Numerical PDE

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Chapter 3

Mixed Method

3.1 Raviart Thomas Mixed Method

$$-\Delta u = f \quad \text{in } \Omega \tag{3.1a}$$
$$u = 0, \quad \text{in } \Gamma \tag{3.1b}$$

Thus we get (3.5) below.

Let us introduce some notations: Given $m \geq 0$ a nonnegative integer,

$$H^m(\Omega) = \{v \in L^2(\Omega) : \partial^\alpha v \in L^2(\Omega), |\alpha| \leq m\}$$

is the usual Sobolev space of order m with the semi norm and norm

$$|v|_{m,\Omega} = \left(\sum_{|\alpha|=m} \int_\Omega |\partial^\alpha v|^2 dx \right)^{1/2}, \quad \|v\|_{m,\Omega} = \left(\sum_{|\alpha|\leq m} \int_\Omega |\partial^\alpha v|^2 dx \right)^{1/2}$$

Given a vector valued functions $\mathbf{q} \in (H^m(\Omega))^n$ set

$$|\mathbf{q}|_{m,\Omega} = \left(\sum_{i=1}^{n} |q_i|^2_{m,\Omega} \right)^{1/2}, \quad \|\mathbf{q}\|_{m,\Omega} = \left(\sum_{i=1}^{n} \|q_i\|^2_{m,\Omega} \right)^{1/2}.$$

3.2 Mixed Formulation

Introduce the space

$$\mathbf{H}(\operatorname{div}; \Omega) = \{\mathbf{q} \in (L^2(\Omega))^n : \operatorname{div} \mathbf{q} \in L^2(\Omega)\} \tag{3.2}$$

with the norm equipped with

$$\|\mathbf{q}\|_{H(\operatorname{div};\Omega)} = \{\|\mathbf{q}\|_{0,\Omega}^2 + \|\operatorname{div}\mathbf{q}\|_{0,\Omega}^2\}^{1/2} \tag{3.3}$$

Given $\mathbf{q} \in \mathbf{H}(\operatorname{div};\Omega)$ we can define its normal components $\mathbf{q}\cdot\nu \in H^{-1/2}(\Gamma)$ where $H^{-1/2}(\Gamma)$ is the dual space of $H^{1/2}(\Gamma)$ and ν is the unit outward normal along Γ. By Green's formula, we see

$$\int_\Omega (\nabla v\cdot\mathbf{q} + v\operatorname{div}\mathbf{q})\,dx = \int_\Gamma v\mathbf{q}\cdot\nu\,d\gamma, \qquad v \in H^1(\Omega) \tag{3.4}$$

where the integral $\int_\Gamma$ represent the duality between the spaces $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

Define the problem (**P**)

Definition 3.2.1. Find $(\mathbf{p}, u)$ in $\mathbf{H}(\operatorname{div};\Omega)\times L^2(\Omega)$ such that

$$\int_\Omega \mathbf{p}\cdot\mathbf{q}\,dx + \int_\Omega u\operatorname{div}\mathbf{q}\,dx = 0, \forall\mathbf{q}\in\mathbf{H}(\operatorname{div};\Omega) \tag{3.5}$$

$$\int_\Omega v(\operatorname{div}\mathbf{p} + f)\,dx = 0, \forall v \in L^2(\Omega) \tag{3.6}$$

Theorem 3.2.2. *The problem (P) ha a unique solution* $(\mathbf{p}, u)$ *in* $\mathbf{H}(\,div;\Omega)\times L^2(\Omega)$*. In addition u is the solution of (3.1) and we have*

$$\mathbf{p} = \nabla u. \tag{3.7}$$

Proof. Uniqueness: Assume $f = 0$ in (3.6). Then we see $\operatorname{div}\mathbf{p} = 0$. Taking $\mathbf{q} = \mathbf{p}$, in (3.5) we obtain $\mathbf{p} = 0$. Therefore

$$\int_\Omega u\operatorname{div}\mathbf{q}\,dx = 0, \forall\mathbf{q}\in\mathbf{H}(\operatorname{div};\Omega) \tag{3.8}$$

Let $w \in H^1(\Omega)$ be such that

$$\Delta w = u \text{ in } \Omega$$

Then by choosing $\mathbf{q} = \nabla w$ in (3.8), we get $u = 0$. It remains to show that the pair $(\mathbf{p}, u) = (\nabla u, u)$ is a solution of (P). We see

$$\operatorname{div}\mathbf{p} + f = \nabla u + f = 0$$

while, by Green's formula

$$\int_\Omega (\mathbf{p}\cdot\mathbf{q} + u\operatorname{div}\mathbf{q})\,dx = \int_\Gamma u\mathbf{q}\cdot\nu\,d\gamma = 0$$

which is (3.5). □

Remark 3.2.3. One can check that the solution of the problem (P) may be characterized as the unique saddle point of the functional

$$L(\mathbf{q}, v) = I(\mathbf{q}) + \int_\Omega v(\operatorname{div}\mathbf{q} + f)\, dx$$

over the space $\mathbf{H}(\operatorname{div};\Omega) \times L^2(\Omega)$, i.e.,

$$L(\mathbf{p}, v) \leq L(\mathbf{p}, u) \leq L(\mathbf{q}, u), \forall \mathbf{q} \in \mathbf{H}(\operatorname{div};\Omega), v \in L^2(\Omega)$$

Hence u is the Lagrange multiplier associated with the constraint $\operatorname{div}\mathbf{p} + f = 0$.

First inequality:

$$\int_\Omega (\operatorname{div}\mathbf{q} + f)(v - u)\, dx \leq 0, \quad \forall v \Rightarrow \int_\Omega (\operatorname{div}\mathbf{q} + f)\, dxv = 0$$

Second inequality:

$$DL(\mathbf{p}, u) \cdot \mathbf{q} = \int \mathbf{p} \cdot \mathbf{q} + u \operatorname{div}\mathbf{q}\, dx = 0 \quad \forall \mathbf{q} \Rightarrow \int_\Omega (\mathbf{p} - \nabla u) \cdot \mathbf{q}\, dx = 0 \Rightarrow \mathbf{p} = \nabla u$$

Discretization

Given two finite dimensional spaces $\mathbf{Q}_h$ and V_h such that

$$\mathbf{Q}_h \subset \mathbf{H}(\operatorname{div};\Omega), V_h \subset L^2(\Omega) \tag{3.9}$$

$$\int_\Omega (\mathbf{p}_h \cdot \mathbf{q}_h + u_h \operatorname{div}\mathbf{q}_h)\, dx = 0, \quad \mathbf{q}_h \in \mathbf{Q}_h \tag{3.10}$$

$$\int_\Omega v_h(\operatorname{div}\mathbf{p}_h + f)\, dx = 0, \quad v_h \in V_h \tag{3.11}$$

The following result is from Brezzi([3]).

Theorem 3.2.4. *Assume*

$$\begin{cases} \mathbf{q}_h \in \mathbf{Q}_h \\ \int_\Omega v_h\, div\,\mathbf{q}_h\, dx = 0, \quad \forall v_h \in V_h \end{cases} \Rightarrow\ div\,\mathbf{q}_h = 0 \tag{3.12}$$

and that there exists a constant $c > 0$ such that

$$\inf_{v_h \in V_h} \sup_{\mathbf{q}_h \in \mathbf{Q}_h} \frac{\int_\Omega v_h\, div\,\mathbf{q}_h\, dx}{\|\mathbf{q}_h\|_{\mathbf{H}(\,div;\Omega)}} \geq c\|v_h\|_{0,\Omega} \tag{3.13}$$

Then the problem (P_h) has a unique solution $(\mathbf{p}_h, u_h) \in \mathbf{Q}_h \times V_h$ and there exists a constant $\tau > 0$ which depends only on c such that

$$\begin{cases} \|\mathbf{p} - \mathbf{p}_h\|_{\mathbf{H}(\,div;\Omega)} + \|u - u_h\|_{0,\Omega} \leq \\ \tau \left\{ \inf_{\mathbf{q}_h \in \mathbf{Q}_h} \|\mathbf{p} - \mathbf{q}_h\|_{\mathbf{H}(\,div;\Omega)} + \inf_{v_h \in V_h} \|u - v_h\|_{0,\Omega} \right\} \end{cases} \tag{3.14}$$

Remark 3.2.5. Define $\nabla_h \in L(V_h, \mathbf{Q}_h)$ by

$$\int_\Omega \nabla_h v_h \cdot \mathbf{q}_h \, dx = -\int_\Omega v_h \operatorname{div} \mathbf{q}_h \, dx, \quad \forall v_h \in V_h, \mathbf{q}_h \in \mathbf{Q}_h. \tag{3.15}$$

This operator ∇_h is clearly an approximation of ∇. Now the function u_h may be characterized as the unique solution of the following problem:

$$\int_\Omega \nabla u_h \cdot \nabla v_h dx = \int_\Omega f v_h dx, \forall v_h \in V_h. \tag{3.16}$$

Clearly (3.16) has a unique solution u_h and the pair $(\nabla u_h, u_h)$ is the solution of problem (P_h). In general, $V_h \not\subset H_0^1(\Omega)$.

It remains to construct finite dimensional spaces $\mathbf{Q}_h$ and V_h so that they satisfy a good approximation property and compatibility conditions (3.12) and (3.13).

From here and thereafter, we shall assume, for convenience, that a bounded polygon and triangulation $\mathcal{T}_h$ consists of triangles and parallelograms whose diameters are $\leq h$.

Lemma 3.2.6. *A function $\mathbf{q} \in (L^2(\Omega))^2$ belongs to $\mathbf{H}(\,div; \Omega)$ if and only if the following conditions hold:*

(i) for all $K \in \mathcal{T}_h$, the restriction $\mathbf{q}|_K$ of $\mathbf{q}$ to the set K belongs to $\mathbf{H}(\,div; K)$.

(ii) for any pair of adjacent elements $K_1, K_2 \in \mathcal{T}_h$, we have the following reciprocity condition

$$\mathbf{q} \cdot \nu_{K_1} + \mathbf{q} \cdot \nu_{K_2} = 0 \quad \text{on } e = K_1 \cap K_2 \tag{3.17}$$

where ν_{K_i} is the unit outward normal vector along the boundary of K_i, $i = 1, 2$.

Proof. Without loss of generality, we may assume $\Omega = K_1 \cup K_2$. Necessity is trivial. For sufficiency, let $\mathbf{q} \in (L^2(\Omega))^2$ satisfy two conditions (i) and (ii). Then for any $v \in C_0^\infty(\Omega)$,

$$\begin{aligned}
\int_{K_1} \operatorname{div} \mathbf{q} v \, dx + \int_{K_2} \operatorname{div} \mathbf{q} v \, dx &= \int_{\partial K_1} \mathbf{q} \cdot \nu v \, ds + \int_{\partial K_2} \mathbf{q} \cdot \nu v \, dx \\
&\quad - \int_{K_1} \mathbf{q} \cdot \nabla v \, ds - \int_{K_2} \mathbf{q} \nabla v \, dx \\
&= \int_{\partial \Omega} \mathbf{q} \cdot \nu v \, ds - \int_\Omega \mathbf{q} \nabla v \, dx
\end{aligned}$$

$$= \quad -\int_\Omega \mathbf{q}\nabla v \, dx$$

where Green's theorem and (ii) were used and this equality defines $\operatorname{div}\mathbf{q}$ weakly(in the distribution sense). i.e, define

$$(\operatorname{div}\mathbf{q})|_{K_i} = \operatorname{div}(\mathbf{q}|_{K_i}), \quad i = 1, 2$$

then $\operatorname{div}\mathbf{q}$ is defined a.e. and by taking a sequence $v_n \to \operatorname{div}(\mathbf{q}|_{K_i})$ on each i, we see that $\operatorname{div}\mathbf{q}$ is in $L^2(\Omega)$.

□

3.2.1 Mixed Triangular Element

In this subsection all $K = \hat{K}$(reference element) and functions $\mathbf{q}, \phi$ are $\hat{\mathbf{q}}, \hat{\phi}$, etc. Let

$$R_k(\partial K) = \{\phi \in L^2(\partial K), \phi_{e_i} \in P_k(e_i), \forall e_i\} \tag{3.18}$$

Assume K is a triangle. Define $\mathbf{Q} \subset \mathbf{H}(\operatorname{div}; K)$ so that

$$(P_k)^2 \subset \mathbf{Q} \tag{3.19}$$
$$dim(\mathbf{Q}) = (k+1)(k+3) \tag{3.20}$$
$$\operatorname{div}\mathbf{q} \in P_k \tag{3.21}$$
$$\mathbf{q}\cdot\nu \in R_k(\partial K) \tag{3.22}$$
$$\mathbf{Q}_0 = \{\mathbf{q} \in \mathbf{Q}, \operatorname{div}\mathbf{q} = 0\} \subset (P_k)^2 \tag{3.23}$$

Lemma 3.2.7. *Assume the condition (3.19)- (3.23) hold. Then a function $\mathbf{q} \in \mathbf{Q}$ is uniquely determined by*

(a) $\int_e \mathbf{q}\cdot\nu p \, ds$ for all $p \in P_k(e)$ on each side e of K. (Equivalently, the values of $\mathbf{q}\cdot\nu$ at $k+1$ points of each side e of K.

(b) The moments of order less than equal to $k-1$ of $\mathbf{q}$, i.e,

$$\int_K \mathbf{q}\cdot\mathbf{p}\, dx, \quad \mathbf{p} \in (P_{k-1})^n$$

Proof. By (3.20), the number of degrees of freedom of (a), (b) is equal to the dimension of $\mathbf{Q}_K$. It suffices to show that the conditions in (a), (b) are indeed independent, Let $\mathbf{q} \in \mathbf{Q}_K$ satisfy $(a) = 0, (b) = 0$. We shall show that $\mathbf{q} = 0$.

The conditions (3.22) and (a) imply $\mathbf{q} \cdot \nu = 0$ on ∂K. Use Green's formula and (b) to see that for all $\phi \in P_k$

$$\int_K \phi \operatorname{div} \mathbf{q} \, dx = -\int_K \nabla\phi \cdot \mathbf{q} \, dx + \int_{\partial K} \phi \mathbf{q} \cdot \nu \, ds = 0$$

Since $\operatorname{div} \mathbf{q} \in P_k$, we get $\operatorname{div} \mathbf{q} = 0$ and hence $\mathbf{q} \in \mathbf{Q}_0$.

From (3.23) there exists $w \in P_{k+1}$(unique up to an additive constant) such that

$$\mathbf{q} = \operatorname{curl} w = (\frac{\partial w}{\partial \eta}, -\frac{\partial w}{\partial \xi})$$

Note that $\mathbf{q} \cdot \nu = \frac{\partial w}{\partial \tau} = 0$ on ∂K, we may assume $w = 0$ on ∂K and write

$$w = \lambda_1 \lambda_2 \lambda_3 z, \quad z \in P_{k-2}(z = 0, \text{ if } k \leq 1)$$

Again by (b), we have for any $\mathbf{r} \in (P_{k-1})^2$

$$0 = \int_K \mathbf{q} \cdot \mathbf{r} \, dx = \int_K \operatorname{curl} w \cdot \mathbf{r} \, dx = \int_K w \operatorname{curl} dx = \int_K \lambda_1 \lambda_2 \lambda_3 z \operatorname{curl} \mathbf{r} \, dx$$

where $\operatorname{curl} \mathbf{r} = \frac{\partial r_2}{\partial \xi} - \frac{\partial r_1}{\partial \eta} \in P_{k-2}$. We can choose $\mathbf{r}$ so that $z = \operatorname{curl} \mathbf{r}$ and then

$$\int_K \lambda_1 \lambda_2 \lambda_3 z \, dx = 0$$

Therefore $z = 0$ and $w = 0$, $\mathbf{q} = \operatorname{curl} w = 0$. □

Remark 3.2.8. The number of above conditions is $3(k+1) + 2 \times_3 \Pi_{k+1} = 2 \times_{k+1} C_{k-1} = (k+1)(k+3)$. There is an alternative way of defining degrees of freedom:

$$\mathbf{P}_k(K) \oplus Span(\mathbf{x}\tilde{P}_k(K))$$

where $\tilde{P}_k(K)$ is the homogeneous polynomial of degree k.

Triangle element by linear mapping

Consider ny triangle K in the plane whose vertices are denoted by $a_i, i = 1, \cdots, 3$. Set

$$h_K = \text{diam } K \qquad (3.24)$$

$$\rho_K = \text{diameter of inscribed circle in } K \qquad (3.25)$$

Let $F_K : \hat{\mathbf{x}} \to F_K(\hat{\mathbf{x}}) = B_K \hat{\mathbf{x}} + \mathbf{b}_K, B_K \in L(\mathbb{R}^2), \mathbf{b}_K \in \mathbb{R}^2$ be the unique affine invertible mapping such that

$$F_K(\hat{a}_i) = a_i, i = 1, 2, 3.$$

If $\hat{\phi}$ is any scalar function defined over $\hat{K}(\partial\hat{K})$, we associate a function on K(pull back) by

$$\phi = \hat{\phi} \circ F_K^{-1}. \tag{3.26}$$

On the other hand, for any vector valued function $\hat{\mathbf{q}} = (\hat{q}_1, \hat{q}_2)$ we associate $\mathbf{q}$ by

$$\mathbf{q} = \frac{1}{J_K} B_K \hat{\mathbf{q}} \circ F_K^{-1} \tag{3.27}$$

where $J_K = \det(B_K)$.

Lemma 3.2.9. *For any* $\hat{\mathbf{q}} \in (H^1(\hat{K}))^2$, *we have*

$$\int_{\hat{K}} \hat{\phi}\, div\, \hat{\mathbf{q}}\, d\hat{x} = \int_K \phi\, div\, \mathbf{q}\, dx, \quad \hat{\phi} \in L^2(\hat{K}) \tag{3.28}$$

$$\int_{\partial\hat{K}} \hat{\phi}\hat{\mathbf{q}} \cdot \hat{\nu}\, d\hat{s} = \int_{\partial K} \phi\mathbf{q} \cdot \nu\, ds, \quad \hat{\phi} \in L^2(\partial\hat{K}) \tag{3.29}$$

Lemma 3.2.10. *For any nonnegative integer* ℓ

$$|\hat{\phi}|_{\ell,\hat{K}} \leq \|B_K\|^{\ell} |J_K|^{-1/2} |\phi|_{\ell,K}, \quad \hat{\phi} \in H^{\ell}(\hat{K}) \tag{3.30}$$

$$|\hat{\mathbf{q}}|_{\ell,\hat{K}} \leq \|B_K\|^{\ell} \|B_K\|^{-1} |J_K|^{1/2} |\mathbf{q}|_{\ell,K}, \hat{\mathbf{q}} \in (H^{\ell}(\hat{K}))^2 \tag{3.31}$$

Now for each K, we associate the space

$$\mathbf{Q}_K = \{\mathbf{q} \in H(\operatorname{div}; K); \hat{\mathbf{q}} \in \hat{\mathbf{Q}}\} \tag{3.32}$$

3.3 Raviart-Thomas Projection

In this section we consider slightly more general equation:

$$-\nabla \cdot \mathbf{a}\nabla u = f \quad \text{in } \Omega \tag{3.33a}$$

$$u = 0, \quad \text{in } \Gamma \tag{3.33b}$$

Let $P_k(K)$ denote the space of polynomials of total degree k.

$$\text{dimension of } P_k(K) = \begin{cases} \frac{1}{2}(k+1)(k+2) \text{ for } n = 2 \\ \frac{1}{6}(k+1)(k+2)(k+3) \text{ for } n = 3 \end{cases}$$

$({}_{n+1}\Pi_k = {}_{n+k}C_k = {}_{n+k}C_n)$ and let $Q_{i,j}(K)(\dim Q_{i,j} = (i+1) \times (j+1))$ denote the space of polynomials of degree $\leq i$ and $\leq j$ in each variable and let $Q_k = Q_{k,k}$

and $\tilde{Q}_k$ be the homogeneous polynomials of degree k. Define the (local) Raviart-Thomas spaces are ($k \geq 0$)

$$\mathbf{V}(K) = \{\mathbf{P}_k(K) \oplus Span(\mathbf{x}\tilde{P}_k(K)\},\ W(K) = P_k(K) \tag{3.34}$$

for triangle and

$$\mathbf{V}(K) = Q_{k+1,k}(K) \times Q_{k,k+1}(K), \quad Q_k(K)$$

for rectangle(parallelogram). Let

$$R_k(\partial K) = \{\phi \in L^2(\partial K), \phi_{e_i} \in P_k(e_i), \forall e_i\} \tag{3.35}$$

$$K_k(\partial K) = R_k(\partial K) \cap C^0(\partial K) \tag{3.36}$$

Let

$$\begin{aligned} \mathbf{V}_h &= \{\mathbf{v} \in \mathbf{H}(\operatorname{div}; \Omega), \mathbf{v}|K \in \mathbf{V}(K), K \in \mathcal{T}_h\} \\ W_h &= \{w \in L^2(\Omega) : w_K \in W(K), K \in \mathcal{T}_h\} \end{aligned}$$

There exists a mapping $\Pi_h : \mathbf{H}^{k+1}(\operatorname{div}; K) \to \mathbf{V}$ such that (Degrees of freedom for local spaces are as follows(Brezzi and Fortin[6] p.126))

$$\begin{cases} \int_{\partial K}(\mathbf{q} - \Pi_h \mathbf{q}) \cdot \nu p_k \, ds = 0, & \forall p_k \in R_k(\partial K), \\ \int_K (\mathbf{q} - \Pi_h \mathbf{q}) \cdot \boldsymbol{\phi}_{k-1} \, dx = 0, & \forall \boldsymbol{\phi}_{k-1} \in (P_{k-1}(K))^2, (k \geq 1) \end{cases} \tag{3.37}$$

for triangles and for rectangles

$$\begin{cases} \int_{\partial K}(\mathbf{q} - \Pi_h \mathbf{q}) \cdot \nu p_k \, ds = 0, & \forall p_k \in R_k(\partial K), \\ \int_K (\mathbf{q} - \Pi_h \mathbf{q}) \cdot \boldsymbol{\phi}_k \, dx = 0, & \forall \boldsymbol{\phi}_k \in Q_{k-1,k}(K) \times Q_{k,k-1}(K), (k \geq 1) \end{cases} \tag{3.38}$$

Define

$$\mathbf{Q}_h = \{\mathbf{q}_h \in \mathbf{H}(\operatorname{div}; \Omega), \mathbf{q}_h|_K \in \mathbf{Q}_K, \forall K\} \tag{3.39}$$

Since $(\operatorname{div} \mathbf{q}_h)_K \in P_k$, a natural choice of V_h is

$$V_h = \{v_h \in L^2(\Omega), v_h|_K \in P_k, \forall K\} \tag{3.40}$$

and (3.12) is automatically satisfied.

inf-sup condition (3.13)

In fact we shall show for any function $v_h \in V_h$, there exists a function $\mathbf{q}_h \in \mathbf{Q}_h$ such that

$$\operatorname{div} \mathbf{q}_h = v_h \tag{3.41}$$

and

$$\|\mathbf{q}_h\|_{\mathbf{H}(\operatorname{div};\Omega)} \le C\|v_h\|_{0,\Omega} \tag{3.42}$$

Next introduce a edge space:

$$M_h = \{\mu_h \in \Pi_K S_k(\partial K); \nu_h|_{\partial K_1} + \nu_h|_{\partial K_2} = 0 \text{ on } K_1 \cap K_2\} \tag{3.43}$$

Theorem 3.3.1. *For any $v_h \in V_h$ there is $\mathbf{q}_h \in \mathbf{Q}_h$ which satisfies (3.41) and (3.42). Hence the inf-sup condition holds.*

Main result.

3.4 Global estimates

Now we consider Dirichlet problem with nonhomogeneous boundary condition $p = -g$ on $\partial\Omega$. Let $\alpha = \mathbf{a}^{-1}$. Then our mixed method is defined by determining $(\mathbf{u}_h, p_h)$ such that

$$(\alpha \mathbf{u}_h, \mathbf{v}) - (\operatorname{div} \mathbf{v}, p_h) = < g, \mathbf{v}\cdot\nu >, \forall \mathbf{v} \in \mathbf{V}_h \tag{3.44a}$$

$$(\operatorname{div} \mathbf{u}_h, w) = (f, w), \forall w \in W_h \tag{3.44b}$$

We shall show p_h and $\mathbf{u}_h$ are of optimal order in L^2 and H^{-s} for $s \le k+1$ provided the domain and the solution p is sufficiently smooth. Indeed we shall prove

$$\|p - p_h\| \le Ch^{k+1}\|p\|_{k+1}, \quad \|\mathbf{u} - \mathbf{u}_h\|_0 \le Ch^{k+1}\|p\|_{k+2} \tag{3.45a}$$

$$\|p - p_h\|_{-k-1} + \|\mathbf{u} - \mathbf{u}_h\|_{-k-1} \le Ch^{2k+2}\|p\|_{k+3}, \tag{3.45b}$$

$$\|\operatorname{div}(\mathbf{u} - \mathbf{u}_h)\|_0 \le Ch^r\|p\|_{r+2}, 0 \le r \le k+1 \tag{3.45c}$$

(PRESENT FALK and Osborns's abstract theory and provide some details) Here triangle includes rectangle also and

$$\operatorname{div} \mathbf{V}_h = W_h \tag{3.46}$$

holds. RT projection is

$$\Pi_h : \mathbf{V} \to \mathbf{V}_h \tag{3.47}$$

such that

(i) P_h is $L^2(\Omega)$ projection

(ii) the following diagram commutes:

$$\begin{array}{ccc} \mathbf{V} & \overset{\text{div}}{\to} & W \\ \Pi_h \downarrow & & \downarrow P_h \\ \mathbf{V}_h & \overset{\text{div}}{\to} & W_h \to 0 \end{array}$$

i,e., $\operatorname{div} \Pi_h = P_h \operatorname{div} : \mathbf{V} \overset{onto}{\to} W_h$.

(iii) The following approximation properties holds

$$\|\mathbf{u} - \Pi_h \mathbf{u}\|_0 \le Ch^r \|\mathbf{u}\|_r, (1 \le r \le k+1) \tag{3.48a}$$
$$\| \operatorname{div} (\mathbf{u} - \Pi_h \mathbf{u})\|_{-s} \le Ch^{r+s} \| \operatorname{div} \mathbf{u}\|_r, (0 \le r, s \le k+1) \tag{3.48b}$$
$$\|p - P_h p\|_{-s} \le Ch^{r+s} \|p\|_r, (0 \le r, s \le k+1) \tag{3.48c}$$
$$\tag{3.48d}$$

Note the following also holds

$$(\operatorname{div} (\mathbf{u} - \Pi_h \mathbf{u}), w_h) = 0, \quad w_h \in W_h \tag{3.49a}$$
$$(\operatorname{div} \mathbf{v}_h, p - P_h p) = 0, \quad \mathbf{v}_h \in \mathbf{V}_h \tag{3.49b}$$

Let

$$\mathbf{e}_h = \mathbf{u} - \mathbf{u}_h, \quad \boldsymbol{\sigma}_h = \Pi_h \mathbf{u} - \mathbf{u}_h \tag{3.50a}$$
$$\eta = p - p_h, \quad \tau = P_h - p_h, \quad \rho = p - P_h p (\rho + \tau = \eta) \tag{3.50b}$$

Then we have error equations

$$(\alpha \mathbf{e}_h, \mathbf{v}) - (\operatorname{div} \mathbf{v}, \eta) = 0, \quad \mathbf{v} \in \mathbf{V}_h \tag{3.51a}$$
$$(\operatorname{div} \mathbf{e}_h, w) = 0, \quad w \in W_h \tag{3.51b}$$

3.4.1 Duality argument-Brezzi-Douglas-Marini for RT element

We start from error equations

$$(\alpha \mathbf{e}_h, \mathbf{v}) - (\operatorname{div} \mathbf{v}, z) = 0, \quad \mathbf{v} \in \mathbf{V}_h = RT_k, k \ge 1 \tag{3.52a}$$
$$(\operatorname{div} \mathbf{e}_h, w) = 0, \quad w \in W_h = P_k \tag{3.52b}$$

where $z = P_h p - p_h$, $\mathbf{e}_h = \mathbf{u} - \mathbf{u}_h$.

Lemma 3.4.1. *For $s \geq 0$,*

$$\|z\|_{-s} \leq Ch^{min(s+1,k+1)}\|\mathbf{e}_h\|_0 + ch^{min(s+2,k+1)}\|\, div\, \mathbf{e}_h\|_0 \tag{3.53}$$

Proof. Let $\phi \in H^{s+2}$ be the solution of dual problem $L^*\phi = \psi$. Then as in Doug-Roberts

$$\begin{aligned}(z,\psi) &= (\alpha\mathbf{e}_h, \mathbf{a}\,\mathrm{grad}\,\phi - \Pi_h\mathbf{a}\,\mathrm{grad}\,\phi) + (\,\mathrm{div}\,\mathbf{e}_h, \phi - P_h\phi)\\ &\leq Ch^{min(s+1,k+1)}\|\mathbf{e}_h\|_0\|\psi\|_s + Ch^{min(s+2,k+1)}\|\,\mathrm{div}\,\mathbf{e}_h\|_0\|\phi\|_{s+2}\end{aligned}$$

□

Lemma 3.4.2. *For $s \geq 0$*

$$\|\, div\, \mathbf{e}_h\|_{-s} \leq Ch^{min(s,k+1)}\|\, div\, \mathbf{e}_h\|_0$$

Proof. Let $\phi \in H^s(\Omega)$. Then

$$(\,\mathrm{div}\,\mathbf{e}_h, \phi) = (\,\mathrm{div}\,\mathbf{e}_h, \phi - w), \quad \forall w \in W_h (= P_k)$$

Taking infimum w.r.t w, we get the result. □

Lemma 3.4.3.

$$\|\mathbf{e}_h\|_{-s} \leq Ch^{min(s,k+1)}\|\mathbf{e}_h\| + Ch^{min(s+1,k+1)}\|\, div\, \mathbf{e}_h\|_0$$

Proof. Let $\boldsymbol{\phi} \in \mathbf{H}^s(\Omega)$. Then from the error equation (3.52b) and property of Π_h,

$$\begin{aligned}(\alpha\mathbf{e}_h, \boldsymbol{\phi}) &= (\alpha\mathbf{e}_h, \Pi_h\boldsymbol{\phi}) + (\alpha\mathbf{e}_h, \boldsymbol{\phi} - \Pi_h\boldsymbol{\phi})\\ &= (\,\mathrm{div}\,\Pi_h\boldsymbol{\phi}, z) + (\alpha\mathbf{e}_h, \boldsymbol{\phi} - \Pi_h\boldsymbol{\phi})\\ &= (\,\mathrm{div}\,\boldsymbol{\phi}, z) + (\alpha\mathbf{e}_h, \boldsymbol{\phi} - \Pi_h\boldsymbol{\phi})\end{aligned}$$

$$|(\alpha\mathbf{e}_h, \boldsymbol{\phi})| \leq \|\boldsymbol{\phi}\|_s\{\|z\|_{-s+1} + Ch^{min(s,k+1)}\|\mathbf{e}_h\|\}$$

Conclusion follows from (3.53) since

$$\begin{aligned}\|\mathbf{e}_h\|_{-s} &\leq \|z\|_{-s+1} + Ch^{min(s,k+1)}\|\mathbf{e}_h\|\\ &\leq Ch^{min(s,k+1)}\|\mathbf{e}_h\|_0 + Ch^{min(s+1,k+1)}\|\,\mathrm{div}\,\mathbf{e}_h\|\end{aligned}$$

□

Theorem 3.4.4. *We have*

$$\|p - p_h\|_0 \le Ch^r \|f\|_r, \quad 1 \le r \le k+1 \tag{3.54}$$

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \le Ch^r \|f\|_{r+1}, \quad 1 \le r \le k+1 \tag{3.55}$$

and

$$\| \, div \, (\mathbf{u} - \mathbf{u}_h)\|_0 \le Ch^r \|f\|_r, \quad 1 \le r \le k+1. \tag{3.56}$$

Proof. Take $\mathbf{v} = \boldsymbol{\sigma}_h = \Pi_h \mathbf{u} - \mathbf{u}_h$ in the error equation. Note

$$\boldsymbol{\sigma}_h = \Pi_h \mathbf{u} - \mathbf{u}_h = \mathbf{u} - \mathbf{u}_h - (\mathbf{u} - \Pi_h \mathbf{u}) = \mathbf{e}_h - (\mathbf{u} - \Pi_h \mathbf{u}).$$

Then by (3.52)

$$(\alpha \mathbf{e}_h, \boldsymbol{\sigma}_h) = (\operatorname{div} \boldsymbol{\sigma}_h, z) = 0$$

Thus

$$(\alpha \boldsymbol{\sigma}_h, \boldsymbol{\sigma}_h) = -(\alpha(\mathbf{u} - \Pi_h \mathbf{u}), \boldsymbol{\sigma}_h)$$

and hence

$$\|\boldsymbol{\sigma}_h\| \le C \|\mathbf{u} - \Pi_h \mathbf{u}\|$$

$$\|\mathbf{u} - \mathbf{u}_h\| \le \|\mathbf{u} - \Pi_h \mathbf{u}\| + \|\Pi_h \mathbf{u} - \mathbf{u}_h\| \le Ch^r \|\mathbf{u}\|_r \text{ for } 1 \le r \le k+1. \tag{3.57}$$

Meanwhile(since $\operatorname{div} \boldsymbol{\sigma}_h = 0$)

$$\| \operatorname{div} \mathbf{e}_h\|_0 = \| \operatorname{div} (\mathbf{u} - \Pi_h \mathbf{u})\|_0 \le Ch^r \| \operatorname{div} \mathbf{u}\|_r, 1 \le r \le k+1$$

From Lemma 3.4.1 and (3.57)

$$\begin{aligned} \|p - p_h\|_0 &\le \|P_h p - p_h\|_0 + \|p - P_h p\|_0 \\ &\le Ch\|\mathbf{e}_h\|_0 + Ch^{min(2,k+1)} \| \operatorname{div} \mathbf{e}_h\|_0 \\ &\le Ch^{1+min(r,k+1)} \|\mathbf{u}\|_r, \\ &\quad + Ch^{min(2,k+1)+min(r,k+1)} \| \operatorname{div} \mathbf{u}\|_r + Ch^{min(r,k+1)} \|p\|_r \end{aligned}$$

Simplify the result, we get (3.54). □

3.5 Hybrid form of mixed methods

The solution of algebraic system associated with mixed formulation can be simplified by the introduction of a Lagrange multiplier to enforce the continuity of normal component of $\mathbf{u}$ across the interelement boundaries. Let

$$M_h = \{m : m_e \in P_k(e) \text{ if } e \subset \Omega, m_e = 0 \text{ if } e \subset \partial\Omega, \} \tag{3.58}$$

Following Fraejis de Veubeke, our problem is to seek $\{\mathbf{u}_h, p_h, m_h\} \in \mathbf{V}_h^k \times W_h^k \times M_h^k$ such that

$$(\alpha \mathbf{u}_h, \mathbf{v}) - \sum_T (\operatorname{div} \mathbf{v}, p_h) + \sum_T < \mathbf{v} \cdot \mathbf{n}_T, m_h >_{\partial T} =< \mathbf{v} \cdot \mathbf{n}, g >, \mathbf{v} \in \mathbf{V}_h^k, \tag{3.59a}$$

$$\sum_T (\operatorname{div} \mathbf{u}_h, w)_T = (f, w), w \in W_h^k \tag{3.59b}$$

$$\sum_T (\mathbf{u}_h, q)_{\partial T} = 0, q \in M_h^k \tag{3.59c}$$

We introduce some norms

$$|m_h|_{0,h}^2 = \sum_e \|m_h\|_{0,e}^2 \tag{3.60a}$$

$$|m_h|_{-1/2,h}^2 = \sum_e |e| \|m_h\|_{0,e}^2 \tag{3.60b}$$

Lemma 3.5.1. *If $\{\mathbf{u}_h, p_h, m_h\} \in \mathbf{V}_h^k \times W_h^k \times M_h^k$ is the solution of (3.59) then*

$$\|m_h - Q_h^k p\|_{0,e} \le C\{h^{1/2} \|\mathbf{u} - \mathbf{u}_h\|_{0,T} + h^{-1/2} \|P_h p - p_h\|_{0,T}\} \tag{3.61}$$

Proof. Let $e \subset \Omega \cap T$ and define $\mathbf{v}$ on T by requiring

$$\mathbf{v} \cdot \mathbf{n}_e = m_h - Q_h p \text{ on } e \tag{3.62a}$$

$$\mathbf{v} \cdot \mathbf{n}_T = 0 \text{ on } \partial T \backslash e \tag{3.62b}$$

$$(\mathbf{v}, \boldsymbol{\phi}_{k-1})_T = 0, \forall \boldsymbol{\phi}_{k-1} \in (P_{k-1}(K))^2. \tag{3.62c}$$

The existence and uniqueness is given by the mixed finite element(RT) construction. A scaling argument gives(Ok, use Piolar transform $\mathbf{v} = \frac{1}{J} B \hat{v} \circ F^{-1}$)

$$h \|\mathbf{v}\|_{1,T} + \|\mathbf{v}\|_{0,T} \le C h^{1/2} \|m_k - Q_h^k p\|_{0,e} \tag{3.63}$$

Take $\mathbf{v}$ as test function in (3.59)

$$(\alpha \mathbf{u}_h, \mathbf{v})_T - (\operatorname{div} \mathbf{v}, p_h)_T + < m_h, m_h - Q_h p >_e = 0$$

Since

$$(\alpha \mathbf{u}, \mathbf{v})_T - (\operatorname{div} \mathbf{v}, p)_T + < u, m_h - Q_h p >_e = 0$$

we have

$$\|m_h - Q_h u\|_{0,e}^2 =< m_h - p, m_h - Qhp >_e = (\alpha(\mathbf{u} - \mathbf{u}_h), \mathbf{v})_T - (\operatorname{div} \mathbf{v}, z_h)_T$$

and the result follows from (3.63) □

Postprocessing introduced by Arnold-Brezzi

Lemma 3.5.2 (Arnold-Brezzi). *Let k be nonnegative even integer. There exists a map*

$$R^{k+1}(T) : \prod_{i=1}^{3} L^2(e_i) \times L^2(T) \to P_{k+1}$$

$p_h^* = R^{k+1}(T)\{m_h, p_h\}$ *is given by*

$$< p_h^* - m_h, q >_{e_i} = 0, q \in P_k(e_i), i = 1,2,3 \tag{3.64a}$$

$$(p_h^* - p_h, q)_T = 0, q \in P_{k-2}(T), \ \textit{for } k \geq 2 \tag{3.64b}$$

Moreover

$$\|p_h^*\|_{0,T} \leq C \left[\|p_h\|_{0,T} + h^{1/2} \sum_i \|m_h\|_{0,e_i} \right] \tag{3.65}$$

The case where $k = 1$ and $k = 3$ have to be treated separately. For $k = 1$, R^2 is determined by

$$< p_h^* - m_h, 1 >_{e_i} = 0, i = 1,2,3 \tag{3.66a}$$

$$(p_h^* - p_h, q)_T = 0, q \in P_1(T). \tag{3.66b}$$

However this case is not used below to extend m_h, p_h when $k = 1$. For $k = 3$, $p_h^* \in P_4(T)$ is determined by

$$< p_h^* - m_h, q >_{e_i} = 0, q \in P_2(e_i), i = 1,2,3 \tag{3.67a}$$

$$(p_h^* - p_h, q)_T = 0, q \in P_2(T). \tag{3.67b}$$

For $k = 0,1,2,3,4,6,8,\cdot$, let R_h^{k+1} denote the extension of corresponding $R^{k+1}(T)$ to all T and let

$$u_h^* = \begin{cases} R_h^1\{m_h, p_h\}, k = 1 \\ R_h^{k+1}\{m_h, p_h\}, k = 2,3,4,6,,8\cdots \end{cases} \tag{3.68}$$

Then

Theorem 3.5.3.

$$\|p - p_h^*\|_0 \leq \begin{cases} Kh^2(\|f\|_1 + |g|_{5/2}), k = 1 \\ Kh^{k+2}(\|f\|_k + |g|_{k+3/2}), k = 2,3,4,6,,8\cdots \end{cases} \tag{3.69}$$

An alternative procedure of postprocessing is a local version of Nitsche: Define $u_h^*|T \in P_{k+1}(T)$, triangle by triangle as the solution of

$$\begin{aligned} A_T(p_h^*, q) &= (a \operatorname{grad} p_h^*, \operatorname{grad} q)_T - \left\langle a\frac{\partial p_h^*}{\partial n}, q\right\rangle_{\partial T} \\ &\quad - \left\langle p_h^*, a\frac{\partial q}{\partial n}\right\rangle_{\partial T} + \sigma h^{-1}\langle p_h^*, q\rangle_{\partial T} \\ &= (f, q)_T - \left\langle m_h, a\frac{\partial q}{\partial n}\right\rangle_{\partial T} + \sigma h^{-1}\langle m_h, q\rangle_{\partial T} \end{aligned} \tag{3.70}$$

for $q \in P_{k+1}(T)$, where σ is some constant to be determined by k, a and shale of triangle only.

Theorem 3.5.4. *Let $k \geq 2$and p_h^* be defined as above. Then*

$$\|p - p_h^*\|_0 \leq Kh^{k+2}(\|f\|_k + |g|_{k+3/2}) \tag{3.71}$$

Proof. Recall A is not coercive over $H^1(T)$, is coercive on $P_{k+1}(T)$ for sufficiently large σ.(on finite dimensional space, $\sigma h^{-1}\langle q, q\rangle_{\partial T}$ dominates $\left\langle a\frac{\partial q}{\partial n}, q\right\rangle_{\partial T}$ for sufficiently large σ. In particular,

$$A(q, q) \geq \rho(|\operatorname{grad} q|_T^2 + h^{-1}\|q\|_{\partial T}^2), q \in P_{k+1}(T) \tag{3.72}$$

Since the error equation is(integrate $(\nabla a \nabla p, q) = (f, q)$ by part and just subtract rhs from (3.70))

$$\begin{aligned} A_T(p - p_h^*, q) &= \left\langle m_h - p, a\frac{\partial p}{\partial n}, q\right\rangle_{\partial T} + \sigma h^{-1}\langle p - m_h^*, q\rangle_{\partial T} \\ &= \left\langle m_h - Q_h^k p, a\frac{\partial p}{\partial n}\right\rangle_{\partial T} + \sigma h^{-1}\langle p - m_h^*, q\rangle_{\partial T} \end{aligned}, q \in P_T^{k+1} \tag{3.73}$$

where Q_h^k is L^2-projection on each edge into $P_k(e)$ with respect to weight a. Take $\phi \in P_{k+1}(T)$. Then

shift p to q^* and choose $q = \phi - p_h^*$:

$$\begin{aligned} A_T(\phi - p_h^*, \phi - p_h^*) &= A_T(p - p_h^*, \phi - p_h^*) + A_T(\phi - p, \phi - p_h^*) \\ &= A_T(\phi - p, \phi - p_h^*) + \left\langle m_h - Q_h^k p, a\frac{\partial}{\partial n}(\phi - p_h^*)\right\rangle_{\partial T} \\ &\quad + \sigma h^{-1}\langle p - m_h, \phi - p_h^*\rangle_{\partial T} \end{aligned} \tag{3.74}$$

By the same inverse inequality used to show (3.72), and proper choice of ϕ,

$$A_T(\phi - p_h^*, \phi - p_h^*) = K[\|p\|_{k+2,T}^2 h^{2k+2} + \|\mathbf{u} - \mathbf{u}_h\|_{0,T}^2 + h^{-2}\|P_h^{k+1}p - p_h\|_{0,T}^2] \tag{3.75}$$

from which it follows that

$$\sum_T \{\|\operatorname{grad}(p - p_h^*)\|_{0,T}^2 + h^{-1}\|p - p_h^*\|_{0,T}^2\} \leq K[\|f\|_k^2 + |g|_{k+3/2}^2]h^{2k+2} \tag{3.76}$$

Now we can use duality argument on each triangle. Let

$$-\operatorname{div}(a \operatorname{grad} \psi) = \phi - p_h^*, \; x \in T \tag{3.77}$$

$$\psi = 0, \; x \in \partial T \tag{3.78}$$

so that $\|\psi\|_{2,T} \leq K\|\phi - p_h^*\|_{0,T}$ and $\|\psi\|_{1,T} \leq Kh\|\psi\|_{2,T}$. Hence

$$\|\phi - p_h^*\|_{0,T}^2 \leq A(\phi - p_h^*, \psi) \leq KhA(\phi - p_h^*, \phi - p_h^*)^{1/2}\|\phi - p_h^*\|_{0,T}$$

□

3.6 Trace estimate

Lemma 3.6.1. *Let $v \in W_p^1(\Omega)$. Then*

$$\|v\|_{L^p(\partial\Omega)} \le C\|v\|_{L^p(\Omega)}^{1-1/p}\|v\|_{W_p^1(\Omega)}^{1/p}$$

See Brenner-Scott's book.

Corollary 3.6.2. *On the reference element,*

$$\|\hat{v}\|_{L^2(\partial\hat{\Omega})}^2 \le C\|\hat{v}\|_{L^2(\hat{\Omega})}(\|\hat{v}\|_{L^2(\hat{\Omega})} + |\hat{v}|_{H^1(\hat{\Omega})})$$

On a finite element K of diameter h, we can show

$$\|v\|_{L^2(K)} \approx Ch\|\hat{v}\|_{L^2(\hat{K})}, \quad |v|_{H^1(K)} \approx |\hat{v}|_{H^1(\hat{K})}, \quad |v|_{L^2(\partial K)} \approx Ch^{1/2}|\hat{v}|_{L^2(\partial\hat{K})}.$$

Transfer to the finite element

$$\|v\|_{L^2(\partial K)}^2 \le Ch^{-1}\|v\|_{L^2(K)}^2 + \|v\|_{L^2(K)}|v|_{H^1(K)}$$

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