Numerical PDE

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November 19, 2007

1.1 The Stokes equations

The Navier-Stokes equations for a viscous incompressible fluid are as follows:

$$\rho\left(\frac{\partial u_i}{\partial t}+\sum_j u_j\frac{\partial u_i}{\partial x_j}\right)-\sum_j\frac{\partial\sigma_{ij}}{\partial x_j}=\rho f_i,\quad 1\le i\le n, \tag{1.1}$$

$$\operatorname{div}\mathbf{u}=\sum_i D_{ii}(\mathbf{u})=0(\text{incompressibility}), \tag{1.2}$$

where

$$\begin{aligned}\sigma_{ij} &= -P\delta_{ij}+2\mu D_{ij}(\mathbf{u})\\ D_{ij}(\mathbf{u}) &= \frac{1}{2}(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}),\quad 1\le i,j\le n.\end{aligned}$$

In these equations, $\mathbf{u}$ is the velocity of the fluid, ρ is the density, $\mu>0$ is the viscosity and P is the pressure; σ_{ij} is the stress tensor and the vector $\mathbf{f}$ represents body forces per unit mass. Let $p=P/\rho$ and $\nu=\mu/\rho$. With these notation, we have the following form:

$$\frac{\partial u_i}{\partial t}+\sum_j u_j\frac{\partial u_i}{\partial x_j}-2\nu\sum_j\frac{\partial D_{ij}(\mathbf{u})}{\partial x_j}+\frac{\partial p}{\partial x_i}=f_i,\quad 1\le i\le n. \tag{1.3}$$

We introduce some notations: For $\mathbf{u}=(u_1,u_2)^T$, let

$$\mathbf{grad}\,\mathbf{u}=\nabla\mathbf{u}=\begin{pmatrix}\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2}\\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2}\end{pmatrix} \tag{1.4}$$

Then $D(\mathbf{u})=\frac{1}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^T)$. We also define vector Laplacian: $\mathbf{\Delta u}=\begin{pmatrix}\Delta u_1\\ \Delta u_2\end{pmatrix}$

Note that if $\operatorname{div}\mathbf{u}=0$, the following identity holds

$$\sum_j\frac{\partial D_{ij}(\mathbf{u})}{\partial x_j}=\frac{1}{2}\sum_j\left(\frac{\partial^2 u_i}{\partial x_j^2}+\frac{\partial^2 u_j}{\partial x_i\partial x_j}\right)=\frac{1}{2}\Delta u_i,\qquad\text{for each } i \tag{1.5}$$

so that it can be written as

$$\begin{cases}\frac{\partial\mathbf{u}}{\partial t}+\sum_j u_j\frac{\partial\mathbf{u}}{\partial x_j}-\nu\mathbf{\Delta u}+\mathbf{grad}\,p=\mathbf{f},\\ \operatorname{div}\mathbf{u}=0.\end{cases} \tag{1.6}$$

Or

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \mathbf{grad}\, p = \mathbf{f}, \\ \operatorname{div}\mathbf{u} = 0. \end{cases} \tag{1.7}$$

where $(\mathbf{u}\cdot\nabla)\mathbf{v} = \mathbf{e}_i \sum_j u_j \frac{\partial v_i}{\partial x_j}$. Here $\mathbf{u}\cdot\nabla$ can be considered as inner product.

We only consider the steady-state case, and assume that $\mathbf{u}$ is so small that we can ignore the non-linear convection term $u_j \frac{\partial u_i}{\partial x_j}$. Thus, we have the Stokes equation:

$$\begin{cases} -2\nu \sum_j \frac{\partial D_{ij}(\mathbf{u})}{\partial x_j} + \frac{\partial p}{\partial x_i} = f_i, \quad 1 \le i \le n, \\ \operatorname{div}\mathbf{u} = 0. \end{cases} \tag{1.8}$$

In vector form, it can be written as

$$\begin{cases} -\nu\Delta\mathbf{u} + \mathbf{grad}\, p &= \mathbf{f} \\ \operatorname{div}\mathbf{u} &= 0. \end{cases}$$

1.1.1 The "Velocity-Pressure" formulation

Let $L_0^2(\Omega)$ be the space of all $L^2(\Omega)$ functions q such that $\int_\Omega q\,dx = 0$. The next theorem is necessary for the stability:

Theorem 1.1.1. *There exists a constant $c > 0$ such that*

$$\sup_{\mathbf{v}\in(H_0^1(\Omega))^n} \frac{(\phi,\, div\,\mathbf{v})}{|\mathbf{v}|_1} \ge c\|\phi\|_0, \quad \forall \phi \in L_0^2(\Omega).$$

Proof . By Cor 2.4 there exists unique $\mathbf{v} \in V^\perp$ such that $\operatorname{div}\mathbf{v} = \phi$ and $|\mathbf{v}|_1 \le c\|\phi\|_0$. Hence

$$\frac{(\phi,\, \operatorname{div}\mathbf{v})}{|\mathbf{v}|_1} = \frac{\|\phi\|_0^2}{|\mathbf{v}|_1} \ge \frac{1}{c}\|\phi\|_0.$$

□

Theorem 1.1.2. *Let $\mathbf{f}$ be given in $(H^{-1}(\Omega))^n$ and $\mathbf{g} \in (H^{1/2}(\Gamma))^n$ resp., such that*

$$\int_\Gamma \mathbf{g}\cdot\mathbf{n}\,d\sigma = 0$$

Then there exists unique pair of functions $(\mathbf{u}, p)$ in $(H^1(\Omega))^n \times L_0^2(\Omega)$ such that

$$\begin{cases} -\nu\Delta\mathbf{u} + \mathbf{grad}\, p &= \mathbf{f} \ \textit{in}\ \Omega \\ \operatorname{div}\mathbf{u} &= 0 \ \textit{in}\ \Omega \\ \mathbf{u} &= \mathbf{g} \ \textit{on}\ \Gamma. \end{cases} \tag{1.9}$$

Proof . There exists a function $\mathbf{u}_g \in H^1(\Omega)^n$ such that

$$\operatorname{div} \mathbf{u}_g = 0 \text{ in } \Omega,\ \mathbf{u}_g = g \text{ on } \Gamma.$$

□

Now let us put problem (1.9) into general framework of chap 4.: We set

$$X = H_0^1(\Omega)^n, \quad M = L_0^2(\Omega).$$

Multiply by $\mathbf{v} \in M$ and integrate by parts,

$$-(\nu\Delta\mathbf{u}, \mathbf{v}) + (\mathbf{grad}\, p, \mathbf{v}) = (\nu\nabla\mathbf{u}, \nabla\mathbf{v}) - (p, \operatorname{div} \mathbf{v}) = (\mathbf{f}, \mathbf{v}).$$

Here $(\nabla\mathbf{u}, \nabla\mathbf{v})$ is interpreted as follows: Write $\mathbf{u} = (u_1, u_2)^T$. Then

$$\nabla\mathbf{u} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{pmatrix}, \quad \nabla\mathbf{v} = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} \end{pmatrix}$$

For two matrices A, B we sometimes write

$$A : B = \sum_{i,j} a_{ij} b_{ij}$$

$$a(\mathbf{u}, \mathbf{v}) = \nu \sum_{i,j=1}^{N} \left(\frac{\partial u_i}{\partial x_j}, \frac{\partial v_i}{\partial x_j} \right) = \nu(\operatorname{grad} \mathbf{u}, \operatorname{grad} \mathbf{v}) = \int \operatorname{grad} \mathbf{u} : \operatorname{grad} \mathbf{v}\, dx \tag{1.10}$$

$$b(\mathbf{v}, q) = -(q, \operatorname{div} \mathbf{v})$$

$$\langle \ell, \mathbf{v} \rangle = \langle \mathbf{f}, \mathbf{v} \rangle - a(\mathbf{u}_g, \mathbf{v}), \chi = 0.$$

and

$$V = \{\mathbf{v} \in H_0^1(\Omega)^n, \operatorname{div} \mathbf{v} = 0\}.$$

$a(\cdot, \cdot)$ satisfies ellipticity and b satisfies inf-sup condition by theorem 1.1.1. We apply Corollary 1.2.5:

Weak formulation There exists a unique pair of functions $(\mathbf{w}, p) \in H_0^1(\Omega)^n \times L_0^2(\Omega)$ such that

$$\begin{cases} a(\mathbf{w}, \mathbf{v}) + b(\mathbf{v}, p) &= \langle \ell, \mathbf{v} \rangle \text{ for all } \mathbf{v} \in H_0^1(\Omega)^n \\ b(\mathbf{w}, q) &= 0 \text{ for all } q \in L_0^2(\Omega). \end{cases}$$

Here $\mathbf{u} = \mathbf{u}_g + \mathbf{w}, \mathbf{w} \in H_0^1(\Omega)$. This is equivalent to Problem (1.9)

Remark 1.1.3. This can be put in an equivalent form as follows:

$$\begin{cases} a(\mathbf{u},\mathbf{v}) - (p, \operatorname{div}\mathbf{v}) & = & \langle \mathbf{f},\mathbf{v}\rangle \text{ for all } \mathbf{v} \in H_0^1(\Omega)^n \\ (q, \operatorname{div}\mathbf{v}) & = & 0 \text{ in for all } q \in L_0^2(\Omega) \\ \mathbf{u} & = & \mathbf{g} \text{ on } \Gamma. \end{cases} \tag{1.11}$$

The choice $M = L_0^2(\Omega)$ is a matter of convenience and we can just as well take $M = M^2(\Omega)/\mathbb{R}$.

1.2 A General result

Let X and M be two Hilbert spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_M$ and let X' and M' be their dual spaces. As usual, we denote $\langle\cdot,\cdot\rangle$ be the duality pairing between X and X' or M and M'

Introduce bilinear forms

$$a(\cdot,\cdot) : X \times X \to \mathbb{R}, \quad b(\cdot,\cdot) : X \times M \to \mathbb{R}$$

with norms

$$\|a\| = \sup_{u,v} \frac{a(u,v)}{\|u\|_X\|v\|_X}, \quad \|b\| = \sup_{v\in X,\mu\in M} \frac{b(v,\mu)}{\|v\|_X\|\mu\|_\mu}.$$

We consider the following two variational problem called problem (Q):

Given $\ell \in X'$ and $\chi \in M'$, find a pair $(u,\lambda) \in X \times M$ such that

$$a(u,v) + b(v,\lambda) = \langle \ell, v\rangle \text{ for all } v \in X \tag{1.12}$$

$$b(u,\mu) = \langle \chi,\mu\rangle \text{ for all } \mu \in M. \tag{1.13}$$

In order to study (Q), we associate two linear operators $A \in \mathcal{L}(X;X')$ and $B \in \mathcal{L}(X;M')$ defined by

$$\langle Au, v\rangle = a(u,v) \text{ for all } u, v \in X \tag{1.14}$$

$$\langle Bv, \mu\rangle = b(u,\mu) \text{ for all } v \in X, \mu \in M. \tag{1.15}$$

Let $B' \in \mathcal{L}(M;X')$ be dual operators defined by

$$\left\langle B'\mu, v\right\rangle = \langle \mu, Bv\rangle = b(v,\mu), v \in X, \mu \in M. \tag{1.16}$$

With these, the problem can be written as

Find $(u,\lambda) \in X \times M$ such that

$$Au + B'\lambda = \ell \text{ in } X' \tag{1.17}$$

$$Bu \quad = \quad \chi \text{ in } M'. \tag{1.18}$$

We set $V = Ker(B)$ and more generally define

$$V(\chi) = \{v \in X; Bv = \chi\}.$$

Note that $V = V(0)$.

Now problem (Q) can be changed into equivalent form (P):

Find $u \in V(\chi)$ such that

$$a(u, v) = \langle \ell, v \rangle\,, \quad v \in V \tag{1.19}$$

Theorem 1.2.1. *The problem (Q) has a unique solution which depends continuously on the given data if there is a constant $\beta > 0$ such that*

$$\inf_{\mu \in M} \sup_{v \in X,} \frac{b(v, \mu)}{\|v\|_X \|\mu\|_\mu} \geq \beta > 0. \tag{1.20}$$

Corollary 1.2.2. *Assume that $a(\cdot, \cdot)$ is coercive on V, i.e. there exists a constant $\alpha > 0$ such that*

$$a(v, v) \geq \alpha \|v\|_X^2, \quad \forall v \in V. \tag{1.21}$$

Then problem (Q) has unique solution b form satisfies inf-sup condition.

1.2.1 Saddle Point Approach

Introduce an *energy functional* $J : X \to \mathbb{R}$ by

$$J(v) = \frac{1}{2} a(v, v) - \langle \ell, v \rangle \tag{1.22}$$

and let

$$\mathcal{L}(v, \mu) = J(v) + b(v, \mu) - \langle \chi, \mu \rangle\,. \tag{1.23}$$

Consider the following problem, called problem (L):

Find a saddle point $(u, \lambda) \in X \times M$ of the Lagrangian functional $\mathcal{L}$, i.e, find a pair $(u, \lambda) \in X \times M$ such that

$$\mathcal{L}(u, \mu) \leq \mathcal{L}(u, \lambda) \leq \mathcal{L}(v, \lambda), \quad v \in X, \mu \in M. \tag{1.24}$$

Theorem 1.2.3. *Assume $a(\cdot, \cdot)$ is symmetric and $a(v, v) \geq 0$. Then the problem (L) has a unique solution (u, λ) which is the solution of problem (Q).*

Proof . The first inequality of (1.24) can be written as

$$b(v, \mu - \lambda) \leq \langle \chi, \mu - \lambda \rangle , \quad \forall \mu \in M.$$

Since μ is arbitrary, this is equivalent to(by taking $\mu \to \pm\mu + \lambda$)

$$b(v, \ \lambda) = \langle \chi, \lambda \rangle , \quad \forall \mu \in M.$$

Next The second inequality of (1.24) is equivalent to

$$\mathcal{L}(u, \lambda) = \inf_{v \in X} \mathcal{L}(v, \lambda).$$

Since $a(\cdot, \cdot)$ is symmetric, we have

$$\frac{d}{dt}\mathcal{L}(u + tv, \lambda)|_{t=0} = a(u, v) - b(v, \lambda) - \langle \ell, v \rangle$$

Furthermore, we have the second derivative

$$\frac{d^2}{dt^2}\mathcal{L}(u + tv, \lambda)|_{t=0} = a(v, v) \geq 0$$

Therefore, $v \to \mathcal{L}$ is convex functional and its minimum u is characterized by the condition $\frac{d}{dt}\mathcal{L}(u + tv, \lambda)|_{t=0} = 0$, i.e.

$$a(u, v) + b(v, \lambda) = \langle \ell, v \rangle , \quad \forall v \in X.$$

Thus (u, λ) is a saddle point of $\mathcal{L}$ iff it is a solution of problem (Q). □

Corollary 1.2.4. *This problem has a unique solution* (u, λ) *which is the solution of problem (Q).*

$$\min_{v \in X} \left(\sup_{\mu \in M} \mathcal{L}(v, \mu) \right) = \mathcal{L}(u, \lambda) = \max_{\mu \in M} \left(\inf_{v \in X} \mathcal{L}(v, \mu) \right). \tag{1.25}$$

1.2.2 Augmented Lagrangian

Introduce *energy functional* $J_r : X \to \mathbb{R}$ by

$$J_r(v) = J(v) + \frac{r}{2} \left\langle C^{-1}(Bv - \chi), Bv - \chi \right\rangle \tag{1.26}$$

and *Augmented Lagrangian functional*

$$\mathcal{L}_r(v) = J_r(v) + b(v, \mu) - \langle \chi, \mu \rangle , \quad r > 0 \tag{1.27}$$

We have

$$J_r(u) = \inf_{v \in V(\chi)} J_r(u).$$

Corollary 1.2.5. *The solution (u, λ) is the unique saddle point problem of augmented lagrangian functional $\mathcal{L}_r$:*

$$\min_{v \in X} \left(\sup_{\mu \in M} \mathcal{L}_r(v, \mu) \right) = \mathcal{L}_r(u, \lambda) = \max_{\mu \in M} \left(\inf_{v \in X} \mathcal{L}_r(v, \mu) \right). \tag{1.28}$$

Standard Uzawa

Let p_h^0 given. With small $\epsilon > 0$, Solve

$$\begin{aligned} a(\mathbf{u}^{m+1}, \mathbf{v}) + b(\mathbf{v}, p_h^m) &= (\mathbf{f}, \mathbf{v}_h) - a(\mathbf{u}_g, \mathbf{v}), \quad \mathbf{v} \in \mathbf{V}_h \\ (p^{m+1} - p^m, q) &= \epsilon b(\mathbf{u}^{m+1}, q), \quad q \in M_h \end{aligned}$$

Stop if $\|p^{m+1} - p^m\|$ is sufficiently small.

Conjugate Gradient for infinite dimensional space

Recall our problem (Q): Consider the following variational problem:

(Q) For ℓ given in X' and $\chi \in M'$, find a pair (u, λ) in $X \times M$ such that

$$a(u, v) + b(v, \lambda) = \langle \ell, v \rangle \quad \forall v \in X \tag{1.29}$$

$$b(u, \mu) = \langle \chi, \mu \rangle \quad \forall \mu \in M. \tag{1.30}$$

p. 78. Define

$$a_r(\mathbf{u}, \mathbf{v}) = a(\mathbf{u}, \mathbf{v}) + r \left\langle C^{-1} B\mathbf{u}, B\mathbf{v} \right\rangle \tag{1.31}$$

Then problem (Q) is equivalent to solving

$$a_r(\mathbf{u}, \mathbf{v}) = \langle \ell, \mathbf{v} \rangle + rb(\mathbf{v}, C^{-1}\chi) - b(\mathbf{v}, \mu) \quad \forall \mathbf{v} \in X \tag{1.32}$$

Given $(\mathbf{u}^0, \lambda^0)$, $\sigma_0 = 0$. For $m \geq 0$, compute $g^m, \omega^m \in M, \mathbf{z}^m \in X, \rho^m, \sigma^m \in \mathbb{R}$ and $(\mathbf{u}^{m+1}, \lambda^{m+1}) \in X \times M$ by

$$\begin{aligned} Cg^m &= \chi - B\mathbf{u}^m \\ \sigma^m &= c(g^m, g^m)/c(g^{m-1}, g^{m-1}) \\ \omega^m &= g^m + \sigma^m \omega^{m-1} \\ A_r \mathbf{z}^m &= B' \omega^m \\ \rho^m &= c(g^m, g^m)/b(\mathbf{z}^m, g^m) \\ \lambda^{m+1} &= \lambda^m - \rho^m \omega^m \\ \mathbf{u}^{m+1} &= \mathbf{u}^m + \rho^m \mathbf{z}^m \end{aligned}$$

Here B' is adjoint of B(In matrix form it is B^t.)

Augmented Lagrangian formulation

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Now change every space to finite dimensional one. Let

$$X_h \subset X, \quad M_h \subset M$$

be finite dimensional subspace with certain approximation properties.

(Q_h) For ℓ given in X_h' and $\chi \in M_h'$, find a pair $(\mathbf{u}_h, \lambda_h)$ in $X_h \times M_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, \lambda_h) = \langle \ell, \mathbf{v}_h \rangle \quad \forall \mathbf{v}_h \in X_h \tag{1.33}$$

$$b(\mathbf{u}_h, \mu) = \langle \chi, \mu \rangle \quad \forall \mu \in M_h. \tag{1.34}$$

Define for some large $r > 0$

$$a_r^h(\mathbf{u}_h, \mathbf{v}_h) = a(\mathbf{u}_h, \mathbf{v}_h) + r \left\langle C^{-1} B\mathbf{u}_h, B\mathbf{v}_h \right\rangle = b(\mathbf{v}_h, C^{-1} B\mathbf{u}_h) \tag{1.35}$$

Problem (Q_h) is equivalent to solving

$$a_r^h(\mathbf{u}_h(\mu_h), \mathbf{v}_h) = \langle \ell, \mathbf{v}_h \rangle + b(\mathbf{v}_h, rC_h^{-1}\chi - \mu_h), \forall \mathbf{v}_h \in X_h$$

Now Uzawa method based on conjugate gradient method is:

Starting from an initial guess $\lambda_h^0 \in M_h$, compute the solution $\mathbf{u}_h^0 \in X_h$ satisfying

$$a_r^h(\mathbf{u}_h^0, \mathbf{v}_h) = \langle \ell, \mathbf{v}_h \rangle + b(\mathbf{v}_h, rC_h^{-1}\chi - \lambda_h^0), \forall \mathbf{v}_h \in X_h$$

Let $\sigma_0 = 0$.

For $m \geq 0$, compute $g_h^m, \omega_h^m \in M_h, \mathbf{z}_h^m \in X_h, \rho_h^m, \sigma_h^m \in \mathbb{R}$ and $(\mathbf{u}_h^{m+1}, \lambda_h^{m+1}) \in X_h \times M_h$ by

$$\begin{aligned}
c(g_h^m, \mu_h) &= \langle \chi_h, \mu_h \rangle - b(\mathbf{u}_h^m, \mu_h), \quad \mu_h \in M_h \text{ residual} \\
\sigma_h^m &= \frac{c(g_h^m, g_h^m)}{c(g_h^{m-1}, g_h^{m-1})} \text{ only if } m \geq 1 \\
\omega_h^m &= g_h^m + \sigma_h^m \omega_h^{m-1}, \quad \omega_h = g_h^0 \\
a_r^h(\mathbf{z}_h^m, \mathbf{v}_h) &= b(\mathbf{v}_h, \omega_h^m), \quad \forall \mathbf{v}_h \in X_h \text{ search direction} \\
\rho_h^m &= \frac{c(g_h^m, g_h^m)}{b(\mathbf{z}_h^m, g_h^m)} \\
\lambda_h^{m+1} &= \lambda_h^m - \rho_h^m \omega_h^m \\
\mathbf{u}_h^{m+1} &= \mathbf{u}_h^m + \rho_h^m \mathbf{z}_h^m
\end{aligned}$$

Take r as large as possible and $c = Id$ and $\rho = r$.

1.2.3 Application to Stokes Equation

The approximate problem is

$$\left\{ \begin{array}{rcl} (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (p_h, \operatorname{div} \mathbf{v}_h) & = & (\mathbf{f}, \mathbf{v}_h) \text{ for all } \mathbf{v}_h \in X_h, \\ (q_h, \operatorname{div} \mathbf{v}_h) & = & 0 \text{ in for all } q_h \in M_h(\Omega) \\ \mathbf{u}_h & = & \mathbf{g} \text{ on } \Gamma. \end{array} \right. \tag{1.36}$$

Then with

$$a(\mathbf{u}_h, \mathbf{v}_h) = (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h), \quad b(\mathbf{v}_h, q) = -(q, \operatorname{div} \mathbf{v}_h)$$

this fits with previous setting. Let $Q_h : L_0^2(\Omega) \to M_h$ is orthogonal projection defined by

$$(Q_h q, \mu) = (q, \mu) \quad \forall \mu \in M_h.$$

Set $\chi = 0$, $C = id$ and $< \ell, \mathbf{v} >=< \mathbf{f}, \mathbf{v} >$ in previous setting. Conjugate gradient method with

$$a_r^h(\mathbf{u}_h, \mathbf{v}_h) = a(\mathbf{u}_h, \mathbf{v}_h) + r \langle Q_h(\operatorname{div} \mathbf{u}_h), Q_h(\operatorname{div} \mathbf{v}_h) \rangle \tag{1.37}$$

is described as follows:

Given an initial guess $p_h^0 \in M_h$, compute the solution $\mathbf{u}_h^0 \in X_h$ satisfying

$$a_r^h(\mathbf{u}_h^0, \mathbf{v}_h) = (p_h^0, \operatorname{div} \mathbf{v}_h) + (\mathbf{f}, \mathbf{v}_h) - a(\mathbf{u}_g, \mathbf{v}), \forall \mathbf{v}_h \in X_h$$

Let $\sigma_0 = 0$.

For $m \geq 0$, compute $(\mathbf{z}_h^m, \omega_h^m) \in X_h \times M_h, \mu_h^m, \sigma_h^m \in \mathbb{R}$ and $(\mathbf{u}_h^{m+1}, p_h^{m+1}) \in X_h \times M_h$ by

$$\begin{array}{rcl} \sigma_h^m & = & \dfrac{\|Q_h(\operatorname{div} \mathbf{u}_h^m)\|_0^2}{\|Q_h(\operatorname{div} \mathbf{u}_h^{m-1})\|_0^2} \\ \omega_h^m & = & Q_h(\operatorname{div} \mathbf{u}_h^m) + \sigma_h^m \omega_h^{m-1}, \quad \omega_h^0 = Q_h(\operatorname{div} \mathbf{u}_h^0) \\ a_r^h(\mathbf{z}_h^m, \mathbf{v}_h) & = & -(\omega_h^m, \operatorname{div} \mathbf{v}_h), \quad \forall \mathbf{v}_h \in X_h, \\ \mu_h^m & = & -\dfrac{\|Q_h(\operatorname{div} \mathbf{u}_h^m)\|_0^2}{(Q_h(\operatorname{div} \mathbf{u}_h^m), Q_h(\operatorname{div} \mathbf{z}_h^m))} \\ p_h^{m+1} & = & p_h^m - \mu_h^m \omega_h^m \\ \mathbf{u}_h^{m+1} & = & \mathbf{u}_h^m + \mu_h^m \mathbf{z}_h^m \end{array}$$

Here the projection is necessary each step because $\operatorname{div} \mathbf{u}_h^m$ does not belong to $L_0^2(\Omega)$.

Finally, add $\mathbf{u}_g$ to $\mathbf{u}_h^\infty$.

1.2.4 Error Estimate

Hypothesis 1

(approximation property of X_h) There exists an operator $\mathbf{r}_h : H^{m+1}(\Omega)^n \cap H_0^1(\Omega)^n \to X_h$ such that

$$\|\mathbf{v} - \mathbf{r}_h \mathbf{v}\|_1 \le Ch^m \|\mathbf{v}\|_{m+1}, \quad \forall \mathbf{v} \in H^{m+1}(\Omega)^n \quad 1 \le m \le l \tag{1.38}$$

Hypothesis 2

(approximation property of M_h) There exists an operator $S_h : L^2(\Omega) \to M_h$ such that

$$\|q - S_h q\|_0 \le Ch^m \|q\|_{m+1}, \quad \forall q \in H^m(\Omega)^n, \quad 0 \le m \le l \tag{1.39}$$

Hypothesis 3

(Uniform inf-sup condition) For each $q_h \in M_h$ there exists a $\mathbf{v}_h \in X_h$ such that

$$(q_h, \operatorname{div} \mathbf{v}_h) = \|q_h\|_0^2, \tag{1.40}$$

$$|\mathbf{v}_h|_1 \le C \|q_h\|_0, \tag{1.41}$$

where $C > 0$ is independent of h, q_h and $\mathbf{v}_h$.

Theorem 1.2.6. *Under Hypothesis 1,2,3, the solution of the problem(1.36) satisfies*

$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \le Ch^m \{\|\mathbf{u}\|_{m+1} + \|p\|_m\}. \tag{1.42}$$

Remark 1.2.7. One can expect one higher order for L^2 error estimate by duality technique.

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \le Ch\{|\mathbf{u} - \mathbf{u}_h|_1 + \inf \|p - p_h\|_0\}. \tag{1.43}$$

The next task is how to construct spaces X_h and M_h which satisfy the hypotheses.

1.2.5 Approximation Spaces X_h and M_h

P_1 nonconforming finite element method

First we introduce a P_1 nonconforming finite element method for $-\Delta u = f$. Given a triangulation of the domain by triangles. Consider the space of all piecewise linear functions which is continuous only at mid point of edges. Here the degree of freedom is located at mid point of edges.

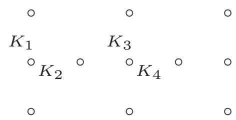

Figure 1.1: Label of elements and vertices

Let N_h be the space of all functions which is linear on each triangle and whose degrees of freedoms are determined

$$\begin{cases} u_h(m)|_L = u_h(m)|_R & \text{when } m \text{ is a mid point of interior edges} \\ u_h(m) = 0 & \text{when } m \text{ is a mid point of boundary edges} \end{cases}$$

Since u_h is discontinuous, the $a(u_h, v_h) = \int_\Omega \nabla u_h \cdot \nabla v_h \, dx$- is not well defined. So we define a discrete form a_h as follows:

$$a_h(u_h, v_h) = \sum_{T \in \mathcal{T}_h} \int_T \nabla u_h \cdot \nabla v_h \, dx \tag{1.44}$$

The solution of this fem with P_1-nonconforming fem is: Find $u_h \in N_h$ such that

$$a_h(u_h, v_h) = f(v_h), \quad \forall v_h \in N_h.$$

Note that in general

$$a_h(u, v_h) \neq f(v_h).$$

Also we define a discrete norm on N_h by

$$\|u_h\|_h = a_h(u_h, u_h)^{1/2}.$$

Theorem 1.2.8 (Second Strang lemma). *Under conditions given above, there exists a constant C independent of v_h such that*

$$\|u - u_h\|_h \;\leq\; C \left(\inf_{v_h \in V_h} \|u - v_h\|_h + \sup_{w_h \in V_h} \frac{|a_h(u, w_h) - f(w_h)|}{\|w_h\|_h} \right). \tag{1.45}$$

Proof . Let v_h be an arbitrary element in V_h. Then

$$\begin{aligned}
\alpha\|u_h - v_h\|_h^2 &\leq a_h(u_h - v_h, u_h - v_h) \\
&= a(u - v_h, u_h - v_h) + f(u_h - v_h) - a_h(u, u_h - v_h) \\
&\leq M\|u - v_h\|_h\|u_h - v_h\|_h + |f(u_h - v_h) - a_h(u, u_h - v_h)|.
\end{aligned}$$

So

$$\begin{aligned}
\alpha\|u_h - v_h\|_h &\leq CM\|u - v_h\|_h + \frac{|f(u_h - v_h) - a_h(u, u_h - v_h)|}{\|w_h\|_h} \\
&\leq CM\|u - v_h\|_h + \sup_{w_h \in V_h} \frac{|f(w_h) - a_h(u, w_h)|}{\|w_h\|_h}
\end{aligned}$$

Now result follows from this and the triangle inequality

$$\|u - v_h\|_h \leq \|u - v_h\|_h + \|u_h - v_h\|_h.$$

□

Stable pair for Stokes equation

For Stokes equation, we need to choose pair of spaces so that inf-sup condition holds. For velocity typically we use P_2 and P_1 for pressure. Another choice is P_1-nonconforming for velocity and P_0 for pressure(Called C-R(Crouzeix-Raviart-1973) element). Let P_0 be the space of all functions which are piecewise constant on each T. Then Hypothesis 1,2,3 hold and it we have

Theorem 1.2.9. *The solution of the Stokes problem(1.36) with $X_h = (N_h)^2$, $M_h = P_0 \cap L_0^2(\Omega)$ satisfies*

$$\|\mathbf{u} - \mathbf{u}_h\|_h + \|p - p_h\|_0 \leq Ch^1\{\|\mathbf{u}\|_2 + \|p\|_1\}. \tag{1.46}$$

1.3 Navier-Stokes equation

Notations: with vector $\mathbf{u}, \mathbf{v}$ and matrix A, B, define

$$\begin{aligned}
\mathbf{grad}\,\mathbf{u} &= \nabla\mathbf{u} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{pmatrix} \\
\mathbf{uu} &= \mathbf{e}_i u_i \mathbf{e}_j u_j \quad \text{dyadic product or outer product} \\
(\mathbf{u} \cdot \nabla)\mathbf{v} &= \mathbf{e}_i \sum_j u_j \frac{\partial v_i}{\partial x_j}
\end{aligned}$$

$$\begin{aligned} \nabla \cdot A &= \mathbf{e}_k \sum_j \frac{\partial A_{jk}}{\partial x_j} \\ A : B &= A_{ij} B_{ij} \quad \text{scalar} \end{aligned}$$

The Navier-Stokes equation(dimensionless form-after some scaling)

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + \sum_j u_j \frac{\partial \mathbf{u}}{\partial x_j} - \nu \Delta \mathbf{u} + \mathbf{grad}\, p = \mathbf{f}, \\ \operatorname{div} \mathbf{u} = 0. \end{cases} \tag{1.47}$$

can be written as

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \mathbf{grad}\, p = \mathbf{f}, \\ \operatorname{div} \mathbf{u} = 0. \end{cases} \tag{1.48}$$

Then we have

$$\nabla \cdot (\mathbf{uu}) = \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{u}(\nabla \cdot \mathbf{u}) \tag{1.49}$$

Exer. Prove it.(First verify)

$$\int_K \nabla \cdot (\mathbf{uu}) dx = \left(\int_{\partial K} (u_1^2 n_1 + u_2 u_1 n_2) ds, \int_{\partial K} (u_1 u_2 n_1 + u_2^2 n_2) ds \right)^T$$

1.3.1 Alternative forms

Divergence form

With (1.49) and divergence free condition, one has another form of NS equation:

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{uu}) - \nu \Delta \mathbf{u} + \mathbf{grad}\, p = \mathbf{f}, \\ \operatorname{div} \mathbf{u} = 0. \end{cases} \tag{1.50}$$

Skew-Symmetric form

We begin with the identity

$$\frac{1}{2}[\nabla \cdot (\mathbf{uu}) + (\mathbf{u} \cdot \nabla)\mathbf{u}] = (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{2}\mathbf{u}(\nabla \cdot \mathbf{u}) \tag{1.51}$$

This gives the following form(Temam)

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{2}\mathbf{u}(\nabla \cdot \mathbf{u}) - \nu \Delta \mathbf{u} + \mathbf{grad}\, p = \mathbf{f}, \\ \operatorname{div} \mathbf{u} = 0. \end{cases} \tag{1.52}$$

Still another form:

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - (\nabla \mathbf{u}) \cdot \mathbf{u} - \nu \Delta \mathbf{u} + \mathbf{grad}\, p = \mathbf{f}, \\ \operatorname{div} \mathbf{u} = 0. \end{cases} \tag{1.53}$$

Symmetric form

Skip. See Incompressible flow and the FEM by P.M Gresho and R.L. Sani, Wiley, 2000

1.4 The “Velocity-Pressure” formulation

Consider homogeneous boundary condition: $\mathbf{u} = 0$ on Γ. Introduce trilinear form

$$a(w; u, v) = \sum_{i,j=1}^{n} \int_{\Omega} w_j \frac{\partial u_i}{\partial x_j} v_i dx \tag{1.54}$$

Lemma 1.4.1. *The trilinear form a is continuous.*

Proof . Use Hölder’s inequality

$$\left| \int_{\Omega} w_j \frac{\partial u_i}{\partial x_j} v_i dx \right| \leq \|w_j\|_{0,4} \|\partial u_i / \partial x_j\|_0 \|v_i\|_{0,4}$$

□

Lemma 1.4.2. *Let* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in H^1(\Omega)^n$ *and let* $div\mathbf{w} = 0$ *and* $\mathbf{w} \cdot \mathbf{n} = 0$ *on boundary. Then*

$$a_1(\mathbf{w}; \mathbf{u}, \mathbf{v}) + a_1(\mathbf{w}; \mathbf{v}, \mathbf{u}) = 0, \tag{1.55}$$

$$a_1(\mathbf{w}; \mathbf{v}, \mathbf{v}) = 0. \tag{1.56}$$

Proof . Check second one. Since

$$a_1(\mathbf{w}; \mathbf{v}, \mathbf{v}) = \frac{1}{2} \sum_{i,j=1}^{n} \int_{\Omega} w_j \frac{\partial v_i^2}{\partial x_j}$$

we have by Green’s formula,

$$a_1(\mathbf{w}; \mathbf{v}, \mathbf{v}) = -\frac{1}{2} \sum_{i,j=1}^{n} \left\{ \int_{\Omega} \operatorname{div} \mathbf{w} v_i + \int_{\Gamma} \mathbf{w} \cdot \mathbf{v} n_i ds \right\} = 0.$$

□

We set

$$a_0(\mathbf{u}, \mathbf{v}) = \nu(\mathbf{grad}\, \mathbf{u}, \mathbf{grad}\, \mathbf{v})$$

and

$$a(\mathbf{w};\mathbf{u},\mathbf{v}) \quad = \quad a_0(\mathbf{u},\mathbf{v}) + a_1(\mathbf{w};\mathbf{u},\mathbf{v}) \tag{1.57}$$

The problem has equivalent form:
Find a pair $(\mathbf{u},p) \in V \times L_0^2(\Omega)$ such that

$$a(\mathbf{u};\mathbf{u},\mathbf{v}) - (p,\, \mathrm{div}\,\mathbf{v}) = (\mathbf{f},\mathbf{v})\,, \quad \forall \mathbf{v} \in H_0^1(\Omega)^n \tag{1.58}$$

Introduce norm of a-form:

$$\mathcal{N} = \sup \frac{a(\mathbf{w};\mathbf{u},\mathbf{v})}{\|\mathbf{w}\|_1\|\mathbf{u}\|_1\|\mathbf{v}\|_1} \tag{1.59}$$

Theorem 1.4.3. *If*

$$\frac{\mathcal{N}}{\nu^2}\|\mathbf{f}\| < 1 \tag{1.60}$$

then problem (1.58) has a unique solution.

1.4.1 Abstract Theory

The nonlinearity is introduced by means of a form

$$a(\cdot;,\cdot,\cdot) : (u,v,w) \in X \times X \times X \to a(w;u,v) \in \mathbb{R}$$

We consider problem (Q):
For ℓ given in X', find a pair (u,λ) in $X \times M$ such that

$$a(u;u,v) + b(v,\lambda) = \langle \ell, v\rangle \quad \forall v \in X \tag{1.61}$$

$$b(u,\mu) = 0 \quad \forall \mu \in M. \tag{1.62}$$

Introduce linear operators $A(w) \in L(X;X')$ for w in X, and $B \in L(X;M')$ by

$$\begin{aligned} A(w)u,v \quad &= \quad a(w;u,v), \quad \forall u,v \in X, \\ \langle Bv,\mu\rangle \quad &= \quad b(v,\mu), \quad \forall \mu \in M. \end{aligned}$$

With these the problem (Q) becomes

$$A(w)u + B'\lambda \quad = \quad \ell, \text{ in } X', \tag{1.63}$$

$$Bu \quad = \quad 0 \text{ in } M'. \tag{1.64}$$

Iterative scheme for continuous case

Starting from $u^0 \in V \equiv Ker(B)$ construct (u_m,λ_m) in $X \times M$ by

$$a(u_m;u_{m+1},v) + b(v,\lambda_{m+1}) = \langle \ell, v\rangle \quad \forall v \in X \tag{1.65}$$

$$b(u_{m+1},\mu) = 0 \quad \forall \mu \in M. \tag{1.66}$$

1.4.2 Numerical method

Let

$$W_{0h} = W_h \cap H_0^1(\Omega)^n, \quad M_h = Q_h \cap L_0^2(\Omega)$$

The approximate problem is : Find $(\mathbf{u}_h, p_h) \in W_{0h} \times M_h(\Omega)$ such that

$$\left\{ \begin{array}{rcl} a(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) - (p_h, \operatorname{div} \mathbf{v}_h) & = & (\mathbf{f}, \mathbf{v}_h) \ \text{ for all } \mathbf{v}_h \in W_{0h}, \\ (q_h, \operatorname{div} \mathbf{v}_h) & = & 0 \text{ in } \ \text{for all } q_h \in M_h(\Omega) \\ \mathbf{u}_h & = & \mathbf{g} \text{ on } \Gamma. \end{array} \right. \tag{1.67}$$

Iterative scheme

$$\left\{ \begin{array}{rcl} a(\mathbf{u}_h^m; \mathbf{u}_h^{m+1}, \mathbf{v}_h) - (p_h^{m+1}, \operatorname{div} \mathbf{v}_h) & = & (\mathbf{f}, \mathbf{v}_h) \ \text{ for all } \mathbf{v}_h \in W_{0h}, \\ (q_h, \operatorname{div} \mathbf{v}_h^{m+1}) & = & 0 \text{ in } \ \text{for all } q_h \in M_h(\Omega) \\ \mathbf{u}_h^{m+1} & = & \mathbf{g} \text{ on } \Gamma. \end{array} \right. \tag{1.68}$$