + \|P_hp-p_h\|_0 \\ &\leq Ch^{k+1}(\|p\|_{k+1} + \|\mathbf{u}\|_{k+1}) \leq Ch^{k+1}\|\mathbf{u}\|_{k+1}.\end{aligned}$$

□

1.4 Auxiliary Spaces

Mixed-HybridBrezz-HybridBrezziBook.tex file in -Education-Graduate mixed folder. Some notations:

$$R_k(\partial K) = \{\phi \in L^2(\partial K), \phi|_e \in P_k(e) \text{ for each edge } e \text{ of } \partial K\}, \tag{1.66}$$

$$T_k(\partial K) = \{\phi \in R_k(\partial K) \cap C^0(\partial K)\}, \tag{1.67}$$

For any subspace $S_k(K)$ of $P_k(K)$, we define

$$\mathcal{L}^s(S_k, \mathcal{T}_h) = \{v \in H^s(\Omega), v|_K \in S_k(K)\}. \tag{1.68}$$

$$\mathcal{L}^s_k = \mathcal{L}^s(P_k, \mathcal{T}_h), \quad \mathcal{L}^s_{[k]} = \mathcal{L}^s(Q_k, \mathcal{T}_h) \tag{1.69}$$

Also we define bubbles.

$$B(S_k) = \mathcal{L}^0_k(\mathcal{S}_k, \mathcal{T}_h) = \mathcal{L}^1(\mathcal{S}_k, \mathcal{T}_h). \tag{1.70}$$

1.5 Nonconforming methods

It is sometimes called an **external approximation** since we consider the problem in a larger space $S \supset \mathbf{V}$ and extend the variational form to S. Consider a variational problem

$$a(u,v) =< f, v >_{V' \times V}, \quad \forall v \in V. \tag{1.71}$$

Let $\tilde{a}$ be a canonical extension to $S \times S$ satisfying

$$\tilde{a}(u,v) = a(u,v), \quad \forall u, v \in V. \tag{1.72}$$

Moreover let $\mathbf{V}_h \subset S$ be a family of f.d. space such that

$$v = \lim_{h\to 0} v_h \Rightarrow v \in V. \tag{1.73}$$

Using standard coerciveness and continuity argument, we get Strang Lemma

$$\|u - u_h\|_S \le C \inf_{v_h\in\mathbf{V}_h} \|u - v_h\|_S + \sup_{v_h\in\mathbf{V}_h} \frac{|\tilde{a}(u, v_h) - (\tilde{f}, v_h)|}{\|v_h\|_S}. \tag{1.74}$$

The second term is called **consistency error**.

Patch Test or Céa-Test

If a nonconforming space contains $P_k(K)$, the first term give an optimal order. Now we have to study the second term. A rule to have an optimal order approximation is *the moment up to degree* $k-1$ *of* u_h *on any interface must be continuous*, that is,

$$\int_e u_h p_{k-1}\, ds, \quad p_{k-1} \in P_{k-1}(e) \tag{1.75}$$

is continuous across e(edge in 2D).

Example 1.5.1.

$$\mathcal{L}_k^{1,nc} = \{v_h \in L^2(\Omega), v_h|_K \in P_k(K), \sum_K \int_{\partial K} u_h\phi\, ds = 0, \forall \phi \in R_{k-1}(\partial K)\} \tag{1.76}$$

Show that if a function in $\mathcal{L}_k^{1,nc}$ is continuous at k Gauss-Legnedre points of each side, then it passes the patch test: If $\sum_{i=1}^k w_i\psi(g_i)$ is the Gauss quadrature, then for $\phi \in P_{k-1}$

$$\int_{\partial K} (u_h^+ + u_h^-)\phi\, ds = \sum_{i=1}^k w_i(u_h^+ + u_h^-)\phi(g_i) = 0$$

since u_h is continuous at k Gauss-Legnedre points and the quadrature is exact up to order $2k-1$. Thus we have the patch test.

1.6 BDM space

Let

$$BDM_k(K) = (P_k(K))^n \tag{1.77}$$

$$dim\, BDM_k(K) = dim\,(P_k(K))^n = \begin{cases} (k+1)(k+2) & \text{for } n=2 \\ \frac{1}{2}(k+1)(k+2)(k+3) & \text{for } n=3. \end{cases} \tag{1.78}$$

For $\mathbf{v} \in BDM_k(K)$, we have $\operatorname{div} \mathbf{v} \in P_{k-1}(K)$ and $\mathbf{v}\cdot\mathbf{n}$ on ∂K belongs to $R_k(\partial K)$. In order to have $\mathbf{v} \in H(\operatorname{div};\Omega)$ it is necessary to ensure continuity of $\mathbf{v}\cdot\mathbf{n}$ on the interfaces. The following shows that three conditions are enough to determine $\mathbf{v}$ locally:

Proposition 1.6.1. *Let $\Phi_k := \{\boldsymbol{\phi} \in (P_k)^n;\, div\, \boldsymbol{\phi}_k = 0,\, \boldsymbol{\phi}_k \cdot \mathbf{n}|_{\partial K} = 0\}$. For $k \geq 1$ and $\mathbf{v} \in BDM_k$ the following imply $\mathbf{v} = 0$.*

$$\int_{\partial K} \mathbf{v}\cdot\mathbf{n} p_k\, ds = 0, \quad 3(k+1),\, \forall p_k \in R_k(\partial K), \tag{1.79}$$

$$\int_K \mathbf{v}\cdot grad\, p_{k-1}\, dx = 0, \quad \frac{1}{2}(k^2+k-2),\, \forall p_{k-1} \in P_{k-1}(K), \tag{1.80}$$

$$\int_K \mathbf{v}\cdot\boldsymbol{\phi}_k\, dx = 0, \quad \frac{1}{2}(k-1)k,\, \forall \boldsymbol{\phi}_k \in \Phi_k. \tag{1.81}$$

By counting

$$dim\, R_k(\partial K) + dim\, P_{k-1}(K) - 1 =$$
$$\begin{cases} 3(k+1) + \frac{1}{2}k(k+1) - 1 = \frac{1}{2}(k^2+7k+4) & \text{for } n=2 \\ 4\frac{1}{2}(k+1)(k+2) + \frac{1}{6}k(k+1)(k+2) - 1 = \frac{1}{6}(k^3+15k^2+38k+18) & \text{for } n=3. \end{cases}$$

From this we can deduce by linear algebra

$$dim\,\Phi_k = \begin{cases} \frac{1}{2}k(k-1) = dim\, P_{k-2}(K), & n=2, k\geq 2 \\ \frac{1}{2}(k^3-k^2) - \frac{(k-2)(k-1)k}{6} = dim\, P^3_{k-2}(K) - dim\, P_{k-3}(K), & n=3. \end{cases}$$

In two dimensional case, the space Φ_k has simple characterization.

$$\Phi_k = \{\boldsymbol{\phi}_k | \boldsymbol{\phi}_k = \operatorname{curl} b_K p_{k-2}, \quad p_{k-2} \in P_{k-2}(K)\}, \tag{1.82}$$

where $b_K = \lambda_1\lambda_2\lambda_3 \in B_3(K)$ is the bubble function.

Now to use above three conditions as dof, it is necessary to check these conditions are linearly independent, in fact, we have

Proposition 1.6.2. *Let $g \in R_k(\partial K)$ and $f \in P_{k-1}(K)$ such that*

$$\int_{\partial K} g\mathbf{v}\cdot\mathbf{n}\, ds + \int_K \mathbf{v}\cdot grad\, f\, dx = 0, \quad \forall \mathbf{v} \in BDM_k(\partial K). \tag{1.83}$$

Then $g = 0$ and f is constant.

BDFM Space

Back to BDM. One can check the by restricting $\mathbf{v}\cdot\mathbf{n}$ to $R_1(\partial K)$ instead of $R_2(\partial K)$ in the definition of BDM_2, Proposition 1.6.1 still holds with the same Φ_k. Thus we get another space called $BDFM_2$ which has same approximation property but lying between BDM_1 and RT_2. The dimension is 9.

1.6.1 BDM - Rectangular case

In this case, the use of reference element is essential. So let $\hat{K} = [-1, 1]^n$ and we build spaces on $\hat{K}$.

Let us consider for $n = 2$

$$BDM[k] = \{\mathbf{v}|\mathbf{v} = \mathbf{p}_k + r\mathrm{curl}\,(x^{k+1}y) + s\mathrm{curl}\,(xy^{k+1}), \mathbf{p}_k \in (P_k)^2\}, \tag{1.84}$$

for $n = 3$ we can consider similarly. Those have been designed so that the following hold.

$$\begin{cases} \mathrm{div}\,\mathbf{v} \in P_{k-1}, \\ \mathbf{v}\cdot\mathbf{n}|_{e_i} \in P_k(e_i). \end{cases} \tag{1.85}$$

For dof we have

Proposition 1.6.3. *For any $\mathbf{v} \in BDM_{[k]}(\hat{K})$, the relations(when $n = 2$)*

$$\int_{e_i} \mathbf{v}\cdot\mathbf{n}\phi_k\,ds = 0, \quad \forall\phi_k \in P_k(e_i) \tag{1.86}$$

$$\int_{\hat{K}} \mathbf{v}\cdot\boldsymbol{\phi}_{k-2}\,dx = 0, \quad \forall\boldsymbol{\phi}_{k-2} \in (P_{k-2})^n \tag{1.87}$$

imply $\mathbf{v} = 0$. For $n = 3$, e_i must be replaced by a face f_i.

Remark 1.6.4. Note that these spaces have the same number of degrees on the sides or faces as $RT_{[k]}$ and still contains $(P_k(K))^n$ so that we have the same order of approximation.

BDFM for rectangular case

As in the triangular case, one can restrict $\mathbf{v}\cdot\mathbf{n}$, $\mathbf{v} \in BDM_{[k+1]}$ to belong to $P_k(e_i)$ instead of $P_{k+1}(e_i)$, we obtain $BDFM_{[k+1]}$. We have (for $n = 2$)

$$BDFM_{[k+1]} = (P_{k+1})^2 \backslash \begin{pmatrix} 0 \\ x^{k+1} \end{pmatrix} \backslash \begin{pmatrix} y^{k+1} \\ 0 \end{pmatrix} \tag{1.88}$$

Compare with

$$BDM_{[k+1]} = (P_{k+1})^2 + r\text{curl}\,(x^{k+2}y) + s\text{curl}\,(xy^{k+2}). \tag{1.89}$$

The difference is 4(one each side). Try to do same thing for $n = 3$.

1.7 Interpolation operator and error estimate

We can use the degrees of freedom in each of the mixed finite element space to define the interpolation operator.(see p.125 Brezzi book.) To define the interpolation operator, we need a slightly more regularity than $H(\text{div}\,;\Omega)$.(It is known that $\mathbf{v}\cdot\mathbf{n} \in H^{-1/2}(\partial K)$ if $\mathbf{v} \in H(\text{div}\,;\Omega)$.) In fact, we need the moment $\mathbf{v}\cdot\mathbf{n}$ up to certain order. Since $R_k(\partial K)$ is not a subspace of $H^{1/2}(\partial K)$,(Even if it is polynomial on each edge, it does not have continuity at vertices), the dof

$$\int_{e_i} \mathbf{v}\cdot\mathbf{n}\phi_k\, ds, \quad \phi_k \in R_k(\partial K)$$

does not make sense. For it to make sense, we have to assure $\mathbf{v}\cdot\mathbf{n} \in H^{1/2}(\partial K)$. Indeed it is possible if $\mathbf{v} \in W(K)$ defined by

$$W(K) = \{\mathbf{v} \in (L^s(K))^n | \text{div}\,\mathbf{v} \in L^2(\Omega)\}, \tag{1.90}$$

for some $s > 2$. Thus, we define Π_K on $W(K) \to M_k(K)$, where $M_k(K)$ is any one of the spaces.

Proposition 1.7.1. *Let K be affine element(i.e, the image of $\hat{K}$ under an affine map $A\mathbf{x}+\mathbf{b}$). Then for $1 \le m \le k+1$, and $s = 0, 1$ and for any $\mathbf{v} \in (H^m(K))^n$, we have($C$ dependent only on k and shape of K)*

$$\|\mathbf{v} - \Pi_K\mathbf{v}\|_{s,K} \le Ch_K^{m-s}|\mathbf{v}|_{m,K}. \tag{1.91}$$

For the error analysis, we need the divergence of each space:

$$\begin{aligned}
\text{div}\,(BDM_k(K)) &= \text{div}\,(BDM_{[k]}(K)) = P_{k-1}(K)\\
\text{div}\,(BDFM_{k+1}(K)) &= \text{div}\,(BDFM_{[k+1]}(K)) = P_k(K)\\
\text{div}\,(RT_{[k]}(K)) &= P_k(K)\\
\text{div}\,(RT_{[k]}(K)) &= \mathcal{F}(Q_k(K)),
\end{aligned}$$

where $\mathcal{F}(v) = \hat{v}\circ\mathbf{F}^{-1}$.

Proposition 1.7.2. *Let K be affine element and let P_K^0 be the L^2 projection on $W_k(K) = \mathrm{div}\,(\mathbf{V}_\mathrm{k})$. Then for $\mathbf{v} \in \mathbf{V}_k(K)$*

$$\mathrm{div}\,(\Pi_\mathrm{K}\mathbf{v}) = \mathrm{P}_\mathrm{K}^0 \,div\, \mathbf{v}. \tag{1.92}$$

In other words,

$$(\mathrm{div}\,(\Pi_\mathrm{K}\mathbf{v}), \phi) = (\mathrm{div}\,\mathbf{v}, \phi), \quad \forall \phi \in \mathrm{W_k(K)}.$$

Proof. For any $\phi \in W_k(K)$

$$\int_K \phi \mathrm{div}\,(\Pi_K \mathbf{v} - \mathbf{v}) dx = -\int_K (\mathbf{v} - \Pi_K \mathbf{v}) \cdot \mathrm{grad}\,\phi\, dx + \int_{\partial K} (\mathbf{v} - \Pi_K \mathbf{v}) \cdot \mathbf{n} \phi\, ds. \tag{1.93}$$

The right hand side vanishes by the definition of the interpolation operator. □

1.7.1 Global estimate- Duality for RT

Theorem 1.7.3. *We have*

$$\|p - p_h\|_0 \le \begin{cases} Ch\|p\|_2, & k = 0 \\ Ch^r\|p\|_r, & k \ge 1, 2 \le r \le k+1 \end{cases} \tag{1.94}$$

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \le Ch^r\|p\|_{r+1}, \quad 1 \le r \le k+1 \tag{1.95}$$

$$\|\mathrm{div}\,(\mathbf{u} - \mathbf{u}_\mathrm{h})\|_0 \le Ch^r\|p\|_{r+2}, \quad 0 \le r \le k+1. \tag{1.96}$$

1.8 BDM-Two family

1.8.1 Duality argument-Brezzi-Douglas-Marini

We start from error equations

$$(\alpha \mathbf{e}_h, \mathbf{v}) - (\mathrm{div}\,\mathbf{v}, z) = 0, \quad \mathbf{v} \in \mathbf{V}_h = (P_k)^2, k \ge 1 \tag{1.97a}$$

$$(\mathrm{div}\,\mathbf{e}_h, w) = 0, \quad w \in W_h = P_{k-1} \tag{1.97b}$$

where $\mathbf{e}_h = \mathbf{u} - \mathbf{u}_h$, $\boldsymbol{\sigma}_h = \Pi_h \mathbf{u} - \mathbf{u}_h$, $z = P_h p - p_h$, $\rho = p - P_h p$.

Lemma 1.8.1. *We have*

$$\|z\|_0 \le Ch\|\mathbf{e}_h\|_0 + ch^{min(2,k)}\|\mathrm{div}\,\mathbf{e}_\mathrm{h}\|_0 \tag{1.98}$$

Proof. Let ϕ be the sol. of dual problem $L^*\phi = \psi$. Then with $\alpha = \mathcal{K}^{-1}$,

$$\begin{aligned}(z, \psi) &= (\alpha \mathbf{e}_h, \mathcal{K}\mathrm{grad}\,\phi - \Pi_h \mathcal{K}\mathrm{grad}\,\phi) + (\mathrm{div}\,\mathbf{e}_h, \phi - P_h\phi) \\ &\le Ch\|\mathbf{e}_h\|_0\|\psi\|_0 + Ch^{min(2,k)}\|\mathrm{div}\,\mathbf{e}_h\|_0\|\phi\|_2.\end{aligned}$$

Dividing by $\|\psi\|_0$ and take supremum, we get the result. □

Theorem 1.8.2. *We have(subtle difference between BDM and RT-See Roberts.)*

$$\|p - p_h\|_0 \leq Ch^r\|f\|_{r-2}, \quad 2 \leq r \leq k+1 \tag{1.99}$$
$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leq Ch^r\|f\|_{r-1}, \quad 1 \leq r \leq k+1 \tag{1.100}$$
$$\|\mathrm{div}\,(\mathbf{u} - \mathbf{u}_\mathrm{h})\|_0 \leq Ch^r\|f\|_r, \quad 0 \leq r \leq k. \tag{1.101}$$

Proof. Take $\mathbf{v} = \boldsymbol{\sigma}_h = \Pi_h\mathbf{u} - \mathbf{u}_h$ in the error equation. Note

$$\boldsymbol{\sigma}_h = \Pi_h\mathbf{u} - \mathbf{u}_h = \mathbf{u} - \mathbf{u}_h - (\mathbf{u} - \Pi_h\mathbf{u}) = \mathbf{e}_h - (\mathbf{u} - \Pi_h\mathbf{u}).$$

Then by (1.97)

$$(\alpha\boldsymbol{\sigma}_h, \boldsymbol{\sigma}_h) = (\alpha\mathbf{e}_h, \boldsymbol{\sigma}_h) - (\alpha(\mathbf{u} - \Pi_h\mathbf{u}), \boldsymbol{\sigma}_h) = -(\alpha(\mathbf{u} - \Pi_h\mathbf{u}), \boldsymbol{\sigma}_h).$$

Hence $\|\boldsymbol{\sigma}_h\| \leq C\|\mathbf{u} - \Pi_h\mathbf{u}\|$ and so

$$\|\mathbf{u} - \mathbf{u}_h\| \leq \|\mathbf{u} - \Pi_h\mathbf{u}\| + \|\Pi_h\mathbf{u} - \mathbf{u}_h\| \leq Ch^r\|\mathbf{u}\|_r \text{ for } 1 \leq r \leq k+1. \tag{1.102}$$

Meanwhile(since $\mathrm{div}\,\boldsymbol{\sigma}_h = 0$ replace $\mathbf{e}_h$ by $\boldsymbol{\sigma}_h$ in (1.97))

$$\|\mathrm{div}\,\mathbf{e}_h\|_0 = \|\mathrm{div}\,(\mathbf{u} - \Pi_h\mathbf{u})\|_0 \leq Ch^r\|\mathrm{div}\,\mathbf{u}\|_r, 1 \leq r \leq k.$$

From Lemma 1.8.1 and (1.102)

$$\begin{aligned}
\|p - p_h\|_0 &\leq \|P_hp - p_h\|_0 + \|p - P_hp\|_0 \\
&\leq Ch\|\mathbf{e}_h\|_0 + Ch^{min(r,k)+min(2,k)}\|\mathrm{div}\,\mathbf{e}_h\|_0 + Ch^{min(r,k)}\|p\|_r \\
&\leq Ch^{1+min(r,k)}\|\mathbf{u}\|_r + Ch^{r+min(2,k)}\|\mathrm{div}\,\mathbf{u}\|_r + Ch^{min(r,k)}\|p\|_r \\
&\leq Ch^r(\|f\|_{r-2} + |g|_{r-1/2})
\end{aligned}$$

for $2 \leq r \leq k+1$. Simplifying the result, we get (1.99). In BDM we lose one order for pressure than RT (See Robert for RT). □

1.9 Hybrid form of mixed methods

The solution of algebraic system associated with mixed formulation can be simplified by the introduction of a Lagrange multiplier to enforce the continuity of normal component of $\mathbf{u}$ across the interelement boundaries. Let

$$M_h = \{m : m_e \in P_k(e) \text{ if } e \subset \Omega, m_e = 0 \text{ if } e \subset \partial\Omega\}. \tag{1.103}$$

Following Fraejis de Veubeke, our problem is to seek $\{\mathbf{u}_h, p_h, m_h\} \in \mathbf{V}_h^k \times W_h^k \times M_h^k$ such that

$$(\alpha\mathbf{u}_h, \mathbf{v}) - \sum_T(\mathrm{div}\,\mathbf{v}, p_h) + \sum_T < \mathbf{v}\cdot\mathbf{n}_T, m_h >_{\partial T} =< \mathbf{v}\cdot\mathbf{n}, g >, \mathbf{v} \in \mathbf{V}_h^k, \tag{1.104a}$$

$$\sum_T (\operatorname{div} \mathbf{u}_h, w)_T = (f, w), w \in W_h^k \quad (1.104b)$$

$$\sum_T (\mathbf{u}_h, q)_{\partial T} = 0, q \in M_h^k. \quad (1.104c)$$

We introduce some norms

$$|m_h|_{0,h}^2 = \sum_e \|m_h\|_{0,e}^2 \quad (1.105a)$$

$$|m_h|_{-1/2,h}^2 = \sum_e |e| \|m_h\|_{0,e}^2. \quad (1.105b)$$

Lemma 1.9.1. *If* $\{\mathbf{u}_h, p_h, m_h\} \in \mathbf{V}_h^k \times W_h^k \times M_h^k$ *is the solution of (1.104) then*

$$\|m_h - Q_h^k p\|_{0,e} \le C\{h^{1/2}\|\mathbf{u} - \mathbf{u}_h\|_{0,T} + h^{-1/2}\|P_h p - p_h\|_{0,T}\}. \quad (1.106)$$

Proof. Let $e \subset \Omega \cap T$ and define $\mathbf{v}$ on T by requiring

$$\mathbf{v} \cdot \mathbf{n}_e = m_h - Q_h p \text{ on } e \quad (1.107a)$$

$$\mathbf{v} \cdot \mathbf{n}_T = 0 \text{ on } \partial T \backslash e \quad (1.107b)$$

$$(\mathbf{v}, \boldsymbol{\phi}_{k-1})_T = 0, \forall \boldsymbol{\phi}_{k-1} \in (P_{k-1}(K))^2. \quad (1.107c)$$

The existence and uniqueness is given by the mixed finite element(RT) construction. A scaling argument gives(Ok, use Piolar transform $\mathbf{v} = \frac{1}{J} B\hat{v} \circ F^{-1}$)

$$h\|\mathbf{v}\|_{1,T} + \|\mathbf{v}\|_{0,T} \le Ch^{1/2}\|m_k - Q_h^k p\|_{0,e}. \quad (1.108)$$

Take $\mathbf{v}$ as test function in (1.104)

$$(\alpha \mathbf{u}_h, \mathbf{v})_T - (\operatorname{div} \mathbf{v}, p_h)_T + < m_h, m_h - Q_h p >_e = 0.$$

Since

$$(\alpha \mathbf{u}, \mathbf{v})_T - (\operatorname{div} \mathbf{v}, p)_T + < u, m_h - Q_h p >_e = 0,$$

we have

$$\|m_h - Q_h u\|_{0,e}^2 = < m_h - p, m_h - Qhp >_e = (\alpha(\mathbf{u} - \mathbf{u}_h), \mathbf{v})_T - (\operatorname{div} \mathbf{v}, z_h)_T$$

and the result follows from (1.108) □

1.10 Trace estimate

Lemma 1.10.1. *Let* $v \in W_p^1(\Omega)$*. Then*

$$\|v\|_{L^p(\partial\Omega)} \le C\|v\|_{L^p(\Omega)}^{1-1/p}\|v\|_{W_p^1(\Omega)}^{1/p}.$$

See Brenner-Scott's book.

Corollary 1.10.2. *On the reference element,*

$$\|\hat{v}\|_{L^2(\partial\hat{\Omega})}^2 \le C\|\hat{v}\|_{L^2(\hat{\Omega})}(\|\hat{v}\|_{L^2(\hat{\Omega})} + |\hat{v}|_{H^1(\hat{\Omega})}).$$

On a finite element K of diameter h, we can show

$$\|v\|_{L^2(K)} \approx Ch\|\hat{v}\|_{L^2(\hat{K})}, \quad |v|_{H^1(K)} \approx |\hat{v}|_{H^1(\hat{K})}, \quad |v|_{L^2(\partial K)} \approx Ch^{1/2}|\hat{v}|_{L^2(\partial\hat{K})}.$$

Transfer to the shape regular finite element(no quasi uniformity necessary), we have for all $v \in H^1(\Omega)$

$$\|v\|_{L^2(\partial K)}^2 \le C(h^{-1}\|v\|_{L^2(K)}^2 + \|v\|_{L^2(K)}|v|_{H^1(K)}).$$

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