Chapter 8

Finite Difference Method

8.1 2nd order linear p.d.e. in two variables

General 2nd order linear p.d.e. in two variables is given in the following form:

$$L[u] = Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

According to the relations between coefficients, the p.d.es are classified into 3 categories, namely,

elliptic	if $AC - B^2 > 0$	i.e., A, C has the same sign
hyperbolic	if $AC - B^2 < 0$	
parabolic	if $AC - B = 0$	

Furthermore, if the coefficients A, B and C are constant, it can be written as

$$[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} + Du_x + Eu_y + Fu = G.$$

Auxiliary condition

$$\begin{cases} \text{B.C.} \\ \text{Interface Cond} \\ \text{I.C.} \end{cases}$$

we say “well posed” if a solution exists. There are basically two class of method to discretize it,

(1) Finite Difference method

(2) Finite Element method

8.2 Finite difference operator

Let $u(x)$ be a function defined on $\Omega \subset \mathbb{R}^n$. Let $U_{i,j}$ be the function defined over discrete domain $\{(x_i, y_j)\}$ such that $U_{i,j} = u(x_i, y_j)$. Such functions are called grid functions. We introduce difference operators on the grid functions.

$$\begin{aligned}
\delta^+ U_i &= \frac{U_{i+1} - U_i}{h_{i+1}}, \quad \text{forward difference} \\
\delta^- U_i &= \frac{U_i - U_{i-1}}{h_i}, \quad \text{backward difference} \\
\delta^0 U_i &= \frac{U_{i+1} - U_{i-1}}{h_i + h_{i+1}}, \quad \text{central difference} \\
\delta^2 U_i &= \frac{2(\delta^+ - \delta^-)}{h_i + h_{i+1}}, \quad \text{central 2nd difference}
\end{aligned}$$

Example 8.2.1. Consider the following second order differential equation :

$$-u''(x) = f(x), u(a) = c, u(b) = d.$$

Given a mesh $a = x_0 < x_1 < \cdots < x_N = b, \Delta x_i = x_{i+1} - x_i = h$, we have

$$-\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} = f_i = f(x_i), \quad i = 1, \cdots N-1, U_0 = c, U_N = d$$

which determines U_i uniquely. We obtain an $(N-1)\times(N-1)$ matrix equations.

$$\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & & \cdot & \cdot & \cdot & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix} \begin{pmatrix} U_1 \\ \cdot \\ \cdot \\ \cdot \\ U_{N-1} \end{pmatrix} = h^2 \begin{pmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_{N-1} \end{pmatrix} + \begin{pmatrix} c \\ 0 \\ 0 \\ 0 \\ d \end{pmatrix}$$

Above equation can be written as $L_h U^h = F^h$, called a difference equation for a given differential equation.

Exercise 8.2.2. Write down a matrix equation for the same problem with second boundary condition changed to the normal derivative condition at b, i.e, $u'(b) = d$. If one uses first order difference for derivative, we lose accuracy.

We need an extra equation in this case. There are several choices:

(1) Use first order backward difference scheme

$$\frac{U_1 - U_{n-1}}{h} = d$$

and append this to the last eq.(first order)

(2) Use central difference equation by assuming a fictitious point U_{N+1} and assume the D.E. holds at the end point: Use

$$-\frac{1}{h^2}(U_{N-1} - 2U_N + U_{N+1}) = f(1) \quad (8.1)$$

$$\frac{1}{2h}(U_{N+1} - U_{N-1}) = b. \quad (8.2)$$

Solve the last eq. and substitute into first eq.

$$\frac{1}{2h}(U_N - U_{N-1}) = -\frac{b}{h} + \frac{1}{2}f(1). \quad (8.3)$$

Eq. 8.3 can be viewed as centered difference approximation to $u'(x_n - \frac{h}{2})$ and rhs as the first two terms of Taylor expansion

$$u'(x_n - \frac{h}{2}) = u'(x_n) - \frac{h}{2}u''(x_n) + \cdots$$

(3) Use

$$u'' \approx \frac{u'_N - u'_{N-1}}{h} = \frac{d - \frac{u_N - u_{N-1}}{h}}{h} = -f(1)$$

so that

$$\frac{u_N - u_{N-1}}{h} = d + hf(1).$$

This is interpreted as using Taylor series of $u'(1-h) = u'(1) + hf(1) = u'(1) - hu''(1)$ But lhs is centered difference to $u(1-\frac{h}{2})$. So not consistent.

(4) Approximate $u'(1)$ by higher order scheme such as

$$\frac{u_{N-2} - 4u_{N-1} + 3u_N}{h} = b.$$

Example 8.2.3 (Heat equation)**.** We consider

$$\begin{aligned} u_t &= \sigma u_{xx}, \quad \text{for } 0 < x < 1, \quad 0 < t < T \\ u(t,0) &= u(t,1) = 0 \\ u(0,x) &= g(x), \quad g(0) = g(1) = 0 \end{aligned}$$

Let $x_i = ih, i = 0, \cdots, N, \Delta x = 1/N$ and $t_n = n\Delta t, \Delta t = \frac{T}{J}$. Then we have the following difference scheme

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \sigma \left[\frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{\Delta x^2} \right],$$

for $i = 1, 2, \cdots, N-1$ and $n = 1, 2, \cdots, M-1$ where $U_i^n \approx u(t_i, x_n)$. From the boundary condition and initial condition we have

$$U_i^0 = g(x_i), U_0^n = 0, U_N^n = 0.$$

$$U_i^{n+1} = U_i^n + \frac{\sigma \Delta t}{\Delta x^2} \left[U_{i-1}^n - 2U_i^n + U_{i+1}^n \right].$$

In vector notation

$$U_h^{n+1} = U_h^n - \frac{\sigma \Delta t}{\Delta x^2} A U_h^n$$

where A is the same matrix as in example 1. If $n = 0$, right hand side is known. Thus

$$U_h^n = (I - \sigma \frac{\Delta t}{\Delta x^2} A)^n G, \quad G = (g(x_1), \cdots, g(x_{N-1}))^T.$$

This is called **forward Euler** or **explicit scheme**. If we change the right hand side to

$$\begin{aligned} \frac{U_i^{n+1} - U_i^n}{\Delta t} &= \sigma \left[\frac{U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}}{\Delta x^2} \right] \\ U_i^{n+1} &= U_i^n + \frac{\sigma \Delta t}{\Delta x^2} \left[U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1} \right]. \end{aligned}$$

$$(I + \sigma \frac{\Delta t}{\Delta x^2} A)^n U_h^n = G, \quad G = (g(x_1), \cdots, g(x_{N-1}))^T.$$

This is called **backward Euler** or **implicit scheme**.

8.2.1 Error of difference operator

For $u \in C^2$, use the Taylor expansion about x_i

$$\begin{aligned} u_{i+1} &= u(x_i + h_i) = u(x_i) + h_i u'(x_i) + \frac{h_i^2}{2} u''(\xi), \quad \xi \in (x_i, x_{i+1}) \\ \therefore \frac{u_{i+1} - u_i}{h_i} - u'(x_i) &= \frac{h_i}{2} u''(\xi). \end{aligned}$$

Expand about x_{i+1},

$$u_i = u_{i+1} - h_i u'(x_i) + \frac{h_i}{2} u''(\theta).$$

These are first order accurate. To derive a second order scheme, expand about $x_{i+1/2}$,

$$\begin{aligned} u_{i+1} &= u_{i+1/2} + \frac{h_i}{2} u'(x_{i+1/2}) + \frac{1}{2}(\frac{h_i}{2})^2 u''(x_{i+1/2}) + \frac{1}{6}(\frac{h_i}{2})^3 u^{(3)}(\xi) \\ u_i &= u_{i+1/2} - \frac{h_i}{2} u'(x_{i+1/2}) + \frac{1}{2}(\frac{h_i}{2})^2 u''(x_{i+1/2}) - \frac{1}{6}(\frac{h_i}{2})^3 u^{(3)}(\xi). \end{aligned}$$

Subtracting,

$$\frac{u_{i+1} - u_i}{h_i} = u'(x_{i+1/2}) + O(h_i^2).$$

Thus we obtain a second order approximation to $u'(x_{i+1/2})$. By translation, we have

$$\frac{u_{i+1} - u_{i-1}}{2h_i} - u'(x_i) = O(h_i^2/6) \quad \text{if } h_i = h_{i+1}.$$

Assume $h_i = h_{i+1}$ and we substitute the solution of differential equation into the difference equation. Using $-u'' = f$ we obtain

$$\begin{aligned} & \frac{(-u_{i-1} + 2u_i - u_{i+1})}{h^2} - f(x_i) \\ = & \frac{1}{h^2}(-u_i + hu_i' - \frac{h^2}{2} u_i'' + \frac{h^3}{6} u^{(3)} - \frac{h^4}{24} u^{(4)}(\theta_1) + 2u_i) \\ & + \frac{1}{h^2}(-u_i - hu_i' - \frac{h^2}{2} u_i'' - \frac{h^3}{6} u^{(3)} - \frac{h^4}{24} u^{(4)}(\theta_2)) - f(x_i) \\ = & -u_i'' - f(x_i) - \frac{h^2}{24}(u^{(4)}(\theta_1) + u^{(4)}(\theta_2)) \text{ truncation error} \\ = & \frac{h^2}{12} \max |u^{(4)}|. \end{aligned}$$

We let $\tau_h = L_h u - F^h$ and call it the **truncation error(discretization**

error).

Definition 8.2.4. We say a difference scheme is **consistent** if the truncation error approaches zero as h approaches zero, in other words, if $L_h u - f \to 0$ in some norm.

Truncation error measures how well the difference equation approximates the differential equation. But it does not measure the actual error in the solution.

Use of different quadrature for f. Instead of $f(x_i)$ we can use

$$\frac{1}{12}[f(x_{i-1}) + 10f(x_i) + f(x_{i+1})] = \frac{5}{6}f(x_i) + \frac{\mu_0}{6}f(x_i)$$

where $\mu_0 f(x_i)$ is the average of f which is $f(x_i) + O(h^2)$.
H.W Let $-u'' = f$. Show the following for uniform grid

$$\frac{-u_{i-1} + 2u_i - u_{i-1}}{h^2} = \frac{1}{12}[f(x_{i-1}) + 10f(x_i) + f(x_{i+1})] + Ch^4 \max |u^{(6)}(x)|.$$

Definition 8.2.5. L_h is said to be **stable** if

$$\|U_h\| \le C\|L_h U^h\| \le C\|F^h\| \quad \text{for all } h > 0$$

where U^h is the solution of the difference equation, $L_h U^h = F^h$. Also note that L_h is stable if and only if L_h^{-1} is bounded.

Definition 8.2.6. A finite difference scheme is said to **converge** if

$$\|U^h - u\| \to 0 \qquad \text{as } h \to 0.$$

$\|U^h - u\|$ is called a **discretization error.**

Theorem 8.2.7 (P. Lax)**.** *Given a consistent scheme, stability is equivalent to convergence.*

Proof. Assume stability. From $L_h u - f = \tau^h, L_h U^h - F^h = 0$, we have $L_h(u - U^h) = \tau^h$. Thus,

$$\|u - U^h\| \le C\|L_h(u - U^h)\| = C\|\tau^h\| \to 0.$$

Hence the scheme converges. Obviously a convergent scheme must be stable.

From the theory of p.d.e, we know $\|u\| \le C\|f\|$. Hence

$$\|U^h\| \le \|U^h - u\| + \|u\| \le O(\tau^h) + C\|f\| \le C\|f\| \le C\|F^h\|.$$

□

8.3 Elliptic equation

8.3.1 Basic finite difference method for elliptic equation

In this chapter, we only consider finite difference method. First consider the following elliptic problem:(Dirichlet problem by Finite Difference Method)

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= g \text{ on } \partial\Omega \end{aligned}$$

(1) Approx. D.E. $-(u_{xx} + u_{yy}) = f$ at each interior mesh pt.

(2) The unknown function is to be approximated by a grid function u

(3) Replace the derivative by difference quotient.

$$\begin{aligned} u(x+h) &= u(x) + hu_x(x) + \tfrac{h^2}{2}u_{xx}(x) + \tfrac{h^3}{6}u_{xxx}(x) + O(h^4) \\ u(x-h) &= \ldots \end{aligned}$$

$$\frac{u(x+h) - 2u(x) + u(x-h)}{h^2} = u_{xx}(x) + O(h^2)$$

$$\begin{aligned} u_{xx}(x,y) &\doteq [u(x+h,y) - 2u(x,y) + u(x-h,y)]/h^2 \\ u_{yy}(x,y) &\doteq [u(x,y+h) - 2u(x,y) + u(x,y-h)]/h^2 \end{aligned}$$

This picture is called, Molecule, Stencil, Star, etc. For each point (interior mesh pt), approx $\nabla^2 u = \Delta u$ by 5-point stencil. By Girshgorin disc theorem, the matrix is nonsingular. $L[u]$ is called **differential operator** while $L_h[u]$ is called **finite difference operator**, e.g.,

$$L_h[u] = \frac{1}{h^2}[-4u(x,y) + u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h)]$$

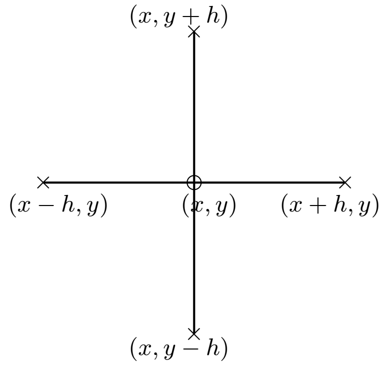

Figure 8.1: 5-point Stencil

or more generally,

$$L[u] = -[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}]\, \mathrm{Diag}\{a_{11}, a_{22}\} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} + cu = -(a_{11}u_x)_x - (a_{22}u_y)_y + cu$$

Uniform meshes

$$\begin{aligned} u_x &\doteq \tfrac{u(x+h)-u(x-h)}{2h} \\ (u_x)_x &\doteq \tfrac{u_x(x+\frac{h}{2})-u_x(x-\frac{h}{2})}{h} \qquad \text{Central difference} \end{aligned}$$

$$\begin{aligned} u_x(x+\tfrac{h}{2}) &= \tfrac{u(x+h)-u(x)}{h} \\ u_x(x-\tfrac{h}{2}) &= \tfrac{u(x)-u(x-h)}{h} \end{aligned}$$

$$(a_{11}u_x)_x = [(a_{11}u_x)(x+\frac{h}{2}) - (a_{11}u_x)(x-\frac{h}{2})]/h$$

Assume the differential operator is of the form(with $c > 0$)

$$L[u] \equiv -(u_{xx} + u_{yy}) + cu = f$$

whose discretized form

$L_h[U] = a_0U(x, y) - a_1U(x + h, y) + \cdots = F(x, y)$

$$\frac{1}{h^2}\begin{pmatrix} 4+ch^2 & -1 & -1 & 0 \\ -1 & 4+ch & 0 & -1 \\ -1 & 0 & 4+ch^2 & -1 \\ 0 & -1 & -1 & 4+rch^2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = \mathbf{f}$$

satisfies

(1) $L_h[u] = L[u] + O(h^2)$ as $h \to 0$. u is true solution.

(2) $AU = F + B.C.$, $Au = [\Delta u - cu + O(h^2)] + B.C.$

$$A(U-u) = O(h^2) = \varepsilon.$$

Then the discretization error $U - u = A$ has the form $^{-1}\varepsilon$(depends on h) and satisfies

$$\|U-u\| \le \|A^{-1}\| \cdot \|\varepsilon\| \le \|A^{-1}\| O(h^2)$$

More generally when the unit square is divided by $n = 1/h$ equal intervals along x-axis and y-axis, then the corresponding matrix A(with $c = 0$) is $(n-1) \times (n-1)$ block-diagonal matrix of the form:

$$A = \frac{1}{h^2}\begin{bmatrix} B & -I & 0 & \cdots & \\ -I & B & -I & 0 & \\ & -I & \ddots & \ddots & \\ & & \ddots & B & -I \\ & \cdots & 0 & -I & B \end{bmatrix} \tag{8.4}$$

where

$$B = \begin{bmatrix} 4 & -1 & 0 & \cdots & \\ -1 & 4 & -1 & 0 & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 4 & -1 \\ & \cdots & 0 & -1 & 4 \end{bmatrix}$$

is $(n-1) \times (n-1)$ matrix. If we put $D = diag A = \{a_{11}, \ldots, a_{nn}\}$, then $D^{-1}A(U-u) = D^{-1}\varepsilon$. Write $D^{-1}A = I + B$, where B is off diagonal. Then we know $\|B\|_\infty = \frac{4}{4+ch^2} < 1$ if $c > 0$. Thus $(D^{-1}A)^{-1} = (I+B)^{-1}$ exists and

$$\|(D^{-1}A)^{-1}\|_\infty = \|(I+B)^{-1}\|_\infty \le \frac{1}{1-\|B\|_\infty} \le \frac{4+ch^2}{ch^2}.$$

Hence

$$\|U-u\|_\infty \le \|(D^{-1}A)^{-1}\|_\infty \cdot \|D^{-1}\varepsilon\|_\infty \le \frac{4+ch^2}{ch^2} \cdot \frac{h^2}{4+ch^2} O(h^2) = O(h^2) \to 0$$

Thus, we have proved the following result.

Theorem 8.3.1. *Let*

(1) $u \in C^4(\Omega)$

(2) $r > 0$

(3) uniform mesh

Then $\|U - u\|_\infty = O(h^2)$ *as* $h \to 0$.

8.4 Parabolic p.d.e's

Consider a heat equation on a bar.

$$u_t = u_{xx}, \quad 0 \le x \le 1, \quad 0 < t \le T.$$

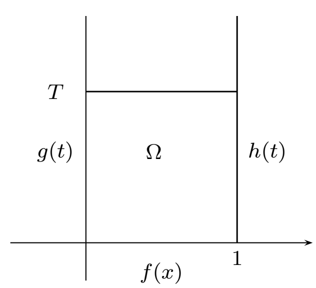

Figure 8.2: Domain

Theorem 8.4.1 (Maximum principle)**.** *If* u *satisfies the above condition for* $t \le T$*, then*

$$\min\{f,g,h\} = m \le \min_{0 \le x \le 1, 0 \le t \le T} u \le \max_{0 \le x \le 1, 0 \le t \le T} u \le M = \max\{f,g,h\}$$

Proof. Put $v = u + Ex^2$, $E > 0$

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = -2E < 0$$

If v attains a maximum at $Q \in \operatorname{int}\Omega$, then

$$\begin{aligned} v_t(Q) &= 0, \\ v_{xx}(Q) &\le 0. \end{aligned}$$

Thus $(v_t - v_{xx})(Q) \ge 0$, a contradiction. Hence v has maximum at a boundary point of Ω

$$u(x,t) \le v(x,t) \le \max v(x,t) \le M + E.$$

Since E was arbitrary, the proof is complete. For minimum, use $-E$ instead of E. □

More general parabolic p.d.e.

$$u_t = Au_{xx} + Du_x + Fu + G$$

$$\text{F.D.M}\begin{cases}\text{Explicit}\cdots\text{write down the values of grid function}\\ \text{Implicit}\cdots\text{ variables implicitly representing the value}\end{cases}$$

Let the grid be given by

$$\begin{aligned} 0 &= x_0 < x_1 < x_2 < \cdots < x_{N+1} = 1, \quad x_i = ih, \quad \text{uniform grid} \\ 0 &= t_0 < t_1 < \cdots, \quad t_j = jk \end{aligned}$$

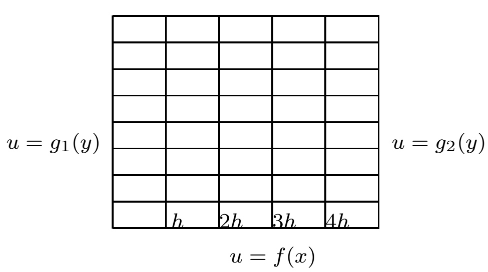

Figure 8.3:

Explicit method

$$\begin{cases} \frac{U_{i,j+1}-U_{i,j}}{k} \doteq u_t \\ \frac{U_{i+1,j}-2U_{i,j}+U_{i-1,j}}{h^2} \doteq u_{xx} \end{cases}$$

$$\therefore \qquad U_{i,j+1} = \lambda U_{i-1,j} + (1-2\lambda)U_{i,j} + \lambda U_{i+1,j}$$

where $\lambda = k/h^2$.

Stability: Error doesn't accumulate. In this case **solution remains bounded** as time goes on.

Theorem 8.4.2. *If u is sufficiently smooth, then*

$$\left| u_{xx} - \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} \right| = O(h^2) \quad as \quad h \to 0$$

and

$$\left| u_t - \frac{u(x,t+k) - u(x,t)}{k} \right| = O(k) \quad as \quad k \to 0$$

Theorem 8.4.3. *Suppose u is sufficiently smooth, and satisfies*

$$\begin{aligned} u_t &= u_{xx} \qquad 0 < x < 1, \quad t > 0 \\ u(x,0) &= f(x) \\ u(0,t) &= g(t) \\ u(1,t) &= h(t). \end{aligned}$$

If $U_{i,j}$ is the solution of the explicit finite difference scheme, then for $0 < \lambda \leq \frac{1}{2}$,

$$\max_{i,j} |u_{i,j} - U_{i,j}| \doteq O(h^2 + k) \quad as \quad h, k \to 0,$$

i.e, finite difference solution converges to the true solution.

Proof. Put $u_{ij} \equiv u(x_i, t_j)$. Then from

(1) $\dfrac{u_{i,j+1} - u_{i,j}}{k} = u_t + O(k)$

(2) $\dfrac{u_{i+1,j} - 2u_{i,j} + u_{i,j}}{h^2} = u_{xx} + O(h^2)$

we get

$$u_{i,j+1} = u_{i,j} + \frac{k}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + k(O(k) + O(h^2)).$$

Hence

$$u_{i,j+1} = \lambda u_{i-1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i+1,j} + Ck(k+h^2).$$

Let the discretization error be $w_{i,j} = u_{ij} - U_{ij}$ so that

$$w_{i,j+1} = \lambda w_{i-1,j} + (1-2\lambda)w_{i,j} + \lambda w_{i+1,j} + O(k^2+kh^2).$$

Since $0 < \lambda \leq \frac{1}{2}$, $0 \leq 1-2\lambda < 1$, three coefficient are positive and their sum is 1. (convex combination) We see

$$|w_{i,j+1}| \leq \lambda|w_{i-1,j}| + (1-2\lambda)|w_{i,j}| + \lambda|w_{i+1,j}| + M(k^2+kh^2) \quad \text{for some} \quad M > 0.$$

If we define $\|w_j\| = \max_{1\leq i\leq N} |w_{i,j}|$, then

$$\begin{aligned} \|w_{j+1}\| &\leq \|w_j\| + M(k^2+kh^2) \\ &\leq \|w_{j-1}\| + 2M(k^2+kh^2) \leq \cdots \leq \|w_0\| + (j+1)M(k^2+kh^2). \end{aligned}$$

Since $\|w_0\| = 0$,

$$\|w_{j+1}\| \leq (j+1)kM(k+h^2) \leq TM(k+h^2), \quad (j+1)k \leq T.$$

In fact,

$$M = \max_{0\leq x\leq 1,\, 0\leq t\leq T}\left(\frac{1}{2}|u_{tt}| + \frac{k^2}{12}|u_{xxxx}|\right).$$

□

Remark 8.4.4. If $\lambda > \frac{1}{2}$, the solution may not converge.

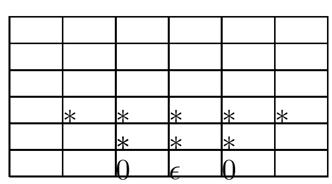

Figure 8.4: Nonzero point

Exercise 8.4.5. Prove the formula is unstable for $\lambda > \frac{1}{2}$. Let

$$u(x,0) = \begin{cases} \varepsilon, & x = \frac{1}{2} \\ 0, & x \neq \frac{1}{2} \end{cases} \quad \text{with} \quad g = h = 0$$

$$\begin{aligned} U_{i,j+1} &= \lambda U_{i+1,j} + (1-2\lambda)U_{i,j} + \lambda U_{i-1,j}, \qquad \lambda = k/h^2 \\ |U_{i,j+1}| &= \lambda |U_{i+1,j}| + (2\lambda - 1)|U_{i,j}| + \lambda |U_{i-1}, j|, \quad 1 \leq i \leq N-1. \end{aligned}$$

Hence

$$\sum_{i=1}^{N-1} |U_{i,j+1}| = \lambda \sum_{i=1}^{N-2} |U_{i+1,j}| + (2\lambda-1) \sum_{i=1}^{N-1} |U_{i,j}| + \lambda \sum_{i=2}^{N} |U_{i-1,j}|,$$

since $U(x_i,t) = 0$, $i = 1, N$.

Let $S(t_j) = \sum_{i=1}^{N} |U(i,j)|$. Then

$$S(t_{j+1}) = (4\lambda - 1)S(t_j) = (4\lambda - 1)^2 S(t_{j-1}) = \cdots = (4\lambda - 1)^{j+1} S(0) = (4\lambda - 1)^{j+1}\varepsilon.$$

Since the number of nonzero $U_{i,j}$ for each j is $2j+1$(Check the numerical scheme, you will see solution is alternating along x-direction dispersing both direction) there is a point (x_p, t_j) such that

$$|U(x_p, t_j)| \geq \frac{1}{2j+1} S(t_j) = \frac{1}{2j+1}(4\lambda - 1)^j \cdot \varepsilon$$

which diverges as $j \to \infty$ since $4\lambda - 1 > 1$.

Considering the alternating sign, one can see the solution alternates: For $j = 1$, we see

$$U_{i,1} = (1-2\lambda)\epsilon, U_{i-1,1} = \lambda\epsilon, U_{i+1,1} = \lambda\epsilon$$

$$U_{i,2} = 2\lambda^2\epsilon + (1-2\lambda)^2\epsilon, U_{i-1,2} = (1-2\lambda)\epsilon + (1-2\lambda)\epsilon = 3\lambda\epsilon(1-2\lambda) < 0.$$

Stability of linear system

$$\begin{pmatrix} U_{1,j+1} \\ \vdots \\ U_{N-1,j+1} \end{pmatrix} = \begin{pmatrix} 1-2\lambda & \lambda & \dots & 0 \\ \lambda & 1-2\lambda & \ddots & \\ 0 & & \ddots & \lambda \\ & & \lambda & 1-2\lambda \end{pmatrix} \begin{pmatrix} U_{1,j} \\ \vdots \\ U_{N-1,j} \end{pmatrix} + \begin{pmatrix} g(t_j) \\ 0 \\ \vdots \\ 0 \\ h(t_j) \end{pmatrix}$$

In vector form, $\mathbf{U_{j+1}} = \mathbf{AU_j} + \mathbf{G_j}$. Assume $\mathbf{G_j} = \mathbf{0}$, $j = 1, 2, \ldots$. Let μ be an eigenvalue of A. Then by G-disk theorem,

$$\begin{aligned}
|1 - 2\lambda - \mu| &\leq 2\lambda \\
-2\lambda &\leq 1 - 2\lambda - \mu \leq 2\lambda \\
-2\lambda &\leq -1 + 2\lambda + \mu \leq 2\lambda \\
1 - 4\lambda &\leq \mu \leq 1
\end{aligned}$$

If $0 < \lambda \leq \frac{1}{2}$, then $-1 \leq \mu \leq 1$, hence stable. If $\lambda > \frac{1}{2}$, then $|\mu| > 1$ is possible. So the scheme may be unstable. The following example show it is actually unstable.

Example 8.4.6 (Issacson, Keller)**.** Try $v(x,t) = \text{Re}(e^{i\alpha x - wt}) = \cos\alpha x \cdot e^{-wt}$.

$$\begin{aligned}
v_t - v_{xx} &\doteq \frac{v(x, t+\Delta t) - v(x,t)}{\Delta t} - \frac{v(x+\Delta x, t) - 2v(x,t) + v(x - \Delta x, t)}{\Delta x^2} \\
&= v(x,t)\left(\frac{e^{-w\Delta t} - 1}{\Delta t}\right) - \frac{\cos(\alpha x + \alpha\Delta x) - 2\cos\alpha x + \cos(\alpha x - \alpha \Delta x)}{\Delta x^2} e^{-wt} \\
&= v(x,t)\left(\frac{e^{-w\Delta t} - 1}{\Delta t} - \frac{2\cos\alpha\Delta x - 2}{\Delta x^2}\right) \\
&= v(x,t)\frac{1}{\Delta t}\{e^{-w\Delta t} - [(1-2\lambda) + 2\lambda\cos\alpha\Delta x]\} \\
&= v(x,t)\frac{1}{\Delta t}\left[e^{-w\Delta t} - \left(1 - 4\lambda\sin^2\frac{\alpha\Delta x}{2}\right)\right]
\end{aligned}$$

Thus v is a solution of the difference equation provided w and α satify $e^{-w\Delta t} = 1 - 4\lambda\sin^2\frac{\alpha\Delta x}{2}$.

With I.C. $v(x, 0) = \cos\alpha x$, solution becomes

$$v(x,t) = \cos\alpha x e^{-wt} = \cos\alpha x\left(1 - 4\lambda\sin^2\frac{\alpha\Delta x}{2}\right)^{\frac{t}{\Delta t}}$$

Clearly, for all $\lambda \leq \frac{1}{2}$, $|v(x,t)| \leq 1$. However, if $\lambda > \frac{1}{2}$, then for some Δx, we have $\left|1 - 4\lambda\sin^2\frac{\alpha\Delta x}{2}\right| > 1$. So $v(x,t)$ becomes arbitrarily large for sufficiently large $t/\Delta t$. Since every even function has a cosine series, we may give any even function $f(x)$ of the form $f(x) = \sum_n \alpha_n\cos(\alpha\pi x)$ to get an unstable problem.

Implicit Finite Difference Method.

Given a heat equation

$$\begin{aligned} u_t &= u_{xx} \\ u(0,t) &= g(t), \quad t > 0 \\ u(1,t) &= h(t) \\ u(x,0) &= f(x), \quad 0 \le x \le 1. \end{aligned}$$

We discretize it by implicit difference method.

$$\frac{U_{i,j+1} - U_{i,j}}{\Delta t} = \frac{U_{i+1,j+1} - 2U_{i,j+1} + U_{i-1,j+1}}{\Delta x^2} \qquad i = 1, \ldots, N-1.$$

Multiply by Δt, then with $\lambda = \Delta t/\Delta x^2$, we have

$$\begin{aligned} U_{i,j+1} - U_{i,j} &= \lambda U_{i+1,j+1} - 2\lambda U_{i,j+1} + \lambda U_{i-1,j+1} \\ -U_{i,j} &= \lambda U_{i+1,j+1} - (1+2\lambda) U_{i,j+1} + \lambda U_{i-1,j+1} \qquad j = 1, \ldots, N-1. \end{aligned}$$

This yields a system of $N-1$ unknowns in $\{U_{i,j+1}\}_{i=1}^{N-1}$.

$$\begin{bmatrix} -\lambda U_{0,j+1} \\ 0 \\ \vdots \\ 0 \\ -\lambda U_{N,j+1} \end{bmatrix} + \begin{bmatrix} -U_{1,j} \\ \vdots \\ -U_{N-1,j} \end{bmatrix} = - \begin{bmatrix} (1+2\lambda) & -\lambda & 0 & & \\ -\lambda & (1+2\lambda) & & -\lambda & \\ 0 & \ddots & & \ddots & \\ & \ddots & \ddots & \ddots & -\lambda \\ & 0 & & -\lambda & (1+2\lambda) \end{bmatrix}$$

$$\times \begin{bmatrix} U_{1,j+1} \\ \vdots \\ U_{N-1,j+1} \end{bmatrix}$$

Theorem 8.4.7. *The implicit finite difference scheme is stable for all $\lambda = \Delta t/\Delta x^2$. (solution remains bounded).*

Proof. For each j, let $U_{k(j),j}$ be chosen so that $|U_{k(j),j}| \ge |U_{i,j}|$, $i = 1, \ldots, N-1$. We choose $i_0 = k(j+1)$ in the following relation.

$$U_{i,j+1} = U_{i,j} + \lambda\{U_{i+1,j+1} - 2U_{i,j+1} + U_{i-1,j+1}\}.$$

Then

$$(1+2\lambda)U_{i_0,j+1} = U_{i_0,j} + \lambda\{U_{i_0+1,j+1} + U_{i_0-1,j+1}\}$$

for $1 \le i \le N-1$. Taking absolute values,

$$(1+2\lambda)|U_{i_0,j+1}| \le |U_{i_0,j}| + \lambda(|U_{i_0+1,j+1}| + |U_{i_0-1,j+1}|) \le |U_{i_0,j}| + 2\lambda|U_{i_0,j+1}|.$$

Thus $|U_{i_0,j+1}| \le |U_{i_0,j}| \le |U_{k(j),j}|$ and hence $|U_{i,j+1}| \le |U_{i_0,j+1}| \le |U_{k(j),j}|$ for $1 \le i \le N-1$, and $|U_{i,j+1}| \le M = \max\{f,g,h\}$, for $i=0$ or N, by boundary condition. Repeat the same procedure until we hit the boundary.

$$|U_{i,j+1}| \le |U_{k(j),j}| \le \cdots \le |U_{k(0),0}| \le M = \max(f,g,h)$$

□

Using the matrix formulation: We check the eigenvalues of the system

$$A\mathbf{U_{j+1}} = \mathbf{U_j} + \mathbf{G_j}$$

Eigenvalue of A satisfies $|\mu + (1+2\lambda)| \le 2\lambda$ by G-disk theorem. From this, we see $|\mu| \ge 1$ and hence the eigenvalues of A^{-1} is less than one in absolute value. Thus

$$U_{j+1} \le A^{-1}(U_j + G_j) = \cdots = A^{-j-1}U_0 + A^{-j-1}G_0 + A^{-j-2}G_1 + \cdots + A^{-1}G_j.$$

$$\|U_{j+1}\| \le \|A^{-j-1}\|\,\|U_0\| + \|A^{-1}\| \cdot \frac{1}{1-\|A^{-1}\|}\max\|G_j\|$$

remain bounded. **Note.** A does not have -1 as eigenvalues and all the eigenvalues are positive real.

Theorem 8.4.8. *For sufficiently smooth u, we have*

$$|u_{ij} - U_{ij}| = \mathcal{O}(h^2 + k) \quad \textit{as} \quad h \quad \textit{and} \quad k \to 0 \qquad \textit{(for all } \lambda\textit{)}$$

Proof. Let $u_{ij} = u(x_i, t_j)$ be the true solution. Then we have

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{1}{h^2}\{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}\} + \mathcal{O}(h^2 + k)$$

Let $w_{i,j} = u_{i,j} - U_{i,j}$ be the discretization error. Then

$$\begin{aligned} w_{i,j+1} &= w_{i,j} + \lambda\{w_{i+1,j+1} - 2w_{i,j+1} + w_{i-1,j+1}\} + \mathcal{O}(kh^2 + k^2) \\ (1+2\lambda)w_{i,j+1} &= w_{i,j} + \lambda w_{i+1,j+1} + \lambda w_{i-1,j+1} + \mathcal{O}(kh^2 + k^2) \end{aligned}$$

Let $\|w_j\| = \max_i |w_{i,j}|$. Then

$$(1+2\lambda)|w_{i,j+1}| \leq \|w_j\| + 2\lambda\|w_{j+1}\| + \mathcal{O}(kh^2 + k^2)$$

and so

$$(1+2\lambda)\|w_{j+1}\| \leq \|w_j\| + 2\lambda\|w_{j+1}\| + \mathcal{O}(kh^2 + k^2).$$

Thus

$$\begin{aligned} \|w_{j+1}\| &\leq \|w_j\| + C(kh^2 + k^2) \\ &\leq \cdots \leq \|w_0\| + C(j+1)k(k+h^2) \\ &\leq \|w_0\| + CT(k+h^2) = CT(k+h^2) \end{aligned}$$

for $t = (j+1)k \leq T$. $\square$